Differential Equations Handbook

A Complete Reference for ODE and Introductory PDE

Contents

1 Classification of Differential Equations

A **differential equation** (DE) is a relation between an unknown function and its derivatives. Differential equations are the language of dynamical systems, appearing in physics, engineering, biology, economics, and throughout the applied sciences.

1.1 Order and Degree

Definition 1.1 (Order)**.** The **order** of a differential equation is the order of the highest derivative appearing in the equation.

Definition 1.2 (Degree)**.** The **degree** of a differential equation is the power of the highest-order derivative, provided the equation is a polynomial in its derivatives.

Worked Example

Determine the order and degree of the following equations:

1. $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} + 2\dfrac{\mathrm{d}y}{\mathrm{d}x} = x$

   The highest derivative is y''', so this is a **third-order** equation. The highest derivative appears to the first power, so the **degree is 1**.

2. $\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2 + \dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

   The highest derivative is y'', so this is **second-order**. The highest derivative is squared, so the **degree is 2**.

3. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^y + \sin x$

   Highest derivative is y', so **first-order**. It appears to the first power, so **degree 1**.

1.2 Linearity

Definition 1.3 (Linearity)**.** An n-th order differential equation is **linear** if the unknown function y and all of its derivatives appear to the first power and are not multiplied together. A linear n-th order DE has the general form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = g(x),$$

where $a_0, a_1, \ldots, a_n$ and g are functions of x only.

If any term violates these conditions, the equation is **nonlinear**.

Worked Example

Classify as linear or nonlinear:

1. $y'' + 3y' + 2y = \cos x$

   All derivatives appear to first power; coefficients depend only on x. **Linear.**

2. $y'' + y\,y' + y = 0$

   The term $y\,y'$ multiplies y by its derivative. **Nonlinear.**

3. $\sin(y) + y' = x$

   The function y appears inside a transcendental function. **Nonlinear.**

1.3 Homogeneity

Definition 1.4 (Homogeneity (linear DEs))**.** A linear differential equation is **homogeneous** if $g(x) = 0$; i.e., every term contains y or a derivative of y. It is **non-homogeneous** (or inhomogeneous) if $g(x) \neq 0$.

Worked Example

1. $y'' - 4y = 0 \quad \longrightarrow$ **Homogeneous linear, second-order.**
2. $y'' - 4y = e^{2x} \quad \longrightarrow$ **Non-homogeneous linear, second-order.**
3. $y' + y^2 = 0 \quad \longrightarrow$ **Nonlinear** (not classified as homogeneous/heterogeneous in the linear sense).

1.4 Classification Summary

Key Result

ODE Classification Checklist:

1. **Order:** What is the highest derivative? (e.g., $y^{(3)} \Rightarrow$ third order)
2. **Degree:** What power is the highest derivative raised to?
3. **Linearity:** Does y and every derivative appear only to the first power, never multiplied together?
4. **Homogeneity (linear only):** Is $g(x) = 0$? If yes, homogeneous.

Table 1: Classification of example differential equations

Equation	Order	Degree	Linear?	Homogeneous?
$y' + 2y = e^x$	1	1	Yes	No
$y'' - 3y' + 2y = 0$	2	1	Yes	Yes
$(y'')^2 + y' = 0$	2	2	No	—
$y' = xy + y^2$	1	1	No	—
$y^{(4)} + y'' = 0$	4	1	Yes	Yes

The remainder of this handbook develops solution methods organized by these classifications; see section 6 for first-order techniques and section 7 for qualitative methods.

2 Initial and Boundary Value Problems

A differential equation by itself typically admits a family of solutions. Additional conditions—initial or boundary values—restrict the family to a specific solution relevant to the physical or mathematical problem.

2.1 Initial Value Problems (IVP)

Definition 2.1 (Initial Value Problem)**.** An **initial value problem** (IVP) consists of a differential equation together with conditions specifying the value of the unknown function and/or its derivatives at a single point x_0:

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \qquad y(x_0) = y_0,\ y'(x_0) = y_0',\ \dots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

An n-th order IVP requires n initial conditions.

Worked Example

IVP examples:

1. First-order IVP:
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x, \qquad y(0) = 3.$$
Integrating: $y(x) = x^2 + C$. The condition $y(0) = 3$ gives $C = 3$, so $y(x) = x^2 + 3$.
2. Second-order IVP:
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x, \qquad y(0) = 1,\ y'(0) = -2.$$
Integrating twice: $y'(x) = 3x^2 + C_1$, $y(x) = x^3 + C_1x + C_2$. The conditions give $C_1 = -2$ and $C_2 = 1$, so $y(x) = x^3 - 2x + 1$.

IVPs model **evolution** problems: given the state of a system at an initial time, predict its future behavior. Typical applications include population growth, radioactive decay, projectile motion, and electrical circuits driven by initial charge/charge rate.

2.2 Boundary Value Problems (BVP)

Definition 2.2 (Boundary Value Problem)**.** A **boundary value problem** (BVP) consists of a differential equation together with conditions at two or more distinct points:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \qquad y(a) = \alpha,\ y(b) = \beta,\ \ldots$$

Worked Example

Second-order BVP:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0, \qquad y(0) = 0,\ y(1) = 1.$$

Integrating: $y(x) = C_1 x + C_2$. The conditions give $C_2 = 0$ and $C_1 = 1$, so $y(x) = x$.

BVPs typically arise in **steady-state** or **spatial** problems: heat distribution along a rod, deflection of a beam, or electrostatic potential between electrodes. Unlike IVPs, BVPs may have no solution, a unique solution, or infinitely many solutions.

2.3 Comparison

Table 2: IVP vs. BVP comparison

	IVP	**BVP**
Conditions at:	Single point x_0	Two or more distinct points
Typical problem:	Time evolution, dynamics	Steady state, spatial distribution
Existence:	Usually guaranteed (Picard–Lindelöf)	Not guaranteed
Uniqueness:	Usually guaranteed	May fail or have multiple solutions
Example:	$y' = y,\ y(0) = 1$	$y'' = 0,\ y(0) = 0,\ y(1) = 1$

3 Direction Fields

A direction field provides a geometric picture of the solution family of a first-order ODE without solving it analytically.

3.1 Construction

For a first-order ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, the direction field is constructed as follows:

1. Choose a rectangular grid of points (x_i, y_j) in the xy-plane.
2. At each grid point, compute the slope $m_{ij} = f(x_i, y_j)$.
3. Draw a short line segment centered at (x_i, y_j) with slope m_{ij}.

Solution curves are trajectories that are everywhere tangent to the direction field. Reading the field visually reveals qualitative features: equilibrium solutions (horizontal segments), asymptotic behavior, and regions of rapid growth or decay.

Hint

Look for **isoslopes**: curves along which $f(x, y)$ is constant. For $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$, isoslopes are the lines $y = x - c$, along which every segment has the same slope c.

3.2 Direction Field Diagram

The following TikZ figure shows the direction field for $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x - y$ over the region $[-2, 2] \times [-2, 2]$, together with two representative solution curves.

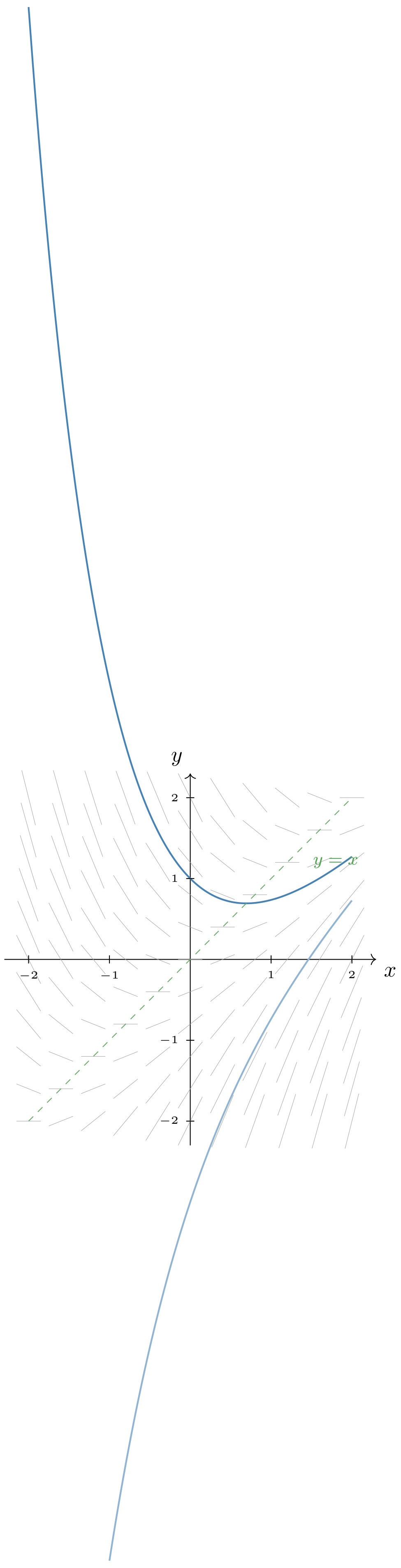


Figure 1: Direction field for $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$. Blue curves are solution trajectories. The dashed green line $y = x$ is the locus of zero slope (equilibrium for the shifted system).

3.3 Sketching Solution Curves

Given a direction field, individual solution curves can be sketched by hand:

1. **Trace the flow:** Start at a point and follow the segments, drawing a smooth curve tangent to the local arrows.
2. **Respect the grid:** The curve should never cross a segment at an angle significantly different from the segment's slope.
3. **Identify special features:** Equilibrium points (zero slope), asymptotes, and inflection points often emerge from the field pattern.

Worked Example

Sketch the solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$ passing through $(0, 1)$.

1. At $(0, 1)$, the slope is $f(0, 1) = 0 - 1 = -1$. Draw a segment with slope -1.
2. Moving right: at $(1, 0.5)$ (approximately), the slope is $1 - 0.5 = 0.5$. The curve bends upward.
3. The line $y = x$ is where the slope is zero. For $y > x$, the slope is negative (curves bend down toward $y = x$). For $y < x$, the slope is positive (curves bend up toward $y = x$).
4. Thus the line $y = x$ acts as an attractor. The solution starting at $(0, 1)$ approaches the line $y = x$ as $x \to \infty$.

Verification: The analytic solution is $y(x) = x - 1 + 2e^{-x}$. Indeed, $\lim_{x\to\infty}(y(x) - x) = -1$, and the transient $2e^{-x}$ decays, consistent with the field.

Direction fields also underpin numerical methods such as Euler's method (see section 7).

4 Existence and Uniqueness

A fundamental question before attempting to solve any differential equation is: *does a solution exist?* and *is it unique?* The Picard–Lindelöf theorem provides the foundational answer for first-order IVPs.

4.1 The Picard–Lindelöf Theorem

Theorem 4.1 (Picard–Lindelöf)**.** *Consider the first-order IVP*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \qquad y(x_0) = y_0.$$

Let $R = \{(x, y) : |x - x_0| \le a,\ |y - y_0| \le b\}$ be a closed rectangle centered at (x_0, y_0) with $a, b > 0$.

If the following two conditions hold:

1. ***Continuity:*** *$f(x, y)$ is continuous on R;*
2. ***Lipschitz condition:*** *There exists a constant $L > 0$ such that for all $(x, y_1), (x, y_2) \in R$,*

$$|f(x, y_1) - f(x, y_2)| \le L\,|y_1 - y_2|.$$

*Then there exists an interval $|x - x_0| \le h > 0$ (with $h \le a$) on which a **unique** solution $y = \varphi(x)$ exists.*

Remark 4.2. A **sufficient** (but not necessary) condition for the Lipschitz condition is that $\frac{\partial f}{\partial y}$ exists and is bounded on R. In that case, one may take $L = \sup_{(x,y)\in R} \left|\frac{\partial f}{\partial y}\right|$.

4.2 The Lipschitz Condition

The Lipschitz condition essentially requires that f does not change too rapidly with respect to y. Geometrically, it prevents solution curves from diverging too quickly from one another.

Worked Example

Verify the Lipschitz condition for $f(x,y) = 2xy$ on the rectangle $R = [-1,1] \times [-1,1]$.
Compute the partial derivative:

$$\frac{\partial f}{\partial y} = 2x.$$

On R, $|x| \leq 1$, so $\left|\frac{\partial f}{\partial y}\right| \leq 2$. Thus f satisfies a Lipschitz condition with $L = 2$ on R. By Theorem 4.1, the IVP $y' = 2xy,\ y(0) = 1$ has a unique solution on some interval around $x = 0$.

Worked Example

Non-uniqueness when the Lipschitz condition fails.
Consider the IVP

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{2/3}, \qquad y(0) = 0.$$

Here $f(y) = y^{2/3}$. The partial derivative with respect to y is

$$\frac{\partial f}{\partial y} = \tfrac{2}{3}\, y^{-1/3},$$

which is **unbounded** as $y \to 0$. The Lipschitz condition fails in any rectangle containing $y = 0$.
Indeed, there are infinitely many solutions:

$$y(x) = 0 \quad \text{(trivial solution)},$$

and for any $c \geq 0$,

$$y(x) = \begin{cases} 0, & 0 \leq x \leq c, \\ \dfrac{(x-c)^3}{27}, & x > c, \end{cases}$$

all satisfy the IVP. The solution branches off from the trivial solution at any chosen point c.

Hint

A quick test for potential non-uniqueness: check whether $\frac{\partial f}{\partial y}$ blows up at or near the initial condition point. If it does, the Lipschitz condition likely fails.

4.3 Interval of Validity

Definition 4.3 (Interval of Validity)**.** The **interval of validity** (or interval of existence) of a solution $y = \varphi(x)$ to an IVP is the **largest** open interval I containing x_0 on which the solution exists and is differentiable.

The interval of validity may be restricted by:

- Singularities in the coefficients of the equation;
- Points where the Lipschitz condition breaks down;
- Points where the solution itself becomes unbounded (blow-up).

Worked Example

Find the interval of validity for $y' = y^2,\ y(0) = 1$.
This is separable: $\dfrac{\mathrm{d}y}{y^2} = \mathrm{d}x \ \Rightarrow\ -\dfrac{1}{y} = x + C$. With $y(0) = 1$, we get $C = -1$, so $y(x) = \frac{1}{1-x}$.
The solution has a vertical asymptote at $x = 1$. Since the initial point is $x_0 = 0$, the interval of validity is

$$(-\infty, 1).$$

The solution blows up at $x = 1$, so it cannot be extended beyond this point.

Worked Example

Interval of validity for $y' = \sqrt{y},\ y(0) = 1$.

Separating: $\dfrac{\mathrm{d}y}{\sqrt{y}} = \mathrm{d}x \Rightarrow 2\sqrt{y} = x + C$. With $y(0) = 1$, we get $C = 2$, so $y(x) = \left(\frac{x+2}{2}\right)^2 = \frac{(x+2)^2}{4}$.

To find the interval of validity, check that the solution satisfies the original ODE: $y' = \frac{x+2}{2}$ while $\sqrt{y} = \frac{|x+2|}{2}$. These are equal only when $x + 2 \geq 0$, i.e., $x \geq -2$. For $x < -2$, we have $y' < 0$ while $\sqrt{y} \geq 0$, so the formula fails the ODE. At $x = -2$, the solution reaches $y = 0$ (the equilibrium). Therefore, the interval of validity is

$$(-2, \infty).$$

Note that although the Lipschitz condition fails at $y = 0$ (since $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ is unbounded there), the particular solution passing through $y(0) = 1$ is valid on $(-2, \infty)$, and uniqueness is preserved on this interval.

5 Summary

Table 3: Chapter 1 summary: key concepts in differential equations

Concept	**Key Definition / Formula**
Order	Highest derivative present (e.g., y''' $\Rightarrow$ 3rd order)
Degree	Power of the highest-order derivative when polynomial in derivatives
Linear DE	$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$
Nonlinear DE	y or derivatives appear to non-integer powers, multiplied together, or inside nonlinear functions
Homogeneous (linear)	$g(x) = 0$ in the linear form
IVP	DE + conditions at a single point: $y(x_0) = y_0, \ldots$
BVP	DE + conditions at two or more distinct points
Direction field	Plot of short line segments with slope $f(x_i, y_j)$ at grid points
Picard–Lindelöf	Continuity + Lipschitz $\Rightarrow$ local existence and uniqueness
Lipschitz condition	$\lvert f(x,y_1) - f(x,y_2)\rvert \leq L\lvert y_1 - y_2\rvert$; sufficient: $\left\lvert\frac{\partial f}{\partial y}\right\rvert \leq L$
Interval of validity	Largest interval containing x_0 where the solution exists and is unique
Non-uniqueness example	$y' = y^{2/3}$, $y(0) = 0$ admits infinitely many solutions

The chapters that follow build directly on these foundations. Solution methods for first-order equations are developed in section 6; qualitative analysis and numerical methods appear in section 7; and linear second-order theory begins in section 8.

6 First-Order Methods

6.1 Separable Equations

A first-order ODE is **separable** when it can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,h(y) \tag{1}$$

i.e. the right-hand side factors into a function of x times a function of y.

Key Result

Separable equations. Rewrite as $\dfrac{\mathrm{d}y}{h(y)} = g(x)\,\mathrm{d}x$ and integrate both sides:

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x + C.$$

Solve for y explicitly when possible; otherwise leave the solution in implicit form.

Hint

If $h(y_0) = 0$, the constant function $y(x) = y_0$ is always a solution (an *equilibrium*). These may be lost when dividing by $h(y)$, so check separately.

Worked Example

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\,y$.

Solution. Separate variables:

$$\frac{\mathrm{d}y}{y} = x\,\mathrm{d}x.$$

Integrate:

$$\ln|y| = \frac{x^2}{2} + C.$$

Exponentiate:

$$|y| = e^C\,e^{x^2/2} \quad \Longrightarrow \quad y = C_1\,e^{x^2/2},$$

where $C_1 = \pm e^C$ is an arbitrary nonzero constant. Including the equilibrium $y = 0$ (lost when dividing by y), the general solution is

$$y(x) = C\,e^{x^2/2}, \qquad C \in \mathbb{R}.$$

Worked Example

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x}{3y^2}$.

Solution. Separate:

$$3y^2\,\mathrm{d}y = 2x\,\mathrm{d}x.$$

Integrate:

$$y^3 = x^2 + C.$$

This implicit form is perfectly acceptable. Solving explicitly gives $y = \sqrt[3]{x^2 + C}$.

6.2 Linear First-Order Equations

A first-order ODE is **linear** if it can be written in the *standard form*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,y = g(x). \tag{2}$$

The coefficient of $\mathrm{d}y/\mathrm{d}x$ **must be** 1; if the equation arrives with a leading coefficient $a(x)$, divide through first.

Key Result

Integrating factor. For equation (2), the integrating factor is

$$\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

Multiplying the equation by $\mu(x)$ produces

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mu(x)\,y\right] = \mu(x)\,g(x),$$

so the general solution is

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(x)\, g(x)\, \mathrm{d}x + C \right).$$

Derivation. With $\mu(x) = \exp\bigl(\int p(x)\, \mathrm{d}x\bigr)$, the chain rule gives $\mu'(x) = \mu(x)\, p(x)$. Multiplying equation (2) by $\mu(x)$:

$$\mu y' + \mu p\, y = \mu g \quad \Longrightarrow \quad \mu y' + \mu' y = \mu g.$$

The left side is exactly the product-rule derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}[\mu y]$, yielding the result.

Hint

Pitfall. If the equation is $a(x)y' + b(x)y = f(x)$ with $a(x) \neq 1$, you *must* divide by $a(x)$ to obtain standard form before computing $\mu(x)$. Forgetting this is the most common error.

Worked Example

Solve $y' + \dfrac{2}{x}\, y = x^3$.

Solution. The equation is already in standard form with $p(x) = \dfrac{2}{x}$ and $g(x) = x^3$. Compute the integrating factor:

$$\mu(x) = \exp\left(\int \frac{2}{x}\, \mathrm{d}x \right) = \exp\bigl(2 \ln |x|\bigr) = |x|^2 = x^2.$$

(We drop the absolute value since $x^2 \geq 0$.)
Multiply the entire equation by x^2:

$$x^2 y' + 2x\, y = x^5 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}[x^2 y] = x^5.$$

Integrate:

$$x^2 y = \frac{x^6}{6} + C.$$

Divide by x^2:

$$y(x) = \frac{x^4}{6} + \frac{C}{x^2}.$$

Worked Example

Solve $2x\, y' + 3y = 6x$.
Solution. The leading coefficient is $2x \neq 1$. Divide through:

$$y' + \frac{3}{2x}\, y = 3.$$

Now $p(x) = \dfrac{3}{2x}$ and $g(x) = 3$. Integrating factor:

$$\mu(x) = \exp\left(\int \frac{3}{2x}\, \mathrm{d}x \right) = \exp\bigl(\tfrac{3}{2} \ln |x|\bigr) = |x|^{3/2}.$$

Work with $x > 0$ for simplicity, so $\mu(x) = x^{3/2}$. Multiply:

$$x^{3/2} y' + \tfrac{3}{2} x^{1/2} y = 3x^{3/2} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{3/2} y\right] = 3x^{3/2}.$$

Integrate:

$$x^{3/2} y = \int 3x^{3/2}\, \mathrm{d}x = 3 \cdot \frac{2}{5} x^{5/2} + C = \frac{6}{5} x^{5/2} + C.$$

Solve for y:

$$y(x) = \frac{6}{5} x + \frac{C}{x^{3/2}}.$$

6.3 Exact Equations

An ODE written in differential form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{3}$$

is **exact** if there exists a scalar function $\psi(x,y)$ such that $\frac{\partial\psi}{\partial x} = M$ and $\frac{\partial\psi}{\partial y} = N$. The solution is then given implicitly by $\psi(x,y) = C$.

Key Result

Exactness test. The equation is exact *if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

When exact, the potential function is

$$\psi(x,y) = \int M\,\mathrm{d}x \;+\; \int \left[N - \frac{\partial}{\partial y}\left(\int M\,\mathrm{d}x\right)\right]\mathrm{d}y.$$

The solution is $\psi(x,y) = C$.

Worked Example

Solve $(2xy + y^3)\,\mathrm{d}x + (x^2 + 3xy^2)\,\mathrm{d}y = 0$.
Solution. Here $M = 2xy + y^3$ and $N = x^2 + 3xy^2$. Check exactness:

$$\frac{\partial M}{\partial y} = 2x + 3y^2, \qquad \frac{\partial N}{\partial x} = 2x + 3y^2.$$

They match, so the equation is exact.
Integrate M with respect to x:

$$\int (2xy + y^3)\,\mathrm{d}x = x^2y + xy^3 + h(y).$$

Differentiate with respect to y and equate to N:

$$\frac{\partial}{\partial y}\left[x^2y + xy^3 + h(y)\right] = x^2 + 3xy^2 + h'(y) \stackrel{!}{=} x^2 + 3xy^2.$$

Thus $h'(y) = 0$, so $h(y)$ is constant. The potential function is

$$\psi(x,y) = x^2y + xy^3.$$

The implicit solution is

$$x^2y + xy^3 = C.$$

Integrating factors for non-exact equations. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation can sometimes be made exact by multiplying by an integrating factor μ.

Key Result

Integrating factors.

- If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ depends *only on* x, then $\mu(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,\mathrm{d}x\right)$.
- If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ depends *only on* y, then $\mu(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,\mathrm{d}y\right)$.

Worked Example

Solve $(3xy + y^2)\,\mathrm{d}x + (x^2 + xy)\,\mathrm{d}y = 0$.

Solution. $M = 3xy + y^2$, $N = x^2 + xy$. Check:

$$\frac{\partial M}{\partial y} = 3x + 2y, \qquad \frac{\partial N}{\partial x} = 2x + y.$$

Not exact. Test for an integrating factor depending on x alone:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(3x+2y) - (2x+y)}{x^2 + xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}.$$

This depends only on x, so

$$\mu(x) = \exp\left(\int \frac{1}{x}\,\mathrm{d}x\right) = \exp(\ln|x|) = x.$$

Multiply the original equation by $\mu = x$:

$$(3x^2y + xy^2)\,\mathrm{d}x + (x^3 + x^2y)\,\mathrm{d}y = 0.$$

Verify: $\frac{\partial \tilde{M}}{\partial y} = 3x^2 + 2xy = \frac{\partial \tilde{N}}{\partial x}$. Exact.
Integrate $\tilde{M}$ with respect to x:

$$\int (3x^2y + xy^2)\,\mathrm{d}x = x^3y + \tfrac{1}{2}x^2y^2 + h(y).$$

Differentiate with respect to y:

$$x^3 + x^2y + h'(y) \overset{!}{=} x^3 + x^2y \quad \Longrightarrow \quad h'(y) = 0.$$

The potential is $\psi(x,y) = x^3y + \dfrac{1}{2}x^2y^2$, and the solution is

$$x^3y + \tfrac{1}{2}x^2y^2 = C.$$

6.4 Bernoulli Equations

A **Bernoulli equation** has the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)\,y = g(x)\,y^n, \qquad n \neq 0, 1. \tag{4}$$

For $n = 0$ the equation is linear; for $n = 1$ it is also linear. The substitution $v = y^{1-n}$ transforms it into a linear equation.

Key Result

Bernoulli substitution. Let $v = y^{1-n}$. Then

$$v' = (1-n)\,y^{-n}\,y',$$

and substituting into the Bernoulli equation yields the *linear* equation in v:

$$v' + (1-n)\,p(x)\,v = (1-n)\,g(x).$$

Worked Example

Solve $y' + \dfrac{1}{x}\,y = x\,y^3$.

Solution. This is Bernoulli with $p(x) = \dfrac{1}{x}$, $g(x) = x$, and $n = 3$. Substitute $v = y^{1-3} = y^{-2}$. Then $v' = -2y^{-3}y'$.
Multiply the original equation by $-2y^{-3}$:

$$-2y^{-3}y' - 2y^{-3} \cdot \frac{1}{x}\,y = -2y^{-3} \cdot x\,y^3.$$

Recognizing $v = y^{-2}$ and $v' = -2y^{-3}y'$:

$$v' - \frac{2}{x}\,v = -2x.$$

This is a linear equation in v. Integrating factor:

$$\mu(x) = \exp\!\left(\int -\frac{2}{x}\,\mathrm{d}x\right) = \exp(-2\ln|x|) = \frac{1}{x^2}.$$

Multiply by μ:

$$\frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{2}{x} \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{v}{x^2}\right] = -\frac{2}{x}.$$

Integrate:

$$\frac{v}{x^2} = -2\ln|x| + C \quad\Longrightarrow\quad v = x^2\bigl(C - 2\ln|x|\bigr).$$

Substitute back $v = y^{-2}$:

$$\frac{1}{y^2} = x^2\bigl(C - 2\ln|x|\bigr) \quad\Longrightarrow\quad y(x) = \pm\frac{1}{x\sqrt{C - 2\ln|x|}}.$$

Worked Example

Solve $y' - \dfrac{2}{x}\,y = x^2\,y^2$.

Solution. Bernoulli with $n = 2$, $p(x) = -\dfrac{2}{x}$, $g(x) = x^2$. Substitute $v = y^{1-2} = y^{-1} = \dfrac{1}{y}$. Then $v' = -y^{-2}y'$. Multiply the original equation by $-y^{-2}$:

$$-y^{-2}y' + \frac{2}{x}\,y^{-1} = -x^2.$$

In terms of v:

$$v' + \frac{2}{x}\,v = -x^2.$$

Integrating factor:

$$\mu(x) = \exp\!\left(\int \frac{2}{x}\,\mathrm{d}x\right) = x^2.$$

Multiply:

$$x^2v' + 2xv = -x^4 \quad\Longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2v\right] = -x^4.$$

Integrate:

$$x^2v = -\frac{x^5}{5} + C \quad\Longrightarrow\quad v = -\frac{x^3}{5} + \frac{C}{x^2}.$$

Since $v = 1/y$:

$$y(x) = \frac{1}{-\dfrac{x^3}{5} + \dfrac{C}{x^2}} = \frac{5x^2}{C - x^5}.$$

6.5 Homogeneous Substitutions

A first-order ODE is **homogeneous** (of degree zero) if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\!\left(\frac{y}{x}\right). \tag{5}$$

That is, the right-hand side is a function of the ratio y/x alone.

Key Result

Homogeneous substitution. Let $y = vx$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}.$$

The equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = F(v) \quad\Longrightarrow\quad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{F(v) - v}{x},$$

which is separable:

$$\int \frac{\mathrm{d}v}{F(v) - v} = \int \frac{\mathrm{d}x}{x} = \ln|x| + C.$$

Worked Example

Solve $y' = \dfrac{x+y}{x-y}$.

Solution. Rewrite the right side:

$$\frac{x+y}{x-y} = \frac{1 + y/x}{1 - y/x} = F\Big(\frac{y}{x}\Big).$$

The equation is homogeneous. Set $y = vx$, so $y' = v + xv'$:

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v}{1-v}.$$

Isolate the derivative:

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+v}{1-v} - v = \frac{1 + v - v(1-v)}{1-v} = \frac{1+v^2}{1-v}.$$

Separate:

$$\frac{1-v}{1+v^2}\,\mathrm{d}v = \frac{\mathrm{d}x}{x}.$$

Split the left integrand:

$$\int \frac{1}{1+v^2}\,\mathrm{d}v - \int \frac{v}{1+v^2}\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

These are standard integrals:

$$\arctan(v) - \tfrac{1}{2}\ln(1+v^2) = \ln|x| + C.$$

Substitute back $v = y/x$:

$$\arctan\Big(\frac{y}{x}\Big) - \tfrac{1}{2}\ln\Big(1 + \frac{y^2}{x^2}\Big) = \ln|x| + C.$$

Using $\ln(1 + y^2/x^2) = \ln\big((x^2+y^2)/x^2\big) = \ln(x^2+y^2) - 2\ln|x|$, the solution simplifies to

$$\arctan\Big(\frac{y}{x}\Big) - \tfrac{1}{2}\ln(x^2+y^2) = C.$$

Worked Example

Solve $y' = \dfrac{xy + y^2}{x^2}$.

Solution. Rewrite:

$$y' = \frac{y}{x} + \frac{y^2}{x^2} = F\Big(\frac{y}{x}\Big).$$

Homogeneous with $F(v) = v + v^2$. Set $y = vx$:

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v + v^2 \quad\Longrightarrow\quad x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2.$$

Separate:

$$\frac{\mathrm{d}v}{v^2} = \frac{\mathrm{d}x}{x}.$$

Integrate:

$$-\frac{1}{v} = \ln|x| + C \quad\Longrightarrow\quad v = -\frac{1}{\ln|x| + C}.$$

Substitute back $v = y/x$:

$$y(x) = -\frac{x}{\ln|x| + C}.$$

The equilibrium solution $y = 0$ (corresponding to $v = 0$) is also valid.

6.6 Summary

The following flowchart guides method selection. Test methods in the order shown; the first applicable method solves the equation.

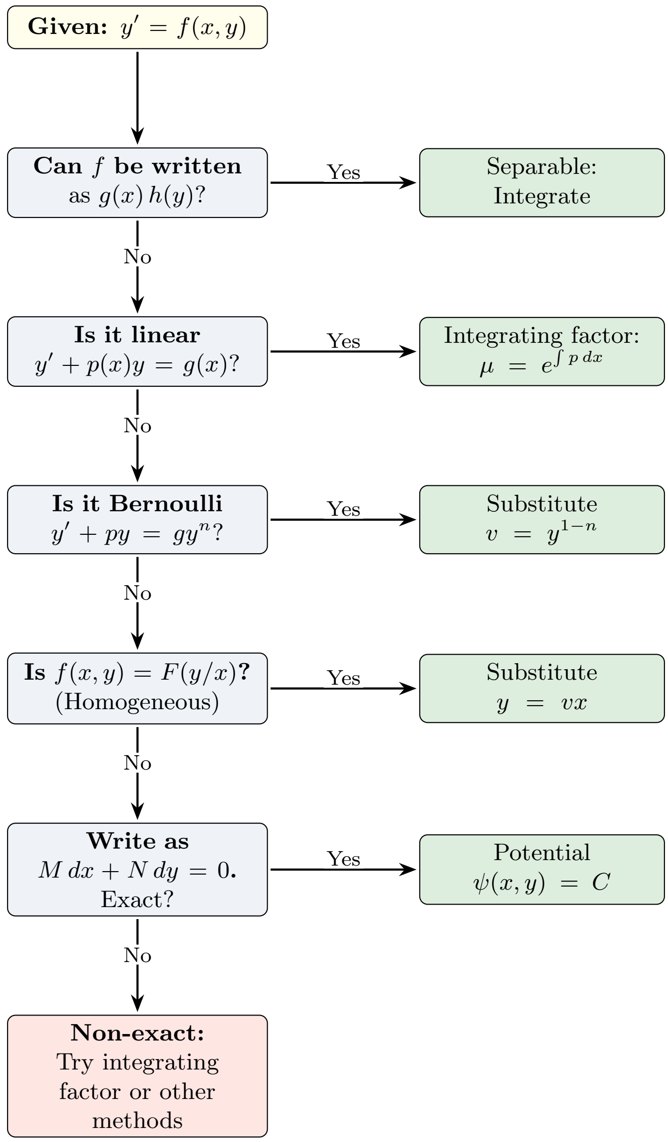


Table 4: First-order solution methods

Type	**Form**	**Method**
Separable	$y' = g(x)h(y)$	Rewrite as $\frac{\mathrm{d}y}{h(y)} = g(x)\,\mathrm{d}x$; integrate both sides
Linear	$y' + p(x)y = g(x)$	Integrating factor $\mu(x) = \exp\left(\int p(x)\,\mathrm{d}x\right)$
Exact	$M\,\mathrm{d}x + N\,\mathrm{d}y = 0$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	Find potential ψ with $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$; $\psi = C$
Non-exact	$M\,\mathrm{d}x + N\,\mathrm{d}y = 0$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$	Multiply by integrating factor $\mu(x)$ or $\mu(y)$
Bernoulli	$y' + p(x)y = g(x)y^n$	Substitute $v = y^{1-n}$; solve resulting linear equation
Homogeneous	$y' = F(y/x)$	Substitute $y = vx$; solve resulting separable equation

7 Qualitative Analysis and Numerical Methods

7.1 Autonomous Equations and Phase Lines

An **autonomous differential equation** is a first-order ODE in which the right-hand side depends only on the dependent variable and *not* explicitly on the independent variable t:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y). \tag{6}$$

Because f has no explicit t-dependence, the direction field is invariant under horizontal translation. Solutions simply shift left or right along the t-axis without changing shape.

Equilibrium solutions. An **equilibrium solution** (or **critical point**) is a constant solution $y(t) \equiv y^*$ that satisfies

$$f(y^*) = 0. \tag{7}$$

At such a point the derivative vanishes and the solution is stationary. Every root of $f(y) = 0$ gives one equilibrium.

Phase line construction. The **phase line** is a one-dimensional sketch of the dynamics of equation (6). It is constructed in three steps:

1. **Identify equilibria:** Solve $f(y) = 0$.
2. **Determine the sign of** $f(y)$ on each interval between consecutive equilibria. If $f(y) > 0$ on an interval, solutions increase (y moves upward on the phase line). If $f(y) < 0$, solutions decrease.
3. **Draw the phase line:** Mark each equilibrium on a vertical line and place arrows on each interval indicating the direction of motion.

Worked Example

Phase line for $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2)$.

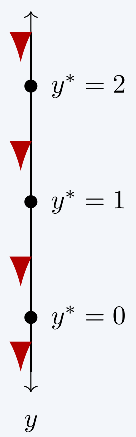
Step 1: Equilibria. Set $f(y) = y(y-1)(y-2) = 0$. The roots are

$$y_1^* = 0, \qquad y_2^* = 1, \qquad y_3^* = 2.$$

Step 2: Sign of $f(y)$ on each interval. Test a sample point in each interval:

Interval	Sample y	$f(y)$
$(-\infty, 0)$	$y = -1$	$(-)(-)(-) = -$ (decreasing)
$(0, 1)$	$y = 0.5$	$(+)(-)(-) = +$ (increasing)
$(1, 2)$	$y = 1.5$	$(+)(+)(-) = -$ (decreasing)
$(2, \infty)$	$y = 3$	$(+)(+)(+) = +$ (increasing)

Step 3: Phase line.



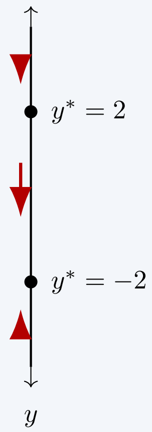
Solutions starting between 0 and 1 approach $y = 0$ from above or $y = 1$ from below as t increases.

Worked Example

Phase line for $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2 - 4$.
Step 1: $y^2 - 4 = 0 \implies y^* = -2,\ +2$.
Step 2: Test intervals:

Interval	Sample y	$f(y)$
$(-\infty, -2)$	$y = -3$	$9 - 4 = +5$ (increasing)
$(-2, 2)$	$y = 0$	$0 - 4 = -4$ (decreasing)
$(2, \infty)$	$y = 3$	$9 - 4 = +5$ (increasing)

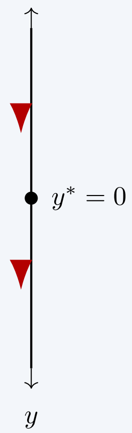
Step 3:



The interval $(-2, 2)$ is a "funnel" where all solutions decrease toward $y = -2$ as $t \to \infty$.

Worked Example

Phase line for $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y^2$.
Step 1: $y^2 = 0 \implies y^* = 0$ (the only equilibrium).
Step 2: For $y \neq 0$, $y^2 > 0$, so $f(y) > 0$ on both $(-\infty, 0)$ and $(0, \infty)$. Solutions increase on both sides of the equilibrium.
Step 3:



Solutions approach $y^* = 0$ from below but depart from above. This is the hallmark of a **semi-stable** equilibrium (discussed in section 7.2).

7.2 Stability Analysis

Linear stability test. Suppose y^* is an equilibrium of $\mathrm{d}y/\mathrm{d}t = f(y)$. Expand f in a Taylor series about y^*:

$$f(y) = f(y^*) + f'(y^*)(y - y^*) + \frac{f''(y^*)}{2}(y - y^*)^2 + \cdots. \tag{8}$$

Since $f(y^*) = 0$, the linearized equation near y^* is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f'(y^*)\,u, \qquad u = y - y^*. \tag{9}$$

The solution is $u(t) = u(0)\,e^{f'(y^*)t}$, which immediately yields three cases.

Key Result

Stability classification for autonomous equations $\mathrm{d}y/\mathrm{d}t = f(y)$**.**
Let y^* be an equilibrium with $f(y^*) = 0$.

Type	**Condition**	**Behavior**
Asymptotically stable (sink)	$f'(y^*) < 0$	Nearby solutions converge to y^* as $t \to \infty$.
Unstable (source)	$f'(y^*) > 0$	Nearby solutions depart from y^* as t increases.
Semi-stable (node)	$f'(y^*) = 0$	Solutions approach y^* from one side and depart on the other.

Worked Example

Classify the equilibria of $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2)$**.**
We have $f(y) = y(y-1)(y-2) = y^3 - 3y^2 + 2y$, so

$$f'(y) = 3y^2 - 6y + 2.$$

Evaluate at each equilibrium:

$$\begin{aligned}
f'(0) &= 3(0)^2 - 6(0) + 2 = 2 > 0 && \implies y^* = 0 \text{ is } \textbf{unstable (source)}.\\
f'(1) &= 3(1)^2 - 6(1) + 2 = -1 < 0 && \implies y^* = 1 \text{ is } \textbf{asymptotically stable (sink)}.\\
f'(2) &= 3(4) - 12 + 2 = 2 > 0 && \implies y^* = 2 \text{ is } \textbf{unstable (source)}.
\end{aligned}$$

These classifications are consistent with the phase line drawn in the previous section: $y = 1$ attracts solutions from both sides, while $y = 0$ and $y = 2$ repel.

Worked Example

Semi-stable equilibrium: $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2$**.**
Here $f(y) = y^2$ and $f'(y) = 2y$. At the equilibrium $y^* = 0$:

$$f'(0) = 0.$$

The linear test is inconclusive (derivative is zero), so we examine the sign of $f(y)$ directly: $f(y) = y^2 > 0$ for all $y \neq 0$. Solutions approach $y^* = 0$ from below but depart to the right from above. This confirms the **semi-stable (node)** classification.

Worked Example

All three types in one equation: $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)^2$**.**
Equilibria: $y^* = 0$ and $y^* = 1$.
Compute $f'(y) = (y-1)^2 + 2y(y-1) = (y-1)(3y-1)$.

$$\begin{aligned}
f'(0) &= (-1)(-1) = 1 > 0 && \implies y^* = 0 \text{ is } \textbf{unstable (source)}.\\
f'(1) &= (0)(2) = 0 && \implies \text{linear test inconclusive.}
\end{aligned}$$

For $y^* = 1$, inspect the sign of $f(y) = y(y-1)^2$: Since $(y-1)^2 \geq 0$ always and $y > 0$ near $y = 1$, we have $f(y) > 0$ on both sides of $y = 1$. Solutions increase on both sides, so $y^* = 1$ is **semi-stable** (approached from below, departed from above).

7.3 The Logistic Equation

Biological motivation. The simplest population model is exponential growth, $\mathrm{d}P/\mathrm{d}t = rP$, which predicts unbounded growth. In reality, resources are finite. The **logistic equation** introduces a carrying capacity $K > 0$ that caps the population:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\,y\left(1 - \frac{y}{K}\right), \qquad r > 0,\ K > 0. \tag{10}$$

The factor $(1-y/K)$ reduces the per-capita growth rate as $y \to K$. When $y \ll K$ the dynamics are approximately exponential; when $y \to K$ the growth rate vanishes.

Phase line analysis. Set $f(y) = ry(1 - y/K)$. The equilibria are:

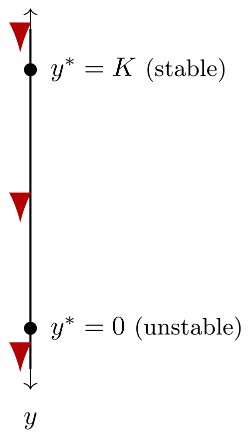
$$f(y) = 0 \implies y\left(1 - \frac{y}{K}\right) = 0 \implies y_1^* = 0, \quad y_2^* = K.$$

Compute the derivative:

$$f'(y) = r - \frac{2r}{K}y.$$

Evaluate at each equilibrium:

$$f'(0) = r > 0 \ \text{(unstable source)}, \qquad f'(K) = r - 2r = -r < 0 \ \text{(asymptotically stable sink)}.$$



All solutions with $y(0) > 0$ approach the carrying capacity K as $t \to \infty$.

Exact solution by separation of variables. We solve equation (10) with initial condition $y(0) = y_0$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\,y\left(1 - \frac{y}{K}\right)$$

$$\frac{\mathrm{d}y}{y\left(1 - \frac{y}{K}\right)} = r\,\mathrm{d}t.$$

Decompose the left-hand side using partial fractions:

$$\frac{1}{y\left(1 - \frac{y}{K}\right)} = \frac{K}{Ky - y^2} = \frac{1}{y} + \frac{1}{K-y}.$$

Verification: $\dfrac{1}{y} + \dfrac{1}{K-y} = \dfrac{K-y+y}{y(K-y)} = \dfrac{K}{y(K-y)}.$

Integrate both sides:

$$\begin{aligned}
\int \left(\frac{1}{y} + \frac{1}{K-y}\right) \mathrm{d}y &= \int r\,\mathrm{d}t \\
\ln|y| - \ln|K-y| &= rt + C \\
\ln\left|\frac{y}{K-y}\right| &= rt + C \\
\frac{y}{K-y} &= A\,e^{rt}, \qquad A = \pm e^{C}.
\end{aligned}$$

Solve for y:

$$\begin{aligned} y &= (K-y)\,A\,e^{rt} \\ y(1+Ae^{rt}) &= KAe^{rt} \\ y(t) &= \frac{KAe^{rt}}{1+Ae^{rt}} = \frac{K}{1+\dfrac{1}{A}e^{-rt}}. \end{aligned}$$

Apply the initial condition $y(0) = y_0$:

$$y_0 = \frac{K}{1+\frac{1}{A}} \implies 1+\tfrac{1}{A} = \frac{K}{y_0} \implies \frac{1}{A} = \frac{K-y_0}{y_0}.$$

Denoting $A' = \dfrac{K-y_0}{y_0}$, the final form is

$$y(t) = \frac{K}{1+A'e^{-rt}}, \qquad A' = \frac{K-y_0}{y_0}. \tag{11}$$

As $t \to \infty$, $e^{-rt} \to 0$ and $y(t) \to K$, confirming the phase line prediction.

Worked Example

Logistic growth with $r = 0.5$, $K = 100$, $y_0 = 10$.
The equation is $\mathrm{d}y/\mathrm{d}t = 0.5\,y(1-y/100)$.
The constant A' is

$$A' = \frac{100-10}{10} = 9.$$

The solution is

$$y(t) = \frac{100}{1+9\,e^{-0.5t}}.$$

At $t = 5$:

$$y(5) = \frac{100}{1+9\,e^{-2.5}} = \frac{100}{1+9\cdot 0.0821} = \frac{100}{1.739} \approx 57.5.$$

The population reaches about 58% of its carrying capacity after 5 time units.

7.4 Euler's Method

Many differential equations do not admit closed-form solutions. **Euler's method** provides a simple numerical procedure to approximate the solution.

Derivation from Taylor expansion. Let $y(t)$ be the exact solution of $y' = f(t,y)$. Expand $y(t_{n+1})$ about t_n using Taylor's theorem:

$$y(t_{n+1}) = y(t_n) + h\,y'(t_n) + \frac{h^2}{2}\,y''(\xi_n), \qquad \xi_n \in (t_n, t_{n+1}), \tag{12}$$

where $h = t_{n+1} - t_n$ is the step size. Since $y'(t_n) = f(t_n, y(t_n))$,

$$y(t_{n+1}) = y(t_n) + h\,f(t_n, y(t_n)) + \frac{h^2}{2}\,y''(\xi_n).$$

Euler's method retains only the first two terms, discarding the $O(h^2)$ remainder.

Key Result

Euler's method. Given $y' = f(t,y)$ with $y(t_0) = y_0$ and step size h:

$$t_{n+1} = t_n + h, \tag{13}$$
$$y_{n+1} = y_n + h\,f(t_n, y_n). \tag{14}$$

Hint

Error analysis.

- **Local truncation error** (error per step):

$$\tau_{n+1} = \frac{h^2}{2}\, y''(\xi_n) = O(h^2).$$

- **Global truncation error** (accumulated error at a fixed t after $N = (t - t_0)/h$ steps):

$$\varepsilon_N = O(h).$$

Halving the step size roughly halves the global error, but doubles the number of steps.

Worked Example

Approximate $y' = y$, $y(0) = 1$ using Euler's method with $h = 0.1$.
The exact solution is $y(t) = e^t$. Here $f(t, y) = y$, so the iteration is

$$y_{n+1} = y_n + 0.1\, y_n = 1.1\, y_n.$$

Starting from $y_0 = 1$:

n	t_n	y_n (Euler)	$y(t_n) = e^{t_n}$ (exact)	Local error τ	Cumulative error
0	0.0	1.0000	1.00000	–	0.0000
1	0.1	1.1000	1.10517	0.00059	0.0052
2	0.2	1.2100	1.22140	0.00061	0.0114
3	0.3	1.3310	1.34986	0.00063	0.0189
4	0.4	1.4641	1.49182	0.00066	0.0277
5	0.5	1.6105	1.64872	0.00069	0.0382
6	0.6	1.7716	1.82212	0.00072	0.0505
7	0.7	1.9487	2.01375	0.00075	0.0650
8	0.8	2.1436	2.22554	0.00079	0.0819
9	0.9	2.3579	2.45960	0.00083	0.1017
10	1.0	2.5937	2.71828	0.00087	0.1246

At $t = 1.0$, Euler's method gives $y_{10} = 2.5937$ while the exact value is $e \approx 2.7183$. The relative error is

$$\frac{|2.71828 - 2.59374|}{2.71828} \approx 4.58\%.$$

The error grows because each step's $O(h^2)$ local error compounds. With the same $h = 0.1$, the error at $t = 1$ is proportional to $h \cdot t \approx 0.1$. To reduce the error by a factor of 10, use $h = 0.01$.

Worked Example

Euler's method on $y' = -2y$, $y(0) = 3$, with $h = 0.5$.
Here $f(t, y) = -2y$, so $y_{n+1} = y_n - h(2y_n) = y_n(1 - 2h)$.
With $h = 0.5$: $y_{n+1} = y_n(1 - 1) = 0$ for $n \geq 1$. This is an **unstable** outcome: any numerical noise would blow up because $|1 - 2h| = 0$ is at the boundary.
The exact solution is $y(t) = 3e^{-2t}$. At $t = 1$: $y(1) = 3e^{-2} \approx 0.406$.
Euler with $h = 0.5$ gives $y_2 = 0$ at $t = 1$, which is far off.
With a smaller step $h = 0.1$: $y_{n+1} = y_n(1 - 0.2) = 0.8\, y_n$.

$$y_{10} = 3 \cdot (0.8)^{10} = 3 \cdot 0.1074 \approx 0.322.$$

This is much closer to the exact 0.406.
Lesson: For stiff equations or large negative eigenvalues, small step sizes are essential for Euler's method.

7.5 Applications

7.5.1 Newton's Law of Cooling

Physical setup. A hot object placed in a cooler environment loses heat at a rate proportional to the temperature difference between the object and the ambient medium.

Model. Let $T(t)$ be the object's temperature and T_a the constant ambient temperature. Newton's law states:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k\,(T - T_a), \qquad k < 0. \tag{15}$$

The equilibrium is $T^* = T_a$ (the object eventually reaches ambient temperature). Since $f'(T_a) = k < 0$, the equilibrium is asymptotically stable.

Worked Example

A cup of coffee cools from $90°$C to $70°$C in 10 minutes in a $20°$C room. How long until it reaches $40°$C?

Step 1: Solve the ODE. Separate variables:

$$\frac{\mathrm{d}T}{T - T_a} = k\,\mathrm{d}t \implies \ln|T - T_a| = kt + C \implies T(t) = T_a + (T_0 - T_a)e^{kt}.$$

With $T_a = 20$ and $T_0 = 90$:

$$T(t) = 20 + 70\,e^{kt}.$$

Step 2: Determine k. At $t = 10$:

$$70 = 20 + 70\,e^{10k} \implies e^{10k} = \frac{50}{70} = \frac{5}{7} \implies k = \frac{1}{10}\ln\left(\frac{5}{7}\right) \approx -0.0336.$$

Step 3: Find the time to reach $40°$C.

$$40 = 20 + 70\,e^{-0.0336t} \implies e^{-0.0336t} = \frac{20}{70} = \frac{2}{7}.$$

$$-0.0336t = \ln\left(\frac{2}{7}\right) \implies t = \frac{\ln(2/7)}{-0.0336} = \frac{-1.253}{-0.0336} \approx 37.3 \text{ minutes}.$$

The coffee reaches $40°$C after approximately **37.3 minutes**.

7.5.2 Mixing Problems

Physical setup. A tank contains a solution of salt and water. Brine flows in at a known concentration and rate, and the well-stirred mixture flows out at a known rate. We wish to track the amount of salt $Q(t)$ in the tank.

Model. Let $V(t)$ be the volume, c_1 the concentration of incoming brine, and r_1, r_2 the inflow and outflow rates. The rate of change of salt is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \underbrace{c_1 r_1}_{\text{rate in}} - \underbrace{\frac{Q(t)}{V(t)}\,r_2}_{\text{rate out}}. \tag{16}$$

If $r_1 = r_2 = r$ then $V(t) = V_0$ is constant, and the equation becomes linear:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{r}{V_0}\,Q = c_1 r. \tag{17}$$

Worked Example

A tank holds 100 L of brine with 20 kg of dissolved salt. Brine containing 0.5 kg/L flows in at 3 L/min. The well-mixed solution flows out at 3 L/min. Find $Q(t)$ and the long-term salt content.

Parameters: $V_0 = 100$ L, $Q(0) = 20$ kg, $c_1 = 0.5$ kg/L, $r = 3$ L/min.

ODE: Since inflow = outflow, $V(t) = 100$ is constant.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = (0.5)(3) - \frac{Q}{100}(3) = 1.5 - \frac{3}{100}Q = 1.5 - 0.03\,Q.$$

Solve: This is a linear first-order equation. The integrating factor is

$$\mu(t) = e^{\int 0.03\,\mathrm{d}t} = e^{0.03t}.$$

Multiply through:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Qe^{0.03t}\right) = 1.5\,e^{0.03t}.$$

Integrate:

$$Q(t)\,e^{0.03t} = \frac{1.5}{0.03}\,e^{0.03t} + C = 50\,e^{0.03t} + C,$$

$$Q(t) = 50 + C\,e^{-0.03t}.$$

Apply $Q(0) = 20$: $20 = 50 + C \implies C = -30$.

$$\boxed{Q(t) = 50 - 30\,e^{-0.03t}}.$$

Long-term behavior: As $t \to \infty$, $Q(t) \to 50$ kg. This makes physical sense: the equilibrium concentration matches the inflow concentration, so the tank eventually holds $0.5 \times 100 = 50$ kg of salt.
Phase line interpretation: The autonomous equation $Q' = 1.5 - 0.03Q$ has one equilibrium at $Q^* = 50$. Since $f'(Q^*) = -0.03 < 0$, it is asymptotically stable.

7.5.3 Falling Body with Air Resistance

Physical setup. A body of mass m falls under gravity. Air resistance opposes the motion. We model the speed $v(t)$ (positive downward).

Model with linear drag. The drag force is proportional to speed: $F_{\text{drag}} = -kv$. Newton's second law gives

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv. \tag{18}$$

The equilibrium (terminal velocity) is found by setting $v' = 0$:

$$mg - kv^* = 0 \implies v^* = \frac{mg}{k}.$$

Since $f'(v^*) = -k/m < 0$, the terminal velocity is asymptotically stable: any initial speed eventually converges to v^*.

Worked Example

A 10 kg object falls under gravity ($g = 9.8$ m/s²) with linear air resistance ($k = 2$ N·s/m). Find the terminal velocity and $v(t)$ if the object is dropped from rest.
Terminal velocity:

$$v^* = \frac{mg}{k} = \frac{10 \cdot 9.8}{2} = 49 \text{ m/s}.$$

Solve the ODE: Divide by $m = 10$:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8 - \frac{2}{10}v = 9.8 - 0.2\,v.$$

Separate:

$$\frac{\mathrm{d}v}{9.8 - 0.2v} = \mathrm{d}t \implies -\frac{1}{0.2}\ln|9.8 - 0.2v| = t + C.$$

With $v(0) = 0$: $-\frac{1}{0.2}\ln(9.8) = C$.

$$\ln|9.8 - 0.2v| = -0.2t + \ln(9.8) \implies 9.8 - 0.2v = 9.8\,e^{-0.2t},$$

$$\boxed{v(t) = 49\left(1 - e^{-0.2t}\right)}.$$

Check: As $t \to \infty$, $v(t) \to 49$ m/s (terminal velocity). At $t = 5$ s:

$$v(5) = 49(1 - e^{-1}) = 49(1 - 0.368) \approx 31.1 \text{ m/s}.$$

Model with quadratic drag. For high-speed motion (e.g., skydiving), the drag force is proportional to v^2:

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv^2. \tag{19}$$

The terminal velocity is $v^* = \sqrt{mg/k}$. The qualitative behavior is the same: solutions converge to v^* asymptotically. The approach, however, is different from the linear case.

7.6 Summary

Table 5: Chapter 3 Summary: Qualitative Analysis and Numerical Methods

Concept	Key Formula/Method
Autonomous equation	$\dfrac{\mathrm{d}y}{\mathrm{d}t} = f(y)$, no explicit t-dependence
Equilibrium	y^* such that $f(y^*) = 0$
Phase line	Vertical line with equilibria marked and arrows showing flow direction
Stability (sink)	$f'(y^*) < 0 \;\Rightarrow$ asymptotically stable
Stability (source)	$f'(y^*) > 0 \;\Rightarrow$ unstable
Semi-stable (node)	$f'(y^*) = 0$; sign of f determines approach/departure
Logistic equation	$\dfrac{\mathrm{d}y}{\mathrm{d}t} = ry\Big(1 - \dfrac{y}{K}\Big)$
Logistic solution	$y(t) = \dfrac{K}{1 + Ae^{-rt}}, \quad A = \dfrac{K - y_0}{y_0}$
Euler's method	$y_{n+1} = y_n + h\,f(t_n, y_n)$
Euler global error	$O(h)$
Newton's cooling	$T(t) = T_a + (T_0 - T_a)e^{kt},\ k < 0$
Mixing (constant V)	$\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{r}{V_0}Q = c_1 r$
Linear drag terminal velocity	$v^* = mg/k$
Quadratic drag terminal velocity	$v^* = \sqrt{mg/k}$

8 Second-Order Homogeneous

8.1 The Characteristic Equation

We consider the **second-order linear homogeneous differential equation with constant coefficients**:

$$a\,y'' + b\,y' + c\,y = 0, \qquad a, b, c \in \mathbb{R},\ \ a \neq 0. \tag{20}$$

This is the simplest class of second-order ODEs that is both tractable analytically and broadly applicable to physical systems (mechanical vibrations, electrical circuits, etc.).

Derivation. The key insight is that the exponential function $y = e^{rx}$ has derivatives proportional to itself, making it a natural candidate for solutions of linear equations with constant coefficients. Substituting the **ansatz**

$$y = e^{rx}, \qquad y' = r\,e^{rx}, \qquad y'' = r^2\,e^{rx}$$

into equation (20) gives

$$a\,r^2 e^{rx} + b\,r\,e^{rx} + c\,e^{rx} = 0.$$

Since $e^{rx} \neq 0$ for all x, we divide through and obtain the **characteristic equation** (also called the **auxiliary equation**):

$$a\,r^2 + b\,r + c = 0. \tag{21}$$

This is an ordinary quadratic equation. Its roots are given by the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{22}$$

The nature of the roots depends on the **discriminant**

$$\Delta = b^2 - 4ac. \tag{23}$$

There are three cases.

Key Result

Three cases for second-order homogeneous equations.

Discriminant	Roots	Solution form
$\Delta > 0$	Two distinct real roots $r_1 \neq r_2$	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
$\Delta = 0$	One repeated real root $r = -b/(2a)$	$y = c_1 e^{rx} + c_2 x\, e^{rx}$
$\Delta < 0$	Complex conjugates $r = \alpha \pm i\beta$	$y = e^{\alpha x}\big[c_1 \cos(\beta x) + c_2 \sin(\beta x)\big]$

We treat each case in detail in the following subsections. First, two quick examples to illustrate the method.

Worked Example

Solve $y'' - 5y' + 6y = 0$.
Solution. The characteristic equation is

$$r^2 - 5r + 6 = 0 \quad \Longrightarrow \quad (r-2)(r-3) = 0.$$

The roots are $r_1 = 2$ and $r_2 = 3$ (two distinct real roots, $\Delta = 25 - 24 = 1 > 0$). The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{3x}.$$

Worked Example

Solve $y'' + 4y = 0$.
Solution. The characteristic equation is

$$r^2 + 4 = 0 \quad \Longrightarrow \quad r^2 = -4 \quad \Longrightarrow \quad r = \pm 2i.$$

Here $\alpha = 0$ and $\beta = 2$, so the general solution is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

This represents undamped oscillations (we will see the physical interpretation in section 8.4).

8.2 Case 1: Distinct Real Roots

When $\Delta = b^2 - 4ac > 0$, the characteristic equation equation (21) has two distinct real roots

$$r_1 = \frac{-b + \sqrt{\Delta}}{2a}, \qquad r_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

Key Result

Distinct real roots. If $r_1 \neq r_2$ are real, then $\{e^{r_1 x}, e^{r_2 x}\}$ is a fundamental set of solutions and the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

Verification. Each exponential satisfies the ODE by construction (that is how the characteristic equation was derived). The general solution follows from the superposition principle (Theorem 8.1): any linear combination of solutions is again a solution.

Worked Example

Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 3$ and $y'(0) = 1$.

Solution. The characteristic equation is

$$r^2 - r - 2 = 0 \quad \implies \quad (r-2)(r+1) = 0.$$

Roots: $r_1 = 2$, $r_2 = -1$. The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Differentiate:

$$y'(x) = 2c_1 e^{2x} - c_2 e^{-x}.$$

Apply initial conditions:

$$\begin{cases} y(0) = c_1 + c_2 = 3, \\ y'(0) = 2c_1 - c_2 = 1. \end{cases}$$

Adding the two equations: $3c_1 = 4$, so $c_1 = \dfrac{4}{3}$. From the first equation: $c_2 = 3 - \dfrac{4}{3} = \dfrac{5}{3}$.
The solution is

$$y(x) = \frac{4}{3}\, e^{2x} + \frac{5}{3}\, e^{-x}.$$

Worked Example

Solve $2y'' + 7y' + 3y = 0$ with $y(0) = 4$ and $y'(0) = -5$.
Solution. The characteristic equation is

$$2r^2 + 7r + 3 = 0.$$

Using the quadratic formula:

$$r = \frac{-7 \pm \sqrt{49 - 24}}{4} = \frac{-7 \pm 5}{4}.$$

So $r_1 = \dfrac{-7+5}{4} = -\dfrac{1}{2}$ and $r_2 = \dfrac{-7-5}{4} = -3$.
The general solution is

$$y(x) = c_1 e^{-x/2} + c_2 e^{-3x}.$$

Differentiate:

$$y'(x) = -\tfrac{1}{2}\, c_1\, e^{-x/2} - 3c_2\, e^{-3x}.$$

Apply initial conditions:

$$\begin{cases} y(0) = c_1 + c_2 = 4, \\ y'(0) = -\tfrac{1}{2}\, c_1 - 3c_2 = -5. \end{cases}$$

From the first equation: $c_1 = 4 - c_2$. Substitute into the second:

$$-\tfrac{1}{2}(4 - c_2) - 3c_2 = -5 \quad \implies \quad -2 + \tfrac{1}{2}c_2 - 3c_2 = -5.$$

$$-\tfrac{5}{2}c_2 = -3 \quad \implies \quad c_2 = \frac{6}{5}, \qquad c_1 = 4 - \frac{6}{5} = \frac{14}{5}.$$

The solution is

$$y(x) = \frac{14}{5}\, e^{-x/2} + \frac{6}{5}\, e^{-3x}.$$

Worked Example

Solve $y'' - 4y = 0$ with $y(0) = 0$ and $y(1) = e^2 - e^{-2}$.
Solution. The characteristic equation is $r^2 - 4 = 0$, so $r = \pm 2$. The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

Apply the first condition:

$$y(0) = c_1 + c_2 = 0 \quad \implies \quad c_2 = -c_1.$$

The solution simplifies to $y(x) = c_1(e^{2x} - e^{-2x})$.
Apply the second condition:

$$y(1) = c_1(e^2 - e^{-2}) = e^2 - e^{-2} \quad \implies \quad c_1 = 1.$$

The solution is

$$y(x) = e^{2x} - e^{-2x} = 2\sinh(2x).$$

8.3 Case 2: Repeated Real Roots

When $\Delta = 0$, the characteristic equation has a single (double) root

$$r = -\frac{b}{2a}.$$

Key Result

Repeated real root. If r is a double root, the general solution is

$$y(x) = c_1 e^{rx} + c_2\, x\, e^{rx}.$$

Why the x factor? With only one root r, the function e^{rx} gives us just one solution. We need a second linearly independent solution to form the general solution of a second-order equation. The method of **reduction of order** provides this second solution.

Derivation via reduction of order. Suppose $y_1(x) = e^{rx}$ is one solution of equation (20) (with $\Delta = 0$). We seek a second solution of the form

$$y_2(x) = v(x)\, y_1(x) = v(x)\, e^{rx},$$

where $v(x)$ is an unknown function to be determined. Compute the derivatives:

$$y_2' = (v' + rv)\, e^{rx}, \qquad y_2'' = (v'' + 2rv' + r^2 v)\, e^{rx}.$$

Substitute y_2 into the ODE $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$ (dividing equation (20) by a):

$$(v'' + 2rv' + r^2 v)e^{rx} + \frac{b}{a}(v' + rv)e^{rx} + \frac{c}{a}\, v\, e^{rx} = 0.$$

Dividing by e^{rx} and collecting terms:

$$v'' + \left(2r + \frac{b}{a}\right) v' + \left(r^2 + \frac{b}{a}r + \frac{c}{a}\right) v = 0.$$

The v-coefficient is exactly the characteristic polynomial evaluated at r, which vanishes because r is a root:

$$r^2 + \frac{b}{a}r + \frac{c}{a} = 0.$$

For a repeated root, $r = -b/(2a)$, so the v'-coefficient is also zero:

$$2r + \frac{b}{a} = 2\left(-\frac{b}{2a}\right) + \frac{b}{a} = 0.$$

We are left with the simple equation

$$v''(x) = 0 \quad \Longrightarrow \quad v(x) = Ax + B.$$

Choosing $A = 1$ and $B = 0$ gives $y_2(x) = x\, e^{rx}$, a second linearly independent solution. The general solution is

$$y(x) = c_1 e^{rx} + c_2 x\, e^{rx}.$$

Hint

Quick check. You can verify that $y = x\, e^{rx}$ satisfies $ay'' + by' + cy = 0$ when $r = -b/(2a)$ and $b^2 = 4ac$ by direct substitution. This is always a safe verification step.

Worked Example

Solve $y'' - 6y' + 9y = 0$.
Solution. The characteristic equation is

$$r^2 - 6r + 9 = 0 \quad \implies \quad (r-3)^2 = 0.$$

Double root: $r = 3$. The general solution is

$$y(x) = c_1 e^{3x} + c_2 x\, e^{3x}.$$

Worked Example

Solve $y'' + 4y' + 4y = 0$ with $y(0) = 2$ and $y'(0) = 0$.
Solution. The characteristic equation is

$$r^2 + 4r + 4 = 0 \quad \implies \quad (r+2)^2 = 0.$$

Double root: $r = -2$. The general solution is

$$y(x) = c_1 e^{-2x} + c_2 x\, e^{-2x}.$$

Differentiate:

$$y'(x) = -2c_1 e^{-2x} + c_2 e^{-2x} - 2c_2 x\, e^{-2x} = \big(-2c_1 + c_2 - 2c_2 x\big)e^{-2x}.$$

Apply initial conditions:

$$\begin{cases} y(0) = c_1 = 2, \\ y'(0) = -2c_1 + c_2 = 0 \end{cases} \implies \quad c_2 = 2c_1 = 4.$$

The solution is

$$y(x) = 2\, e^{-2x} + 4x\, e^{-2x} = 2(1+2x)\, e^{-2x}.$$

8.4 Case 3: Complex Conjugate Roots

When $\Delta = b^2 - 4ac < 0$, the characteristic equation has two complex conjugate roots:

$$r = \alpha \pm i\beta, \qquad \alpha = -\frac{b}{2a}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

Key Result

Complex conjugate roots. If $r = \alpha \pm i\beta$ with $\beta > 0$, the general real-valued solution is

$$y(x) = e^{\alpha x}\Big[c_1 \cos(\beta x) + c_2 \sin(\beta x)\Big].$$

Equivalently, using amplitude–phase form:

$$y(x) = A\, e^{\alpha x} \cos(\beta x - \phi),$$

where $A = \sqrt{c_1^2 + c_2^2}$ and $\phi = \arctan(c_2/c_1)$.

Derivation from Euler's formula. The complex-valued solutions are $e^{(\alpha+i\beta)x}$ and $e^{(\alpha-i\beta)x}$. By Euler's formula,

$$e^{(\alpha+i\beta)x} = e^{\alpha x}\, e^{i\beta x} = e^{\alpha x}\big[\cos(\beta x) + i\, \sin(\beta x)\big].$$

Because the ODE has real coefficients, both the real and imaginary parts are themselves real solutions:

$$\Re\Big[e^{(\alpha+i\beta)x}\Big] = e^{\alpha x} \cos(\beta x),$$
$$\Im\Big[e^{(\alpha+i\beta)x}\Big] = e^{\alpha x} \sin(\beta x).$$

These two functions are linearly independent (one is not a constant multiple of the other), so they form a fundamental set. The general solution is their linear combination, giving the formula above.

Physical interpretation: damped oscillations. The parameter α governs the exponential growth ($\alpha > 0$) or decay ($\alpha < 0$) of the amplitude, while β determines the angular frequency of the oscillation. In mechanical systems, this corresponds to **underdamped motion**: the mass oscillates about equilibrium while the amplitude decays exponentially (if $\alpha < 0$). When $\alpha = 0$ (i.e. $b = 0$), we have pure undamped oscillations with constant amplitude.

Worked Example

Solve $y'' + 2y' + 5y = 0$.
Solution. The characteristic equation is

$$r^2 + 2r + 5 = 0.$$

Discriminant: $\Delta = 4 - 20 = -16 < 0$. The roots are

$$r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i.$$

Here $\alpha = -1$ and $\beta = 2$. The general solution is

$$y(x) = e^{-x}\Big[c_1 \cos(2x) + c_2 \sin(2x)\Big].$$

This represents damped oscillations with amplitude decaying like e^{-x}.

Worked Example

Solve $y'' + 4y' + 13y = 0$ with $y(0) = 2$ and $y'(0) = 3$.
Solution. The characteristic equation is

$$r^2 + 4r + 13 = 0.$$

Discriminant: $\Delta = 16 - 52 = -36 < 0$. The roots are

$$r = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i.$$

So $\alpha = -2$ and $\beta = 3$. The general solution is

$$y(x) = e^{-2x}\Big[c_1 \cos(3x) + c_2 \sin(3x)\Big].$$

Differentiate (product rule):

$$y'(x) = -2\,e^{-2x}\big[c_1 \cos(3x) + c_2 \sin(3x)\big] + e^{-2x}\big[-3c_1 \sin(3x) + 3c_2 \cos(3x)\big].$$

Apply initial conditions:

$$\begin{cases} y(0) = c_1 = 2, \\ y'(0) = -2c_1 + 3c_2 = 3 \quad \Longrightarrow \quad -4 + 3c_2 = 3 \quad \Longrightarrow \quad c_2 = \dfrac{7}{3}. \end{cases}$$

The solution is

$$y(x) = e^{-2x}\left(2\cos(3x) + \frac{7}{3}\sin(3x)\right).$$

8.5 Superposition Principle

The entire theory rests on the linearity of the differential operator.

Theorem 8.1 (Superposition Principle)**.** *Let $L[y] = ay'' + by' + cy$ with constant a, b, c and $a \neq 0$. If $y_1(x)$ and $y_2(x)$ are solutions of $L[y] = 0$, then any linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution, for arbitrary constants $c_1, c_2 \in \mathbb{R}$.

Proof. Compute:

$$L[c_1y_1 + c_2y_2] = a(c_1y_1 + c_2y_2)'' + b(c_1y_1 + c_2y_2)' + c(c_1y_1 + c_2y_2).$$

By linearity of differentiation:

$$= c_1\big(ay_1'' + by_1' + cy_1\big) + c_2\big(ay_2'' + by_2' + cy_2\big) = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Therefore $c_1y_1 + c_2y_2$ satisfies the ODE. □

Definition 8.2 (Fundamental Set of Solutions). Two solutions $\{y_1, y_2\}$ of equation (20) form a **fundamental set of solutions** on an interval I if they are linearly independent on I. The general solution is then

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1, c_2 are arbitrary constants.

Worked Example

Verify that $y_1(x) = e^{3x}$ and $y_2(x) = e^{-2x}$ form a fundamental set for $y'' - y' - 6y = 0$, and write the general solution.
Solution. Check each function:

$$y_1 = e^{3x} \;\Rightarrow\; y_1' = 3e^{3x},\; y_1'' = 9e^{3x}.$$

$$L[y_1] = 9e^{3x} - 3e^{3x} - 6e^{3x} = 0. \quad \checkmark$$

$$y_2 = e^{-2x} \;\Rightarrow\; y_2' = -2e^{-2x},\; y_2'' = 4e^{-2x}.$$

$$L[y_2] = 4e^{-2x} + 2e^{-2x} - 6e^{-2x} = 0. \quad \checkmark$$

Linear independence: the ratio $y_1/y_2 = e^{5x}$ is not constant, so y_1 and y_2 are linearly independent. They form a fundamental set.
The general solution is

$$y(x) = c_1e^{3x} + c_2e^{-2x}.$$

Worked Example

Show that $\{y_1, y_2\} = \{e^{2x}, 2e^{2x}\}$ is *not* a fundamental set for $y'' - 4y = 0$, even though both are solutions.
Solution. Clearly $y_2 = 2y_1$, so the two functions are linearly dependent. Any linear combination gives

$$c_1e^{2x} + c_2(2e^{2x}) = (c_1 + 2c_2)e^{2x},$$

which spans only a one-dimensional solution space. We cannot represent all solutions (we are missing the e^{-2x} component). A proper fundamental set is $\{e^{2x}, e^{-2x}\}$.

8.6 Wronskian and Linear Independence

To systematically determine whether two solutions form a fundamental set, we use the **Wronskian**.

Definition 8.3 (Wronskian). For two differentiable functions $y_1(x)$ and $y_2(x)$, the **Wronskian** is

$$W(y_1, y_2)(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x).$$

Theorem 8.4 (Wronskian Test). *Let y_1 and y_2 be two solutions of equation* (20) *on an interval I. Then:*

1. *If $W(y_1, y_2)(x_0) \neq 0$ for some $x_0 \in I$, then $\{y_1, y_2\}$ is a fundamental set on I.*
2. *If $W(y_1, y_2)(x) = 0$ for all $x \in I$, then y_1 and y_2 are linearly dependent on I.*

In particular, $W(y_1, y_2)(x)$ is either identically zero on I or never zero on I.

The Wronskian gives a practical computational test: evaluate W at any single point; if it is nonzero, the solutions are linearly independent.

Worked Example

Compute the Wronskian of $y_1(x) = e^{3x}$ and $y_2(x) = e^{-2x}$.

Solution. We have $y_1' = 3e^{3x}$ and $y_2' = -2e^{-2x}$.

$$W(y_1, y_2)(x) = e^{3x} \cdot (-2e^{-2x}) - 3e^{3x} \cdot e^{-2x} = -2e^{x} - 3e^{x} = -5e^{x}.$$

Since $W(x) = -5e^x \neq 0$ for all x, the two functions are linearly independent. They form a fundamental set.

Worked Example

Compute the Wronskian of $y_1(x) = e^{rx}$ and $y_2(x) = x\,e^{rx}$ (the repeated-root case).
Solution. We have $y_1' = r\,e^{rx}$ and $y_2' = (1 + rx)\,e^{rx}$.

$$W(y_1, y_2)(x) = e^{rx} \cdot (1 + rx)\,e^{rx} - r\,e^{rx} \cdot x\,e^{rx} = (1 + rx)\,e^{2rx} - rx\,e^{2rx} = e^{2rx}.$$

Since $W(x) = e^{2rx} \neq 0$ for all x, these two functions are linearly independent. This confirms that $\{e^{rx}, x\,e^{rx}\}$ is a valid fundamental set for the repeated-root case.

Remark 8.5. The connection between the Wronskian and fundamental sets is fundamental (pardon the pun): a pair of solutions is a fundamental set *if and only if* their Wronskian is nonzero on the interval. This is the standard criterion used in practice.

8.7 Abel's Identity

Abel's identity provides a formula for the Wronskian without computing derivatives, by relating it directly to the coefficients of the ODE.

Theorem 8.6 (Abel's Identity)**.** *Let y_1 and y_2 be two solutions of the standard-form equation*

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{24}$$

on an interval I, where $p(x)$ and $q(x)$ are continuous. Then the Wronskian satisfies

$$W(x) = W(x_0)\,\exp\!\left(-\int_{x_0}^{x} p(t)\,\mathrm{d}t\right)$$

for any $x_0 \in I$.

Proof. Let $W(x) = y_1 y_2' - y_1' y_2$. Differentiate:

$$W' = y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' = y_1 y_2'' - y_1'' y_2.$$

Since y_1 and y_2 both satisfy $y'' + p(x)y' + q(x)y = 0$, we have

$$y_1'' = -p(x)y_1' - q(x)y_1, \qquad y_2'' = -p(x)y_2' - q(x)y_2.$$

Substitute:

$$\begin{aligned} W' &= y_1\big[-p y_2' - q y_2\big] - \big[-p y_1' - q y_1\big] y_2 \\ &= -p\,y_1 y_2' - q\,y_1 y_2 + p\,y_1' y_2 + q\,y_1 y_2 \\ &= -p\,(y_1 y_2' - y_1' y_2) = -p(x)\,W(x). \end{aligned}$$

Thus $W' + p(x)W = 0$, a first-order separable equation. The solution is

$$W(x) = W(x_0)\,\exp\!\left(-\int_{x_0}^{x} p(t)\,\mathrm{d}t\right).$$

□

Remark 8.7. For the constant-coefficient equation equation (20), dividing by a puts it in standard form with $p(x) = b/a$ (constant). Abel's identity then gives

$$W(x) = W(x_0)\,\exp\!\left(-\frac{b}{a}(x - x_0)\right).$$

Since $W(x_0) \neq 0$ for a fundamental set, this confirms that $W(x)$ is never zero (exponential of a real number is always positive).

Worked Example

Use Abel's identity to find the Wronskian of two solutions of $y'' + 3y' + 2y = 0$, given that $W(0) = 5$.
Solution. In standard form, $p(x) = 3$. By Abel's identity:

$$W(x) = W(0)\,\exp\Big(-\int_0^x 3\,\mathrm{d}t\Big) = 5\,e^{-3x}.$$

The Wronskian is $W(x) = 5e^{-3x}$, which is always positive (confirming the solutions are linearly independent).

Worked Example

Verify Abel's identity for $y'' - 4y = 0$ using the fundamental set $\{e^{2x}, e^{-2x}\}$.
Solution. In standard form, $p(x) = 0$. Abel's identity predicts

$$W(x) = W(x_0)\,\exp\Big(-\int_{x_0}^x 0\,\mathrm{d}t\Big) = W(x_0).$$

The Wronskian should be constant. Compute it directly:

$$W(e^{2x}, e^{-2x}) = e^{2x}(-2e^{-2x}) - (2e^{2x})e^{-2x} = -2 - 2 = -4.$$

Indeed, $W(x) = -4$ is constant, in agreement with Abel's identity.

Hint

Practical tip. Abel's identity is especially useful when you know one solution and need to find another. It tells you the Wronskian *without* computing derivatives, and combined with the formula $v' = W/y_1^2$ from reduction of order (section 8.8), it directly yields the second solution.

8.8 Reduction of Order

Reduction of order is a systematic method for finding a second linearly independent solution when one solution $y_1(x)$ is already known. We saw this technique in section 8.3 to derive the $x\,e^{rx}$ factor; here we state it in full generality.

Derivation. Consider the standard-form equation $y'' + p(x)y' + q(x)y = 0$. Suppose $y_1(x)$ is a known solution. We seek a second solution of the form

$$y_2(x) = v(x)\,y_1(x), \tag{25}$$

where $v(x)$ is to be determined. Differentiate:

$$\begin{aligned} y_2' &= v'y_1 + vy_1', \\ y_2'' &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned}$$

Substitute into the ODE:

$$\big(v''y_1 + 2v'y_1' + vy_1''\big) + p(x)\big(v'y_1 + vy_1'\big) + q(x)\,v\,y_1 = 0.$$

Collect terms:

$$v''y_1 + v'\big(2y_1' + p(x)y_1\big) + v\big(y_1'' + p(x)y_1' + q(x)y_1\big) = 0.$$

The v-term vanishes because y_1 is a solution. We obtain a first-order equation in $u = v'$:

$$y_1\,u' + \big(2y_1' + p(x)y_1\big)u = 0.$$

This is separable:

$$\frac{u'}{u} = -\frac{2y_1'}{y_1} - p(x) = -2\frac{d}{dx}\big(\ln|y_1|\big) - p(x).$$

Integrating:

$$\ln|u| = -2\ln|y_1| - \int p(x)\,\mathrm{d}x + C,$$

so

$$u = \frac{v'}{y_1^2}\cdot y_1^2 = \frac{C}{y_1^2}\,\exp\Big(-\int p(x)\,\mathrm{d}x\Big).$$

Noting that $W(x) = W(x_0)\exp\big(-\int p\,\mathrm{d}x\big)$ by Abel's identity, we can write this as

$$v'(x) = \frac{W(x)}{y_1(x)^2}. \tag{26}$$

Key Result

Reduction of order formula. Given one solution $y_1(x)$ of $y'' + p(x)y' + q(x)y = 0$, a second linearly independent solution is

$$y_2(x) = y_1(x)\int \frac{1}{y_1(x)^2}\exp\left(-\int p(x)\,\mathrm{d}x\right)\mathrm{d}x.$$

Worked Example

Given that $y_1(x) = e^{3x}$ is a solution of $y'' - 6y' + 9y = 0$, find a second linearly independent solution using reduction of order.
Solution. The equation in standard form has $p(x) = -6$. By Abel's identity (or direct computation):

$$\exp\left(-\int p(x)\,\mathrm{d}x\right) = \exp\left(-\int(-6)\,\mathrm{d}x\right) = e^{6x}.$$

Using equation (26):

$$v'(x) = \frac{e^{6x}}{(e^{3x})^2} = \frac{e^{6x}}{e^{6x}} = 1.$$

Integrate: $v(x) = x$ (choosing the simplest antiderivative with $C = 0$). Therefore

$$y_2(x) = v(x)\,y_1(x) = x\,e^{3x}.$$

This recovers the repeated-root solution we derived earlier. The general solution is

$$y(x) = c_1e^{3x} + c_2x\,e^{3x}.$$

Worked Example

Given that $y_1(x) = x$ is a solution of $x^2y'' - xy' + y = 0$, find a second solution.
Solution. Write the equation in standard form by dividing by x^2:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0.$$

Here $p(x) = -\dfrac{1}{x}$. Compute the integrating factor:

$$\exp\left(-\int p(x)\,\mathrm{d}x\right) = \exp\left(-\int\left(-\frac{1}{x}\right)\mathrm{d}x\right) = \exp\!\big(\ln|x|\big) = |x|.$$

For $x > 0$, this is simply x. Now apply the reduction of order formula:

$$v'(x) = \frac{x}{y_1(x)^2} = \frac{x}{x^2} = \frac{1}{x}.$$

Integrate: $v(x) = \ln|x|$. The second solution is

$$y_2(x) = y_1(x)\,v(x) = x\,\ln|x|.$$

The general solution (for $x > 0$) is

$$y(x) = c_1\,x + c_2\,x\,\ln x.$$

Table 6: Second-order linear homogeneous equations with constant coefficients

Discriminant	Roots	Solution	Key idea
$\Delta > 0$	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	Two distinct real exponentials
$\Delta = 0$	$r = -\dfrac{b}{2a}$ (double)	$y = c_1 e^{rx} + c_2 x e^{rx}$	Extra x from reduction of order
$\Delta < 0$	$\alpha \pm i\beta$	$e^{\alpha x}[c_1 \cos(\beta x) + c_2 \sin(\beta x)]$	Damped oscillation via Euler's formula

Table 7: Theory results: Wronskian, Abel's identity, reduction of order

Concept	Key formula
Wronskian	$W(y_1, y_2)(x) = y_1 y_2' - y_1' y_2$
Wronskian test	$W(x_0) \neq 0 \;\Rightarrow\;$ fundamental set on I
Abel's identity	$W(x) = W(x_0)\,\exp\left(-\int_{x_0}^{x} p(t)\,\mathrm{d}t\right)$
Abel (constant coeff.)	$W(x) = W(x_0)\,\exp\left(-\dfrac{b}{a}(x - x_0)\right)$
Reduction of order	$y_2(x) = y_1(x) \displaystyle\int \frac{e^{-\int p(x)\,\mathrm{d}x}}{y_1(x)^2}\,\mathrm{d}x$
Superposition	$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2] = 0$

8.9 Summary

Hint

Problem-solving workflow.

1. Write the ODE in standard form (divide by a if needed).
2. Form the characteristic equation $ar^2 + br + c = 0$.
3. Compute the discriminant $\Delta = b^2 - 4ac$.
4. Apply the appropriate case from table 6.
5. If initial/boundary conditions are given, determine c_1 and c_2.

9 Second-Order Nonhomogeneous

We now extend the theory from section 8 to the nonhomogeneous case. The general equation is

$$a\,y'' + b\,y' + c\,y = g(x), \qquad a, b, c \in \mathbb{R}, \;\; a \neq 0, \tag{27}$$

where $g(x)$ is a given **forcing function** (also called the **source term** or **nonhomogeneous term**). When $g(x) \equiv 0$, we recover the homogeneous equation equation (20). The presence of $g(x)$ models external driving forces in physical systems — a periodically driven pendulum, a forced electrical circuit, or a building shaken by an earthquake.

The general strategy in all cases is the same: first solve the associated homogeneous equation, then find one particular solution of the nonhomogeneous equation. The two parts combine to give the complete solution.

9.1 General Solution Structure

Theorem 9.1 (Structure of the General Solution)**.** *Let $L[y] = ay'' + by' + cy$ with constant a, b, c and $a \neq 0$. If $y_h(x)$ is the general solution of the associated homogeneous equation $L[y] = 0$, and $y_p(x)$ is* any *particular solution of the nonhomogeneous equation $L[y] = g(x)$, then the general solution of $L[y] = g(x)$ is*

$$y(x) = y_h(x) + y_p(x).$$

Proof. Define the linear operator $L[y] = ay'' + by' + cy$. We need to show that $y = y_h + y_p$ satisfies $L[y] = g(x)$ for *every* choice of the constants in y_h.

By linearity of differentiation (the **superposition principle**, Theorem 8.1):

$$L[y_h + y_p] = L[y_h] + L[y_p].$$

Since y_h solves the homogeneous equation, $L[y_h] = 0$. Since y_p is a particular solution, $L[y_p] = g(x)$. Therefore:

$$L[y_h + y_p] = 0 + g(x) = g(x).$$

Thus $y_h + y_p$ is a solution of the nonhomogeneous equation. Conversely, if y is any solution of $L[y] = g(x)$, then $L[y - y_p] = L[y] - L[y_p] = g(x) - g(x) = 0$, so $y - y_p = y_h$ for some homogeneous solution y_h, meaning $y = y_h + y_p$. □

The homogeneous part $y_h(x)$ retains two arbitrary constants c_1, c_2 from the general solution found in section 8. The particular solution $y_p(x)$ contains no free parameters — we only need *one* specific function that works.

Hint

Two-step workflow for every nonhomogeneous problem.

1. **Solve the homogeneous equation** $ay'' + by' + cy = 0$ using the characteristic equation method from section 8. Obtain $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$.
2. **Find one particular solution** $y_p(x)$ using either the method of undetermined coefficients or variation of parameters.

Combine: $y(x) = y_h(x) + y_p(x)$.

We now develop the two main methods for finding $y_p(x)$.

9.2 Method of Undetermined Coefficients

The method of **undetermined coefficients** (UC) is an algebraic technique that works when the forcing function $g(x)$ has a **simple form**. The idea is straightforward: guess the *shape* of y_p based on the form of $g(x)$, but leave the coefficients unknown. Then substitute the guess into the ODE and solve for those coefficients.

When to use. Undetermined coefficients applies only when $g(x)$ is one of the following (or a sum/product of them):

- A polynomial $P_n(x)$ of degree n.
- An exponential $e^{\alpha x}$.
- A sine or cosine function $\sin(\beta x)$ or $\cos(\beta x)$.
- Products of the above, such as $e^{\alpha x} P_n(x)$ or $e^{\alpha x}\sin(\beta x)$.

If $g(x) = \ln(x)$, $1/x$, $\tan(x)$, or any other function outside this family, undetermined coefficients **will not work**. Use variation of parameters instead (section 9.3).

Hint

Limitation warning. Undetermined coefficients is an *educated guessing* method. It exploits the fact that polynomials, exponentials, and trigonometric functions are closed under differentiation. Functions like $\ln x$ and $\tan x$ do not share this property, so the method breaks down.

9.2.1 Guess Table

The following table gives the initial guess for y_p based on the form of $g(x)$. The constants A_k, B_k are the "undetermined coefficients" to be found by substitution.

Key Result

Guess table for the method of undetermined coefficients.

Form of $g(x)$	**Initial guess for** $y_p(x)$
$P_n(x) = a_n x^n + \cdots + a_0$	$A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$
$e^{\alpha x}$	$A\,e^{\alpha x}$
$\sin(\beta x)$ **or** $\cos(\beta x)$	$A\sin(\beta x) + B\cos(\beta x)$
$e^{\alpha x}P_n(x)$	$e^{\alpha x}(A_n x^n + \cdots + A_0)$
$e^{\alpha x}\sin(\beta x)$ **or** $e^{\alpha x}\cos(\beta x)$	$e^{\alpha x}\big[(A_n x^n + \cdots + A_0)\cos(\beta x)$ $+(B_n x^n + \cdots + B_0)\sin(\beta x)\big]$
Product of two forms	Multiply the individual guesses
Sum of two forms $g_1 + g_2$	Sum the individual guesses

9.2.2 Modification Rule

There is one important caveat: if any term of your initial guess **already appears in the homogeneous solution** y_h, the guess will fail (substitution leads to $0 = g(x)$, a contradiction). In this case, you must **modify** the guess.

Hint

Modification rule. If a term in the initial guess for y_p is also a solution of the homogeneous equation:

1. Multiply that term by x.
2. If the modified term *still* appears in y_h, multiply by x again (i.e. by x^2).

For a second-order equation, x^2 is always sufficient, because the homogeneous solution space has dimension 2.

Why this works. The modification multiplies the guess by x (or x^2), which introduces new functional forms (like xe^{rx} or x^2e^{rx}) that are *not* solutions of the homogeneous equation, while preserving the algebraic structure needed for substitution.

9.2.3 Worked Examples

Example 1: Polynomial forcing.

Worked Example

Solve $y'' - 3y' + 2y = 4x + 1$.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 - 3r + 2 = 0 \quad\Longrightarrow\quad (r-1)(r-2) = 0.$$

Roots: $r_1 = 1$, $r_2 = 2$. The homogeneous solution is

$$y_h(x) = c_1e^x + c_2e^{2x}.$$

Step 2: Particular solution. Here $g(x) = 4x + 1$, a first-degree polynomial. From the guess table, we try

$$y_p(x) = Ax + B.$$

No term overlaps with y_h (exponentials vs. polynomials), so no modification is needed. Differentiate:

$$y_p' = A, \qquad y_p'' = 0.$$

Substitute into the ODE:

$$0 - 3A + 2(Ax + B) = 4x + 1.$$

Collect terms:

$$2A\,x + (2B - 3A) = 4x + 1.$$

Equate coefficients of like powers of x:

$$\begin{cases} 2A = 4 \quad\Longrightarrow\quad A = 2, \\ 2B - 3A = 1 \quad\Longrightarrow\quad 2B - 6 = 1 \quad\Longrightarrow\quad B = \dfrac{7}{2}. \end{cases}$$

The particular solution is $y_p(x) = 2x + \frac{7}{2}$.
Step 3: General solution.

$$y(x) = c_1 e^x + c_2 e^{2x} + 2x + \frac{7}{2}.$$

Example 2: Exponential forcing.

Worked Example

Solve $y'' + y' - 6y = 12e^{3x}$.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 + r - 6 = 0 \implies (r-2)(r+3) = 0.$$

Roots: $r_1 = 2$, $r_2 = -3$. Thus

$$y_h(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

Step 2: Particular solution. Here $g(x) = 12e^{3x}$. From the guess table, we try

$$y_p(x) = A\,e^{3x}.$$

Check for overlap: e^{3x} is not in y_h (which contains e^{2x} and e^{-3x}), so no modification needed. Differentiate:

$$y_p' = 3A\,e^{3x}, \qquad y_p'' = 9A\,e^{3x}.$$

Substitute:

$$9A\,e^{3x} + 3A\,e^{3x} - 6A\,e^{3x} = 12e^{3x}.$$

Factor out e^{3x} (never zero):

$$(9 + 3 - 6)A = 12 \implies 6A = 12 \implies A = 2.$$

So $y_p(x) = 2e^{3x}$.
Step 3: General solution.

$$y(x) = c_1 e^{2x} + c_2 e^{-3x} + 2e^{3x}.$$

Example 3: Trigonometric forcing.

Worked Example

Solve $y'' + 4y = 8\cos(2x)$.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 + 4 = 0 \implies r = \pm 2i.$$

Thus

$$y_h(x) = c_1 \cos(2x) + c_2 \sin(2x).$$

Step 2: Particular solution. Here $g(x) = 8\cos(2x)$. From the guess table, we try

$$y_p(x) = A\cos(2x) + B\sin(2x).$$

Overlap detected! Both $\cos(2x)$ and $\sin(2x)$ appear in y_h. We must apply the modification rule: multiply the entire guess by x:

$$y_p(x) = x\big[A\cos(2x) + B\sin(2x)\big] = Ax\cos(2x) + Bx\sin(2x).$$

Check again: $x\cos(2x)$ and $x\sin(2x)$ are *not* in y_h, so we are good. Differentiate (product rule):

$$\begin{aligned}
y_p' &= A\cos(2x) - 2Ax\sin(2x) + B\sin(2x) + 2Bx\cos(2x),\\
y_p'' &= -2A\sin(2x) - 2A\sin(2x) - 4Ax\cos(2x) + 2B\cos(2x) + 2B\cos(2x) - 4Bx\sin(2x)\\
&= -4A\sin(2x) - 4Ax\cos(2x) + 4B\cos(2x) - 4Bx\sin(2x).
\end{aligned}$$

Substitute into $y'' + 4y = 8\cos(2x)$:

$$\big[-4A\sin(2x) - 4Ax\cos(2x) + 4B\cos(2x) - 4Bx\sin(2x)\big] + 4\big[Ax\cos(2x) + Bx\sin(2x)\big] = 8\cos(2x).$$

The x-terms cancel:

$$-4A\sin(2x) + 4B\cos(2x) = 8\cos(2x).$$

Equate coefficients:

$$\begin{cases} -4A = 0 \implies A = 0, \\ 4B = 8 \implies B = 2. \end{cases}$$

So $y_p(x) = 2x\sin(2x)$.
Step 3: General solution.

$$y(x) = c_1\cos(2x) + c_2\sin(2x) + 2x\sin(2x).$$

Example 4: Product of exponential and trigonometric.

Worked Example

Solve $y'' - 2y' + 5y = 10e^x\cos(2x)$.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 - 2r + 5 = 0.$$

Discriminant: $\Delta = 4 - 20 = -16 < 0$. Roots:

$$r = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

Thus

$$y_h(x) = e^x\left[c_1\cos(2x) + c_2\sin(2x)\right].$$

Step 2: Particular solution. Here $g(x) = 10e^x\cos(2x)$. From the guess table for $e^{\alpha x}\cos(\beta x)$ with $\alpha = 1, \beta = 2$, we try

$$y_p(x) = e^x\left[A\cos(2x) + B\sin(2x)\right].$$

Overlap detected! Both $e^x\cos(2x)$ and $e^x\sin(2x)$ appear in y_h. Apply the modification rule: multiply by x:

$$y_p(x) = x\,e^x\left[A\cos(2x) + B\sin(2x)\right].$$

Now differentiate. Let $y_p = x\,e^x\,(A\cos(2x) + B\sin(2x))$. Set $u = x\,e^x$ and $v = A\cos(2x) + B\sin(2x)$ for clarity. Then $u' = e^x(1+x)$ and $v' = -2A\sin(2x) + 2B\cos(2x)$.

$$\begin{aligned} y_p' &= u'v + uv' = e^x(1+x)\left[A\cos(2x) + B\sin(2x)\right] + x\,e^x\left[-2A\sin(2x) + 2B\cos(2x)\right], \\ y_p'' &= e^x(1+x)\left[-2A\sin(2x) + 2B\cos(2x)\right] + e^x\left[A\cos(2x) + B\sin(2x)\right] \\ &\quad + e^x(1+x)\left[-2A\sin(2x) + 2B\cos(2x)\right] + x\,e^x\left[-4A\cos(2x) - 4B\sin(2x)\right] \\ &= 2e^x(1+x)\left[-2A\sin(2x) + 2B\cos(2x)\right] + e^x\left[A\cos(2x) + B\sin(2x)\right] \\ &\quad - 4x\,e^x\left[A\cos(2x) + B\sin(2x)\right]. \end{aligned}$$

Substitute y_p, y_p', and y_p'' into the ODE $y'' - 2y' + 5y$. Collect the coefficient of $e^x\cos(2x)$ and $e^x\sin(2x)$ (the x-terms will cancel due to the modification rule):
Coefficient of $e^x\cos(2x)$: $4B + A - 2A + 5A = 4B + 4A$.
Coefficient of $e^x\sin(2x)$: $-4A + B - 2B + 5B = -4A + 4B$.
Set equal to the RHS $10e^x\cos(2x)$:

$$\begin{cases} 4A + 4B = 10, \\ -4A + 4B = 0 \implies A = B. \end{cases}$$

From the second equation, $A = B$. Substituting into the first: $8A = 10$, so $A = \frac{5}{4}$ and $B = \frac{5}{4}$.
The particular solution is

$$y_p(x) = \frac{5}{4}\,x\,e^x\left[\cos(2x) + \sin(2x)\right].$$

Step 3: General solution.

$$y(x) = e^x\Big[c_1\cos(2x) + c_2\sin(2x)\Big] + \frac{5}{4}\,x\,e^x\left[\cos(2x) + \sin(2x)\right].$$

Example 5: Modification rule (repeated-root overlap).

Worked Example

Solve $y'' - 4y' + 4y = 6e^{2x}$.

Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 - 4r + 4 = 0 \quad \Longrightarrow \quad (r-2)^2 = 0.$$

Double root: $r = 2$. Thus

$$y_h(x) = c_1 e^{2x} + c_2 x\, e^{2x}.$$

Step 2: Particular solution. Here $g(x) = 6e^{2x}$. The initial guess is $y_p = A\, e^{2x}$.

Overlap check: e^{2x} is in y_h. Modify: multiply by $x \Rightarrow y_p = Ax\, e^{2x}$.

Overlap check again: $x\, e^{2x}$ is *also* in y_h. Modify again: multiply by x once more $\Rightarrow y_p = Ax^2 e^{2x}$.

Now $x^2 e^{2x}$ is *not* in y_h, so this is our final guess. Differentiate (product rule):

$$\begin{aligned} y_p &= Ax^2 e^{2x}, \\ y_p' &= A(2xe^{2x} + 2x^2 e^{2x}) = 2Ax\, e^{2x} + 2Ax^2 e^{2x}, \\ y_p'' &= 2Ae^{2x} + 4Ax\, e^{2x} + 4Ax\, e^{2x} + 4Ax^2 e^{2x} \\ &= 2Ae^{2x} + 8Ax\, e^{2x} + 4Ax^2 e^{2x}. \end{aligned}$$

Substitute into $y'' - 4y' + 4y = 6e^{2x}$:

$$\begin{aligned} &\big[2Ae^{2x} + 8Ax\, e^{2x} + 4Ax^2 e^{2x}\big] - 4\big[2Ax\, e^{2x} + 2Ax^2 e^{2x}\big] + 4\big[Ax^2 e^{2x}\big] \\ &= 2Ae^{2x} + (8A - 8A)x\, e^{2x} + (4A - 8A + 4A)x^2 e^{2x} \\ &= 2Ae^{2x}. \end{aligned}$$

Set equal to $6e^{2x}$:

$$2A = 6 \quad \Longrightarrow \quad A = 3.$$

The particular solution is $y_p(x) = 3x^2 e^{2x}$.

Step 3: General solution.

$$y(x) = c_1 e^{2x} + c_2 x\, e^{2x} + 3x^2 e^{2x}.$$

Example 6: Polynomial times exponential.

Worked Example

Solve $y'' + 2y' + y = 3x\, e^{-x}$.

Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 + 2r + 1 = 0 \quad \Longrightarrow \quad (r+1)^2 = 0.$$

Double root: $r = -1$. Thus

$$y_h(x) = c_1 e^{-x} + c_2 x\, e^{-x}.$$

Step 2: Particular solution. Here $g(x) = 3x\, e^{-x} = e^{-x} P_1(x)$ where $P_1(x) = 3x$ is a first-degree polynomial. From the guess table, the initial guess would be

$$y_p(x) = e^{-x}(Ax + B).$$

Overlap check: e^{-x} and $x\, e^{-x}$ are both in y_h. We must multiply the entire guess by x^2:

$$y_p(x) = x^2 e^{-x}(Ax + B) = Ax^3 e^{-x} + Bx^2 e^{-x}.$$

Differentiate:

$$\begin{aligned} y_p &= (Ax^3 + Bx^2)e^{-x}, \\ y_p' &= (3Ax^2 + 2Bx)e^{-x} - (Ax^3 + Bx^2)e^{-x} = (-Ax^3 + (3A - B)x^2 + 2Bx)\, e^{-x}, \\ y_p'' &= (-3Ax^2 + 2(3A - B)x + 2B)e^{-x} - (-Ax^3 + (3A - B)x^2 + 2Bx)e^{-x} \\ &= \big[Ax^3 + (-3A + 2(3A - B) - (3A - B))x^2 + \dots\big]e^{-x}. \end{aligned}$$

Computing y_p'' more carefully with the product rule:

$$\begin{aligned} y_p' &= e^{-x}\big(-Ax^3 + (3A - B)x^2 + 2Bx\big), \\ y_p'' &= e^{-x}\Big(Ax^3 - (6A - B)x^2 + (6A - 4B)x + 2B\Big). \end{aligned}$$

Now substitute into $y'' + 2y' + y$:

$$\begin{aligned} y_p'' + 2y_p' + y_p &= e^{-x}\Big[\big(Ax^3 - (6A - B)x^2 + (6A - 4B)x + 2B\big) \\ &\quad + 2\big(-Ax^3 + (3A - B)x^2 + 2Bx\big) + \big(Ax^3 + Bx^2\big)\Big] \\ &= e^{-x}\Big[(A - 2A + A)x^3 + \big(-(6A - B) + 2(3A - B) + B\big)x^2 \\ &\quad + \big((6A - 4B) + 4B\big)x + 2B\Big] \\ &= e^{-x}\Big[0 \cdot x^3 + 0 \cdot x^2 + 6Ax + 2B\Big]. \end{aligned}$$

Set equal to $g(x) = 3x\,e^{-x}$:

$$\begin{cases} 6A = 3 & \Longrightarrow \quad A = \dfrac{1}{2}, \\ 2B = 0 & \Longrightarrow \quad B = 0. \end{cases}$$

So $y_p(x) = \dfrac{1}{2}x^3 e^{-x}$.

Step 3: General solution.

$$y(x) = c_1 e^{-x} + c_2 x\,e^{-x} + \frac{1}{2}x^3 e^{-x}.$$

9.3 Variation of Parameters

Variation of parameters (VoP) is a **universal** method for finding a particular solution. Unlike undetermined coefficients, it works for *any* forcing function $g(x)$, provided we can evaluate the resulting integrals. The trade-off is that VoP typically involves more computation.

9.3.1 Full Derivation

Setup. Consider the nonhomogeneous equation in **normalized form**:

$$y'' + p(x)\,y' + q(x)\,y = g(x). \tag{28}$$

Let $\{y_1(x), y_2(x)\}$ be a fundamental set of solutions for the associated homogeneous equation $y'' + p(x)y' + q(x)y = 0$. Their Wronskian is

$$W(x) = W(y_1, y_2)(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x) \neq 0.$$

Ansatz. We look for a particular solution of the form

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x), \tag{29}$$

where $u_1(x)$ and $u_2(x)$ are unknown functions to be determined. This is called "variation of parameters" because we have replaced the constant coefficients c_1, c_2 of the homogeneous solution with variable functions $u_1(x), u_2(x)$.

Differentiate y_p:

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'. \tag{30}$$

We have two unknown functions but only one equation (the ODE). To resolve this underdetermination, we impose an **auxiliary condition**:

$$u_1' y_1 + u_2' y_2 = 0. \tag{31}$$

This is not a restriction — it is a convenient choice that simplifies the algebra. With this constraint, the first derivative simplifies to

$$y_p' = u_1 y_1' + u_2 y_2'.$$

Differentiate once more:

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''. \tag{32}$$

Substitution. Substitute y_p, y_p', and y_p'' into the ODE equation (28):

$$\left[u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''\right] + p(x)\left[u_1y_1' + u_2y_2'\right] + q(x)\left[u_1y_1 + u_2y_2\right] = g(x).$$

Group terms by u_1 and u_2:

$$u_1'\,y_1' + u_2'\,y_2' + u_1\left[y_1'' + p(x)y_1' + q(x)y_1\right] + u_2\left[y_2'' + p(x)y_2' + q(x)y_2\right] = g(x).$$

Since y_1 and y_2 are solutions of the homogeneous equation, the bracketed terms vanish. We are left with

$$u_1'\,y_1' + u_2'\,y_2' = g(x).$$

System of equations. Combining this with the constraint equation (31), we have a 2×2 linear system for u_1' and u_2':

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = g(x). \end{cases} \tag{33}$$

Solution via Cramer's rule. Write the system in matrix form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

The determinant of the coefficient matrix is the Wronskian:

$$\det = y_1y_2' - y_1'y_2 = W(x).$$

Since $\{y_1, y_2\}$ is a fundamental set, $W(x) \neq 0$ and the system is solvable. By Cramer's rule:

$$u_1'(x) = \frac{\det\begin{pmatrix} 0 & y_2 \\ g(x) & y_2' \end{pmatrix}}{W(x)} = \frac{-y_2(x)\,g(x)}{W(x)}, \tag{34}$$

$$u_2'(x) = \frac{\det\begin{pmatrix} y_1 & 0 \\ y_1' & g(x) \end{pmatrix}}{W(x)} = \frac{y_1(x)\,g(x)}{W(x)}. \tag{35}$$

Integrate to find u_1 and u_2 (we may choose any antiderivatives; we set the constants of integration to zero since we need only *one* particular solution):

$$u_1(x) = -\int \frac{y_2(x)\,g(x)}{W(x)}\,\mathrm{d}x, \tag{36}$$

$$u_2(x) = \int \frac{y_1(x)\,g(x)}{W(x)}\,\mathrm{d}x. \tag{37}$$

Key Result

Variation of parameters formula.
Given a fundamental set $\{y_1, y_2\}$ of the homogeneous equation associated with equation (28), a particular solution is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)\,g(x)}{W(x)}\,\mathrm{d}x + y_2(x)\int \frac{y_1(x)\,g(x)}{W(x)}\,\mathrm{d}x, \tag{38}$$

where $W(x) = y_1y_2' - y_1'y_2$.
CRITICAL: The equation *must* be in normalized form (coefficient of y'' equals 1) before applying this formula. For $a\,y'' + b\,y' + c\,y = \tilde{g}(x)$, first divide by a so that the right-hand side becomes $g(x) = \tilde{g}(x)/a$.

Hint

Normalized form warning. A very common mistake is to apply the VoP formula to $a\,y'' + b\,y' + c\,y = \tilde{g}(x)$ directly, using $\tilde{g}(x)$ as $g(x)$. This is **incorrect**. The derivation assumed the coefficient of y'' is 1, so you *must* divide the entire equation by a first, making $g(x) = \tilde{g}(x)/a$.

9.3.2 Worked Examples

Worked Example

Solve $y'' + y = \sec(x)$ on the interval $(-\pi/2, \pi/2)$.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is $r^2 + 1 = 0$, so $r = \pm i$. Thus

$$y_h(x) = c_1 \cos x + c_2 \sin x.$$

We take $y_1(x) = \cos x$ and $y_2(x) = \sin x$.
Step 2: Wronskian.

$$W(x) = \cos x \cdot \cos x - (-\sin x) \cdot \sin x = \cos^2 x + \sin^2 x = 1.$$

The Wronskian is constant, which simplifies the integrals.
Step 3: Apply VoP. The equation is already in normalized form ($g(x) = \sec x$). Compute:

$$u_1(x) = -\int \frac{y_2(x)\, g(x)}{W(x)}\, \mathrm{d}x = -\int \frac{\sin x \cdot \sec x}{1}\, \mathrm{d}x = -\int \tan x \, \mathrm{d}x = \ln|\cos x|,$$

$$u_2(x) = \int \frac{y_1(x)\, g(x)}{W(x)}\, \mathrm{d}x = \int \frac{\cos x \cdot \sec x}{1}\, \mathrm{d}x = \int 1 \, \mathrm{d}x = x.$$

Step 4: Particular solution.

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x) = \ln|\cos x| \, \cos x + x \, \sin x.$$

Step 5: General solution.

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x \, \ln|\cos x| + x \, \sin x.$$

Worked Example

Solve $y'' - 4y' + 3y = \dfrac{e^{2x}}{1 + e^x}$ using variation of parameters.
Solution. *Step 1: Homogeneous solution.* The characteristic equation is

$$r^2 - 4r + 3 = 0 \quad \Longrightarrow \quad (r-1)(r-3) = 0.$$

Roots: $r_1 = 1$, $r_2 = 3$. Thus

$$y_h(x) = c_1 e^x + c_2 e^{3x}.$$

Take $y_1(x) = e^x$ and $y_2(x) = e^{3x}$.
Step 2: Wronskian.

$$W(x) = e^x \cdot 3e^{3x} - e^x \cdot e^{3x} = 3e^{4x} - e^{4x} = 2e^{4x}.$$

Step 3: Apply VoP. The equation is in normalized form with $g(x) = \dfrac{e^{2x}}{1 + e^x}$. Compute:

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)\, g(x)}{W(x)}\, \mathrm{d}x = -\int \frac{e^{3x} \cdot \dfrac{e^{2x}}{1+e^x}}{2e^{4x}}\, \mathrm{d}x \\ &= -\frac{1}{2} \int \frac{e^{5x}}{e^{4x}(1+e^x)}\, \mathrm{d}x = -\frac{1}{2} \int \frac{e^x}{1+e^x}\, \mathrm{d}x. \end{aligned}$$

Substituting $u = 1 + e^x$, $\mathrm{d}u = e^x \, \mathrm{d}x$:

$$u_1(x) = -\frac{1}{2} \int \frac{1}{u}\, \mathrm{d}u = -\frac{1}{2} \ln|u| = -\frac{1}{2} \ln(1 + e^x).$$

Next:

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)\, g(x)}{W(x)}\, \mathrm{d}x = \int \frac{e^x \cdot \dfrac{e^{2x}}{1+e^x}}{2e^{4x}}\, \mathrm{d}x \\ &= \frac{1}{2} \int \frac{e^{3x}}{e^{4x}(1+e^x)}\, \mathrm{d}x = \frac{1}{2} \int \frac{1}{e^x(1+e^x)}\, \mathrm{d}x. \end{aligned}$$

Use partial fractions: $\dfrac{1}{e^x(1+e^x)} = \dfrac{1}{e^x} - \dfrac{1}{1+e^x} = e^{-x} - \dfrac{1}{1+e^x}$.

$$\begin{aligned} u_2(x) &= \frac{1}{2}\int \left(e^{-x} - \frac{1}{1+e^x}\right) \mathrm{d}x \\ &= \frac{1}{2}\Big(-e^{-x} - x + \ln(1+e^x)\Big). \end{aligned}$$

(The second integral: $\int \frac{1}{1+e^x}\mathrm{d}x = x - \ln(1+e^x)$, obtained by writing $\frac{1}{1+e^x} = 1 - \frac{e^x}{1+e^x}$.)
Step 4: Particular solution.

$$\begin{aligned} y_p(x) &= u_1(x)\,y_1(x) + u_2(x)\,y_2(x) \\ &= -\frac{1}{2}\ln(1+e^x)\,e^x + \frac{1}{2}\Big(-e^{-x} - x + \ln(1+e^x)\Big)e^{3x} \\ &= -\frac{1}{2}e^x\ln(1+e^x) - \frac{1}{2}e^{2x} + \frac{1}{2}e^{3x}\ln(1+e^x) - \frac{1}{2}x\,e^{3x}. \end{aligned}$$

Step 5: General solution.

$$y(x) = c_1e^x + c_2e^{3x} - \frac{1}{2}e^x\ln(1+e^x) + \frac{1}{2}e^{3x}\ln(1+e^x) - \frac{1}{2}e^{2x} - \frac{1}{2}x\,e^{3x}.$$

Worked Example

Solve $2y'' + 4y' + 2y = e^{-x}$ using variation of parameters. Note that this equation is **not** in normalized form.
Solution. *Step 0: Normalize the equation.* Divide the entire equation by 2:

$$y'' + 2y' + y = \frac{1}{2}\,e^{-x}.$$

Now $g(x) = \dfrac{1}{2}e^{-x}$.
Step 1: Homogeneous solution. The characteristic equation is

$$r^2 + 2r + 1 = 0 \quad\Longrightarrow\quad (r+1)^2 = 0.$$

Double root $r = -1$. Take $y_1(x) = e^{-x}$ and $y_2(x) = x\,e^{-x}$.
Step 2: Wronskian.

$$W(x) = e^{-x}\big(e^{-x} - x\,e^{-x}\big) - (-e^{-x})(x\,e^{-x}) = e^{-2x}(1-x) + x\,e^{-2x} = e^{-2x}.$$

Step 3: Apply VoP.

$$u_1(x) = -\int \frac{y_2(x)\,g(x)}{W(x)}\,\mathrm{d}x = -\int \frac{x\,e^{-x}\cdot\frac{1}{2}e^{-x}}{e^{-2x}}\,\mathrm{d}x = -\frac{1}{2}\int x\,\mathrm{d}x = -\frac{1}{4}x^2,$$

$$u_2(x) = \int \frac{y_1(x)\,g(x)}{W(x)}\,\mathrm{d}x = \int \frac{e^{-x}\cdot\frac{1}{2}e^{-x}}{e^{-2x}}\,\mathrm{d}x = \frac{1}{2}\int 1\,\mathrm{d}x = \frac{1}{2}x.$$

Step 4: Particular solution.

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x) = -\frac{1}{4}x^2\,e^{-x} + \frac{1}{2}x^2\,e^{-x} = \frac{1}{4}x^2\,e^{-x}.$$

This is consistent with the undetermined coefficients result from Example 5 (the modification rule also gave x^2e^{-x} for this case).
Step 5: General solution.

$$y(x) = c_1e^{-x} + c_2x\,e^{-x} + \frac{1}{4}x^2\,e^{-x}.$$

9.4 Method Comparison

Both undetermined coefficients (UC) and variation of parameters (VoP) find particular solutions, but they have different strengths and limitations.

Table 8: Undetermined coefficients vs. variation of parameters

Aspect	Undetermined Coefficients	Variation of Parameters
Applicable $g(x)$	Polynomials, $e^{\alpha x}$, $\sin / \cos(\beta x)$, sums/products	*Any* continuous $g(x)$ (in principle)
Requires y_h?	Only for overlap check	Yes — need a full fundamental set
Requires $W(x)$?	No	Yes
Normalized form?	Not required	**Required** (coefficient of y'' must be 1)
Computation	Algebra (solve linear system)	Integration (may be difficult or non-elementary)
Speed (simple $g(x)$)	Fast — algebraic, no integrals	Slower — integrals needed
Speed (complex $g(x)$)	**N/A** — method fails	Always applicable (if integrals exist)

Hint

Decision guide.

1. Is $g(x)$ a polynomial, exponential, sine/cosine, or a product of these? **Yes** $\Rightarrow$ use undetermined coefficients (simpler algebra).
2. Is $g(x)$ anything else ($\ln x$, $\tan x$, rational functions, etc.)? **Yes** $\Rightarrow$ use variation of parameters.
3. If you are unsure, variation of parameters is always a safe fallback (provided you can compute the integrals).

Example comparison. Consider $y'' + y' - 2y = 4x$.

Using **undetermined coefficients**: guess $y_p = Ax + B$, substitute, solve the linear system. Takes about 5 lines of algebra.

Using **variation of parameters**: find y_h, compute $W(x)$, evaluate two integrals involving $\int xe^{x}\,\mathrm{d}x$ and $\int xe^{-2x}\,\mathrm{d}x$ (by parts). Takes about 15 lines of computation.

Both give the same answer, but UC is clearly faster here. However, if $g(x) = \ln x$, only VoP can be applied.

9.5 Summary

Table 9: Nonhomogeneous second-order equations: complete reference

Concept	Key formula / rule
General solution	$y(x) = y_h(x) + y_p(x)$ (Theorem 9.1)
Homogeneous part	From section 8: $y_h = c_1 y_1 + c_2 y_2$
UC: polynomial $P_n(x)$	$y_p = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_0$
UC: $e^{\alpha x}$	$y_p = A\,e^{\alpha x}$
UC: $\sin(\beta x)$ **or** $\cos(\beta x)$	$y_p = A\sin(\beta x) + B\cos(\beta x)$
UC: $e^{\alpha x}P_n(x)$	$y_p = e^{\alpha x}(A_n x^n + \cdots + A_0)$
UC: $e^{\alpha x}\sin(\beta x)$ **or** $e^{\alpha x}\cos(\beta x)$	$y_p = e^{\alpha x}\big[(A_n x^n + \cdots + A_0)\cos(\beta x) + (B_n x^n + \cdots + B_0)\sin(\beta x)\big]$
UC modification rule	If guess term $\in y_h$, multiply by x; if still $\in y_h$, multiply by x^2
VoP formula	$y_p = -y_1 \displaystyle\int \frac{y_2 g}{W}\,\mathrm{d}x + y_2 \int \frac{y_1 g}{W}\,\mathrm{d}x$
VoP constraint	Equation must be normalized: $y'' + py' + qy = g$ (coeff. of $y'' = 1$)
VoP system	$u_1' y_1 + u_2' y_2 = 0,\ \ u_1' y_1' + u_2' y_2' = g(x)$
VoP Cramer's rule	$u_1' = -\dfrac{y_2 g}{W},\ \ u_2' = \dfrac{y_1 g}{W}$

Hint

Problem-solving workflow for nonhomogeneous equations.

1. **Identify the form of $g(x)$.** If it is a polynomial, exponential, trig function, or their products/sums, prefer undetermined coefficients. Otherwise, use variation of parameters.
2. **Solve the homogeneous equation** $ay'' + by' + cy = 0$ and obtain $y_h(x)$.
3. **If using UC:** Write the guess based on the table, check for overlap with y_h, modify if needed, substitute, and solve for coefficients.
4. **If using VoP:** *First* normalize the equation (divide by a if needed). Compute $W(x)$, then evaluate the two integrals for u_1 and u_2.
5. **Combine:** $y(x) = y_h(x) + y_p(x)$. Apply initial conditions if given.

10 Mechanical Applications

Many real-world systems are modeled by second-order linear differential equations. In this chapter, we apply the theory developed in sections 8 and 9 to mechanical vibrations and the closely related RLC electrical circuits. Every model follows the same pipeline: **real-world setup** → **force analysis** → **ODE** → **solution**.

10.1 Spring-Mass Systems

We begin with the fundamental mechanical system: a mass attached to a spring, possibly with a damper and an external driving force.

Physical setup. Consider a mass m attached to a vertical spring with spring constant k. The spring is fixed at its top end. We define the **equilibrium position** as the point where the mass hangs at rest with no motion. We measure the displacement $u(t)$ from this equilibrium position, with the **downward direction taken as positive**.

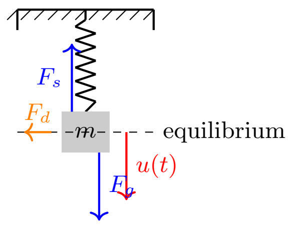


Figure 2: Spring-mass-damper system. The mass m is displaced by $u(t)$ from equilibrium. Downward is positive.

Force analysis. We identify four forces acting on the mass (positive direction = downward):

1. **Gravity:** $F_g = mg$.
2. **Spring force (Hooke's law):** The spring exerts a restoring force proportional to its stretch from the natural length. If L is the stretch at equilibrium, then the total stretch is $L + u(t)$. The spring force (upward) is $F_s = -k(L + u)$.
3. **Damping force:** A dashpot/damper provides a force proportional to velocity, opposing motion: $F_d = -c\,u'$, where $c > 0$ is the damping coefficient.
4. **External force:** An applied driving force $F(t)$.

Newton's Second Law. Summing forces and applying $F_{\text{net}} = ma$:

$$m\,u'' = mg - k(L + u) - c\,u' + F(t). \tag{39}$$

At equilibrium (the mass hangs at rest with $u = 0$ and $u' = u'' = 0$):

$$0 = mg - kL \quad \Longrightarrow \quad mg = kL. \tag{40}$$

Substituting $mg = kL$ into equation (39):

$$m\,u'' + c\,u' + k\,u = F(t). \tag{41}$$

Key Result

Spring-mass-damper equation. The displacement $u(t)$ of a mass-spring-damper system from equilibrium satisfies

$$m\,u''(t) + c\,u'(t) + k\,u(t) = F(t),$$

where $m > 0$ is the mass, $c \geq 0$ is the damping coefficient, $k > 0$ is the spring constant, and $F(t)$ is the external force.

When $F(t) \equiv 0$, the equation is homogeneous and describes **free vibrations**. When $F(t) \neq 0$, we have **forced vibrations**. Both cases are analyzed below. The ODE equation (41) has the same mathematical form as the general second-order linear equation equation (27) from section 9, so all the solution methods from sections 8 and 9 apply directly.

Hint

Key insight. The displacement $u(t)$ is measured from *equilibrium*, not from the spring's natural length. This is what eliminates the mg and kL terms and gives us the clean equation equation (41). Always set up your coordinate system this way.

Worked Example

A spring stretches 0.3 m when a 2 kg mass is attached. The mass is then pulled down an additional 0.2 m from equilibrium and released with an upward velocity of 1 m/s. There is no damping and no external force. Set up the initial value problem (IVP) and solve for $u(t)$.

Solution. *Step 1: Determine parameters.* The mass is $m = 2\,\text{kg}$. At equilibrium, $mg = kL$, so

$$k = \frac{mg}{L} = \frac{2 \cdot 9.8}{0.3} = \frac{19.6}{0.3} = \frac{196}{3}\,\text{N/m}.$$

There is no damping: $c = 0$. No external force: $F(t) = 0$.

Step 2: Form the IVP. The ODE is

$$2\,u'' + \frac{196}{3}\,u = 0 \quad\Longrightarrow\quad u'' + \frac{98}{3}\,u = 0.$$

Initial conditions: the mass is pulled down 0.2 m, so $u(0) = 0.2$. It is released with upward velocity 1 m/s; since downward is positive, $u'(0) = -1$.

Step 3: Solve. The characteristic equation is

$$r^2 + \frac{98}{3} = 0 \quad\Longrightarrow\quad r = \pm i\sqrt{\frac{98}{3}} = \pm i\,\frac{7\sqrt{2}}{\sqrt{3}}.$$

Let $\omega_0 = \sqrt{98/3} \approx 5.715$. The general solution is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t).$$

Differentiate:

$$u'(t) = -\omega_0 c_1 \sin(\omega_0 t) + \omega_0 c_2 \cos(\omega_0 t).$$

Apply initial conditions:

$$\begin{cases} u(0) = c_1 = 0.2, \\ u'(0) = \omega_0 c_2 = -1 \end{cases} \quad\Longrightarrow\quad c_2 = -\frac{1}{\omega_0} = -\frac{\sqrt{3}}{7\sqrt{2}} \approx -0.175.$$

The solution is

$$u(t) = 0.2\,\cos(\omega_0 t) - \frac{1}{\omega_0}\,\sin(\omega_0 t), \qquad \omega_0 = \sqrt{\frac{98}{3}}.$$

10.2 Free Vibrations: Damping Cases

We now analyze the homogeneous equation

$$m\,u'' + c\,u' + k\,u = 0, \tag{42}$$

which models free vibrations (no external driving force). Dividing by m:

$$u'' + \frac{c}{m}\,u' + \frac{k}{m}\,u = 0. \tag{43}$$

The characteristic equation is

$$m\,r^2 + c\,r + k = 0, \tag{44}$$

with roots

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

The behavior depends entirely on the sign of the discriminant $\Delta = c^2 - 4mk$. We define the **natural frequency**

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and the **critical damping coefficient**

$$c_{\text{cr}} = 2\sqrt{mk} = 2m\omega_0. \tag{45}$$

The three cases correspond to $c < c_{\text{cr}}$, $c = c_{\text{cr}}$, and $c > c_{\text{cr}}$.

10.2.1 Underdamped Case ($c^2 < 4mk$)

When $c^2 < 4mk$ (equivalently, $c < c_{\text{cr}}$), the discriminant is negative and the roots are complex conjugates:

$$r = -\frac{c}{2m} \pm i\,\omega_d, \qquad \omega_d = \frac{\sqrt{4mk - c^2}}{2m} = \sqrt{\omega_0^2 - \left(\frac{c}{2m}\right)^2}.$$

Key Result

Underdamped solution. When $c^2 < 4mk$, the general solution is

$$u(t) = e^{-ct/(2m)}\Big[c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\Big],$$

where $\omega_d = \sqrt{\omega_0^2 - (c/(2m))^2}$ is the **damped natural frequency**. Equivalently, in amplitude–phase form:

$$u(t) = A\,e^{-ct/(2m)} \cos(\omega_d t - \phi),$$

with $A = \sqrt{c_1^2 + c_2^2}$ and $\phi = \arctan(c_2/c_1)$.

Physical interpretation. The mass oscillates at frequency ω_d (slightly lower than the natural frequency ω_0), while the exponential envelope $e^{-ct/(2m)}$ causes the amplitude to decay over time. The system is "underdamped" because the damping is not strong enough to prevent oscillation.

Hint

Identifying the damping regime. Compute the ratio c/c_{cr}. If $c/c_{\text{cr}} < 1$, the system is underdamped; if equal to 1, critically damped; if greater than 1, overdamped. The dimensionless ratio $\zeta = c/c_{\text{cr}}$ is called the **damping ratio**.

10.2.2 Critically Damped Case ($c^2 = 4mk$)

When $c^2 = 4mk$ (equivalently, $c = c_{\text{cr}}$), the characteristic equation has a double root:

$$r = -\frac{c}{2m} = -\omega_0.$$

Key Result

Critically damped solution. When $c^2 = 4mk$, the general solution is

$$u(t) = c_1 e^{-ct/(2m)} + c_2\, t\, e^{-ct/(2m)}.$$

Physical interpretation. The mass returns to equilibrium **as fast as possible without oscillating**. This is often the desired behavior in engineering systems such as door closers and shock absorbers: you want the system to settle quickly but without bouncing. The $t\, e^{-ct/(2m)}$ factor is a direct application of the repeated-root solution from section 8.3.

10.2.3 Overdamped Case ($c^2 > 4mk$)

When $c^2 > 4mk$ (equivalently, $c > c_{\text{cr}}$), the discriminant is positive and the roots are two distinct real numbers:

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}.$$

Both roots are negative (since $\sqrt{c^2 - 4mk} < c$ when $k > 0$).

Key Result

Overdamped solution. When $c^2 > 4mk$, the general solution is

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

with $r_1 < 0$ and $r_2 < 0$.

Physical interpretation. The mass returns to equilibrium without oscillation, but more slowly than the critically damped case. The heavy damping "overdamps" the system, causing sluggish motion. The displacement is a sum of two decaying exponentials.

10.2.4 Damping Cases Comparison

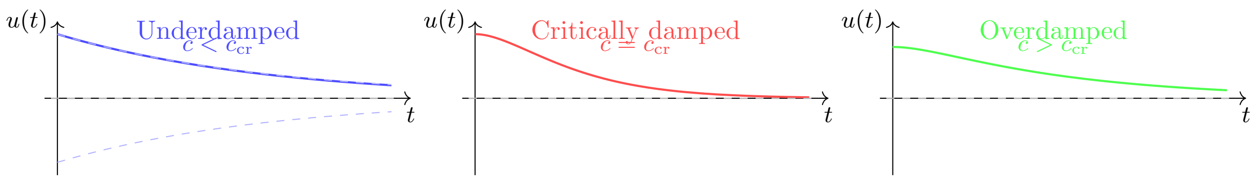


Figure 3: Displacement vs. time for the three damping regimes (all with $u(0) > 0$, $u'(0) = 0$). Underdamped: oscillatory decay. Critically damped: fastest non-oscillatory return. Overdamped: slow non-oscillatory decay.

10.2.5 Worked Examples

Worked Example

(Underdamped IVP) A 4 kg mass is attached to a spring with constant $k = 64\,\text{N/m}$. The damping coefficient is $c = 8\,\text{N}\cdot\text{s/m}$. The mass is pulled down 0.5 m from equilibrium and released with zero initial velocity. Find $u(t)$.

Solution. *Step 1: Determine the damping regime.*

$$c^2 = 64, \qquad 4mk = 4 \cdot 4 \cdot 64 = 1024.$$

Since $c^2 = 64 < 1024 = 4mk$, the system is **underdamped**.

Step 2: Compute parameters.

$$\frac{c}{2m} = \frac{8}{8} = 1, \qquad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{64}{4}} = 4,$$

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{c}{2m}\right)^2} = \sqrt{16 - 1} = \sqrt{15}.$$

Step 3: General solution.

$$u(t) = e^{-t}\Big[c_1 \cos(\sqrt{15}\,t) + c_2 \sin(\sqrt{15}\,t)\Big].$$

Step 4: Apply initial conditions. $u(0) = 0.5$ and $u'(0) = 0$.

$$u(0) = c_1 = 0.5.$$

Differentiate:

$$u'(t) = -e^{-t}\big[c_1 \cos(\sqrt{15}\,t) + c_2 \sin(\sqrt{15}\,t)\big] + e^{-t}\big[-\sqrt{15}\,c_1 \sin(\sqrt{15}\,t) + \sqrt{15}\,c_2 \cos(\sqrt{15}\,t)\big].$$

$$u'(0) = -c_1 + \sqrt{15}\,c_2 = 0 \quad\Longrightarrow\quad c_2 = \frac{c_1}{\sqrt{15}} = \frac{0.5}{\sqrt{15}} = \frac{1}{2\sqrt{15}}.$$

The solution is

$$u(t) = e^{-t}\left(\frac{1}{2}\cos(\sqrt{15}\,t) + \frac{1}{2\sqrt{15}}\sin(\sqrt{15}\,t)\right).$$

The mass oscillates with damped frequency $\sqrt{15} \approx 3.87\,\mathrm{rad/s}$ while the amplitude decays like e^{-t}.

Worked Example

(Overdamped IVP) A $2\,\mathrm{kg}$ mass is attached to a spring with $k = 10\,\mathrm{N/m}$ and a damper with $c = 14\,\mathrm{N \cdot s/m}$. The mass is displaced $1\,\mathrm{m}$ above equilibrium and released with downward velocity $3\,\mathrm{m/s}$. Find $u(t)$.
Solution. *Step 1: Determine the damping regime.*

$$c^2 = 196, \qquad 4mk = 4 \cdot 2 \cdot 10 = 80.$$

Since $c^2 = 196 > 80 = 4mk$, the system is **overdamped**.
Step 2: Compute roots.

$$r = \frac{-14 \pm \sqrt{196 - 80}}{4} = \frac{-14 \pm \sqrt{116}}{4} = \frac{-14 \pm 2\sqrt{29}}{4} = \frac{-7 \pm \sqrt{29}}{2}.$$

So $r_1 = \dfrac{-7 + \sqrt{29}}{2} \approx -0.191$ and $r_2 = \dfrac{-7 - \sqrt{29}}{2} \approx -6.809$.
Step 3: General solution.

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Step 4: Apply initial conditions. $u(0) = -1$ (above equilibrium, so negative) and $u'(0) = 3$ (downward is positive).

$$u(0) = c_1 + c_2 = -1.$$

$$u'(0) = r_1 c_1 + r_2 c_2 = 3.$$

From the first equation, $c_2 = -1 - c_1$. Substituting into the second:

$$r_1 c_1 + r_2(-1 - c_1) = 3 \quad\Longrightarrow\quad (r_1 - r_2)c_1 = 3 + r_2.$$

$$c_1 = \frac{3 + r_2}{r_1 - r_2} = \frac{3 + \frac{-7-\sqrt{29}}{2}}{\sqrt{29}} = \frac{6 - 7 - \sqrt{29}}{2\sqrt{29}} = \frac{-1 - \sqrt{29}}{2\sqrt{29}},$$

$$c_2 = -1 - c_1 = -1 + \frac{1 + \sqrt{29}}{2\sqrt{29}} = \frac{-2\sqrt{29} + 1 + \sqrt{29}}{2\sqrt{29}} = \frac{1 - \sqrt{29}}{2\sqrt{29}}.$$

The solution is

$$u(t) = \frac{-1 - \sqrt{29}}{2\sqrt{29}}\, e^{r_1 t} + \frac{1 - \sqrt{29}}{2\sqrt{29}}\, e^{r_2 t}.$$

Both exponential terms decay to zero. The term with $r_2 \approx -6.809$ decays very rapidly, so for large t, the behavior is dominated by the slower-decaying $r_1 \approx -0.191$ term.

10.3 Forced Vibrations

We now consider the nonhomogeneous equation

$$m\,u'' + c\,u' + k\,u = F_0 \cos(\omega t), \tag{46}$$

where a periodic external force $F_0 \cos(\omega t)$ drives the system. The forcing frequency ω may or may not coincide with the system's natural frequency ω_0.

The general solution is $u(t) = u_h(t) + u_p(t)$, where $u_h(t)$ is the homogeneous solution (free vibration, from section 10.2) and $u_p(t)$ is a particular solution forced by the driving term.

Finding the steady-state solution. Since $g(t) = F_0 \cos(\omega t)$ is a trigonometric function, we can use the method of undetermined coefficients from section 9.2. The guess is

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t).$$

(Note: if $\omega = \omega_0$ and $c = 0$, this guess overlaps with the homogeneous solution and requires the modification rule; this is the pure resonance case discussed in section 10.4.)

Differentiating:

$$\begin{aligned} u_p'(t) &= -\omega A\sin(\omega t) + \omega B\cos(\omega t), \\ u_p''(t) &= -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t). \end{aligned}$$

Substitute into equation (46):

$$\left[-m\omega^2 A + c\omega B + kA\right]\cos(\omega t) + \left[-m\omega^2 B - c\omega A + kB\right]\sin(\omega t) = F_0\cos(\omega t).$$

Equating coefficients:

$$\begin{cases} (k - m\omega^2)A + c\omega B = F_0, \\ -c\omega A + (k - m\omega^2)B = 0. \end{cases}$$

Solving this system (e.g., by Cramer's rule):

$$\det = (k - m\omega^2)^2 + (c\omega)^2,$$

$$A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + (c\omega)^2}, \qquad B = \frac{F_0\, c\omega}{(k - m\omega^2)^2 + (c\omega)^2}.$$

It is more illuminating to write the solution in amplitude–phase form. Define

$$C = \sqrt{A^2 + B^2}, \qquad \delta = \arctan\left(\frac{B}{A}\right) = \arctan\left(\frac{c\omega}{k - m\omega^2}\right).$$

Then

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \delta).$$

Key Result

Steady-state (particular) solution for forced vibrations. For $m\,u'' + c\,u' + k\,u = F_0\cos(\omega t)$, the steady-state solution is

$$u_p(t) = C\cos(\omega t - \delta),$$

where

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}, \qquad \delta = \arctan\left(\frac{c\omega}{k - m\omega^2}\right).$$

The amplitude C and phase shift δ depend on the forcing frequency ω, the system parameters, and the forcing strength F_0.

Amplitude vs. frequency. The amplitude $C(\omega)$ as a function of the forcing frequency is called the **amplitude response** or **frequency response**. Its shape reveals the resonance behavior of the system.

Key observations:

- When $\omega \ll \omega_0$, the amplitude is approximately F_0/k (the static deflection).
- As ω approaches ω_0, the amplitude grows significantly.
- When $\omega \gg \omega_0$, the amplitude decays like $1/\omega^2$ (the mass cannot keep up with the rapid forcing).
- With damping ($c > 0$), the peak is finite. Without damping ($c = 0$), the amplitude goes to infinity at $\omega = \omega_0$ (pure resonance).

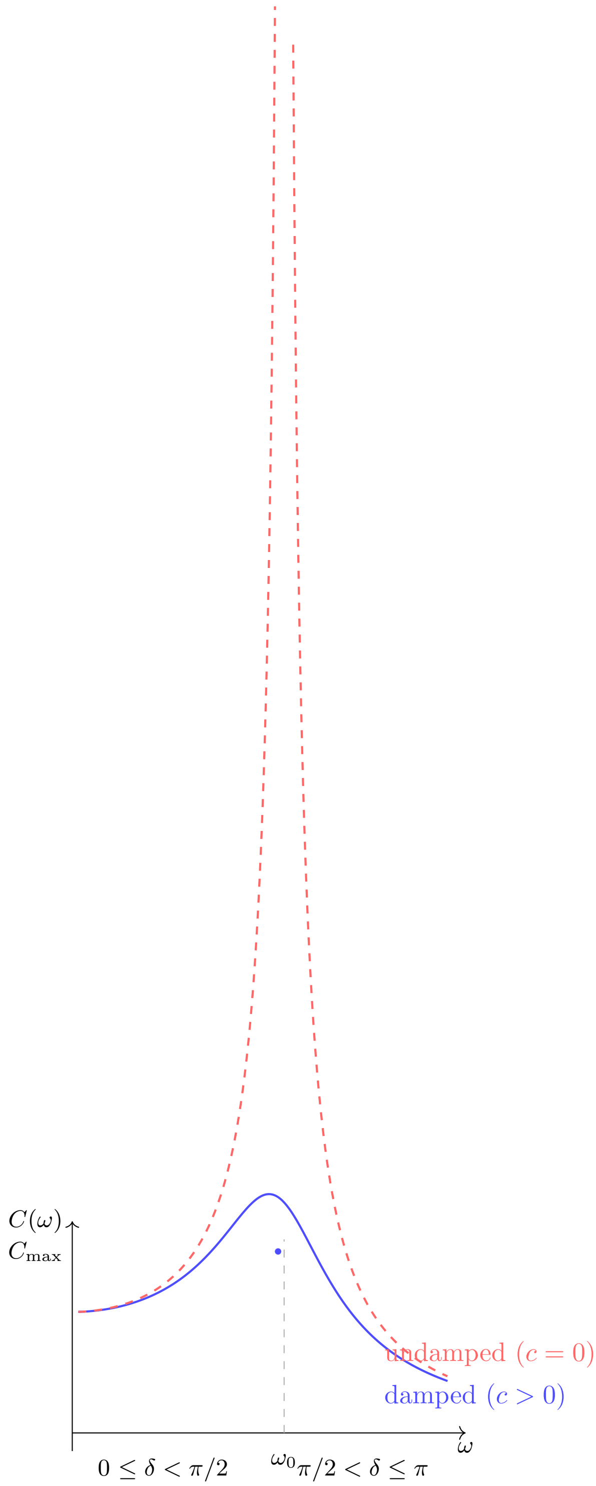


Figure 4: Amplitude response $C(\omega)$ for forced vibrations. The damped system (blue) has a finite peak near ω_0. The undamped system (red dashed) has a vertical asymptote at $\omega = \omega_0$ (pure resonance).

Phase shift. The phase shift δ describes how much the response lags behind the forcing:

- When $\omega \ll \omega_0$: $\delta \approx 0$ (response is in phase with forcing).
- When $\omega = \omega_0$: $\delta = \pi/2$ (response lags by 90°).
- When $\omega \gg \omega_0$: $\delta \approx \pi$ (response is nearly out of phase).

Hint

Two components of the full solution. The complete solution is $u(t) = u_h(t) + u_p(t)$. The homogeneous part $u_h(t)$ (transient response) decays to zero when $c > 0$, leaving only the steady-state response $u_p(t)$ at large times. This is why engineers focus primarily on $u_p(t)$ for long-term behavior.

Worked Example

Consider a mass-spring-damper system with $m = 1\,\text{kg}$, $k = 4\,\text{N/m}$, and $c = 2\,\text{N}\cdot\text{s/m}$. The system is driven by $F(t) = 8\cos(2t)$. Find the steady-state solution.
Solution. The forcing frequency is $\omega = 2$. The natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. Since $\omega = \omega_0$, this is the resonant case (though damped, so no unbounded growth).
The amplitude is

$$C = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{8}{\sqrt{(4 - 1\cdot 4)^2 + (2\cdot 2)^2}} = \frac{8}{\sqrt{0 + 16}} = \frac{8}{4} = 2.$$

The phase shift:

$$\delta = \arctan\left(\frac{c\omega}{k - m\omega^2}\right) = \arctan\left(\frac{4}{0}\right) = \frac{\pi}{2}.$$

The steady-state solution is

$$u_p(t) = 2\cos\left(2t - \frac{\pi}{2}\right) = 2\sin(2t).$$

(We used $\cos(\theta - \pi/2) = \sin(\theta)$.) The amplitude is $2\,\text{m}$ and the response is a pure sine wave, lagging the cosine forcing by 90°.

Worked Example

Using the same system ($m = 1$, $k = 4$, $c = 2$), suppose the forcing is instead $F(t) = 8\cos(t)$. Find the steady-state solution and compare the amplitude to the resonant case.
Solution. Here $\omega = 1$, which is below resonance ($\omega_0 = 2$).

$$C = \frac{8}{\sqrt{(4 - 1\cdot 1)^2 + (2\cdot 1)^2}} = \frac{8}{\sqrt{9 + 4}} = \frac{8}{\sqrt{13}} \approx 2.22.$$

$$\delta = \arctan\left(\frac{2\cdot 1}{4 - 1\cdot 1}\right) = \arctan\left(\frac{2}{3}\right) \approx 0.588\,\text{rad} \approx 33.7°.$$

The steady-state solution is

$$u_p(t) \approx 2.22\cos(t - 0.588).$$

Interestingly, the amplitude at $\omega = 1$ is slightly larger than the amplitude at resonance $\omega = 2$ (where it was 2.0). This is because the *practical resonance* peak (see section 10.4) shifts slightly below ω_0 when damping is present. Here $c^2 = 4$ and $2mk = 8$, so $c^2 < 2mk$ and practical resonance exists at $\omega_{\text{pr}} = \sqrt{4 - 4/2} = \sqrt{2} \approx 1.414$, between $\omega = 1$ and $\omega = 2$.

10.4 Resonance

Resonance is the phenomenon where a periodic driving force causes dramatically amplified oscillations. It is one of the most important concepts in mechanical engineering and has both beneficial applications and catastrophic failure modes.

10.4.1 Pure Resonance (Undamped, $c = 0$)

Consider the undamped forced equation

$$m\,u'' + k\,u = F_0\cos(\omega t).$$

The natural frequency is $\omega_0 = \sqrt{k/m}$. When the forcing frequency ω differs from ω_0, the method of undetermined coefficients gives the steady-state solution

$$u_p(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t),$$

which has a finite amplitude $F_0/|k - m\omega^2|$.

But when $\omega = \omega_0$, the coefficient $k - m\omega^2 = 0$ and the amplitude would be infinite — the guess $A\cos(\omega_0 t)$ fails because it is a solution of the homogeneous equation. We must apply the **modification rule** from section 9.2.2: multiply the guess by t.

Try $u_p(t) = t\,[A\cos(\omega_0 t) + B\sin(\omega_0 t)]$. Differentiating:

$$\begin{aligned} u_p' &= A\cos(\omega_0 t) + B\sin(\omega_0 t) + \omega_0 t[-A\sin(\omega_0 t) + B\cos(\omega_0 t)], \\ u_p'' &= -2\omega_0 A\sin(\omega_0 t) + 2\omega_0 B\cos(\omega_0 t) - \omega_0^2 t[A\cos(\omega_0 t) + B\sin(\omega_0 t)]. \end{aligned}$$

Substitute into $u'' + \omega_0^2 u = (F_0/m)\cos(\omega_0 t)$:

$$-2\omega_0 A\sin(\omega_0 t) + 2\omega_0 B\cos(\omega_0 t) = \frac{F_0}{m}\cos(\omega_0 t).$$

Equating coefficients:

$$-2\omega_0 A = 0 \implies A = 0, \qquad 2\omega_0 B = \frac{F_0}{m} \implies B = \frac{F_0}{2m\omega_0}.$$

Key Result

Pure resonance solution. For $m\,u'' + k\,u = F_0\cos(\omega_0 t)$ (undamped, forcing at natural frequency), the particular solution is

$$u_p(t) = \frac{F_0}{2m\omega_0}\,t\,\sin(\omega_0 t).$$

The general solution is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0}\,t\,\sin(\omega_0 t).$$

The amplitude grows **linearly with time**, leading to unbounded oscillation.

Physical interpretation. In pure resonance, energy is fed into the system at exactly the right rate to build oscillations without limit. The $t\,\sin(\omega_0 t)$ term represents oscillations whose envelope grows linearly. In reality, some damping is always present, so true pure resonance does not occur, but it provides a useful theoretical limit.

Catastrophic examples.

- **Tacoma Narrows Bridge collapse (1940):** Wind-induced oscillations excited the bridge's natural frequency, leading to destructive resonance and eventual collapse.
- **Operational military marching:** Soldiers are ordered to break step when crossing bridges to avoid exciting the bridge's natural frequency.
- **Breaking glass with sound:** An opera singer hitting the right note can shatter a wine glass through resonance.

Beneficial applications.

- **Tuning forks and musical instruments:** Resonance amplifies the sound at specific frequencies.
- **MRI machines:** Nuclear magnetic resonance exploits resonance at the atomic level.
- **Radio receivers:** Tuned circuits resonate at the frequency of the desired broadcast.

Worked Example

(Pure resonance) A mass $m = 1\,\text{kg}$ is attached to a spring with $k = 4\,\text{N/m}$. There is no damping. The system is driven by $F(t) = 8\cos(2t)$. The mass starts at equilibrium with zero velocity: $u(0) = 0$, $u'(0) = 0$. Find $u(t)$.

Solution. The natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. The forcing frequency is $\omega = 2$, so $\omega = \omega_0$: this is **pure resonance**.
The ODE is

$$u'' + 4u = 8\cos(2t).$$

The homogeneous solution is

$$u_h(t) = c_1\cos(2t) + c_2\sin(2t).$$

Using the pure resonance formula (or undetermined coefficients with modification), the particular solution is

$$u_p(t) = \frac{F_0}{2m\omega_0}\,t\,\sin(\omega_0 t) = \frac{8}{2\cdot 1\cdot 2}\,t\,\sin(2t) = 2t\sin(2t).$$

The general solution is

$$u(t) = c_1\cos(2t) + c_2\sin(2t) + 2t\sin(2t).$$

Apply initial conditions:

$$u(0) = c_1 = 0.$$

$$u'(t) = -2c_1\sin(2t) + 2c_2\cos(2t) + 2\sin(2t) + 4t\cos(2t),$$

$$u'(0) = 2c_2 = 0 \implies c_2 = 0.$$

The solution is

$$u(t) = 2t\sin(2t).$$

The displacement grows without bound, with an oscillation frequency of 2 rad/s and an amplitude that increases linearly as $2t$. After 10 s, the amplitude reaches 20 m.

10.4.2 Practical Resonance (Damped, $c > 0$)

When damping is present, the amplitude $C(\omega)$ is always finite, but it still has a maximum. This maximum defines **practical resonance**.

Recall the amplitude formula:

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}.$$

To find the frequency at which $C(\omega)$ is maximized, we minimize the denominator. Equivalently, we minimize the square of the denominator:

$$D(\omega) = (k - m\omega^2)^2 + (c\omega)^2.$$

Differentiating with respect to ω:

$$D'(\omega) = 2(k - m\omega^2)(-2m\omega) + 2c^2\omega = -4mk\omega + 4m^2\omega^3 + 2c^2\omega = 2\omega(2m^2\omega^2 - 2mk + c^2).$$

Setting $D'(\omega) = 0$ (and discarding $\omega = 0$):

$$2m^2\omega^2 = 2mk - c^2 \implies \omega^2 = \frac{2mk - c^2}{2m^2} = \frac{k}{m} - \frac{c^2}{2m^2}.$$

Key Result

Practical resonance frequency. For the damped forced system $m\,u'' + c\,u' + k\,u = F_0\cos(\omega t)$, the amplitude $C(\omega)$ is maximized at

$$\omega_{\mathrm{pr}} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}.$$

Practical resonance exists (i.e., ω_{pr} is real and positive) only when

$$c^2 < 2mk.$$

If $c^2 \geq 2mk$, the amplitude $C(\omega)$ is monotonically decreasing and no resonance peak exists.

Hint

Key comparison. The practical resonance frequency ω_{pr} is always *below* the natural frequency ω_0 (when damping is present). As damping increases, the peak shifts further below ω_0 and becomes lower. When damping is strong enough ($c^2 \geq 2mk$), the peak disappears entirely.

10.5 Beats

Beats occur when a system is forced at a frequency close to, but not exactly equal to, its natural frequency. This phenomenon is particularly clear in the undamped case.

Consider the undamped forced equation

$$u'' + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t),$$

with initial conditions $u(0) = 0$, $u'(0) = 0$, and $\omega \neq \omega_0$.

The general solution (from undetermined coefficients) is

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t).$$

Applying the initial conditions:

$$u(0) = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0 \implies c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}.$$

$$u'(0) = \omega_0 c_2 = 0 \implies c_2 = 0.$$

So

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left[\cos(\omega t) - \cos(\omega_0 t)\right].$$

Using the trigonometric identity

$$\cos(A) - \cos(B) = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right),$$

we rewrite:

$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 + \omega}{2}t\right)\sin\left(\frac{\omega_0 - \omega}{2}t\right).$$

Key Result

Beats formula. For the undamped forced oscillator $u'' + \omega_0^2 u = (F_0/m)\cos(\omega t)$ with zero initial conditions and $\omega \neq \omega_0$,

$$u(t) = \underbrace{\left(\frac{2F_0}{m(\omega_0^2 - \omega^2)}\right)}_{\text{constant}} \cdot \underbrace{\sin\left(\frac{\omega_0 + \omega}{2}t\right)}_{\text{fast oscillation}} \cdot \underbrace{\sin\left(\frac{\omega_0 - \omega}{2}t\right)}_{\text{slow envelope}}.$$

The system oscillates at the **average frequency** $(\omega_0 + \omega)/2$ with an amplitude that varies at the **beat frequency** $|\omega_0 - \omega|/2$.

Physical interpretation. The solution is the product of a fast oscillation (at approximately the natural frequency) and a slow envelope. When ω is close to ω_0, the slow envelope varies very gradually, creating a pattern of rapid oscillations that periodically wax and wane in amplitude. This is the **beats** phenomenon.

The **beat frequency** is $f_{\text{beat}} = |\omega_0 - \omega|/(2\pi)$ (in Hz), or the angular beat frequency is $|\omega_0 - \omega|/2$ (in rad/s). One complete beat (from maximum to maximum amplitude) takes time $T_{\text{beat}} = 2\pi/|\omega_0 - \omega|$.

Worked Example

(Beats) An undamped mass-spring system with $m = 1\,\text{kg}$ and $k = 100\,\text{N/m}$ is driven by $F(t) = 10\cos(9t)$. The mass starts at equilibrium with zero velocity. Find $u(t)$ and describe the beats.

Solution. The natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{100} = 10$. The forcing frequency is $\omega = 9$, which is close to but not equal to ω_0.

The ODE is

$$u'' + 100u = 10\cos(9t).$$

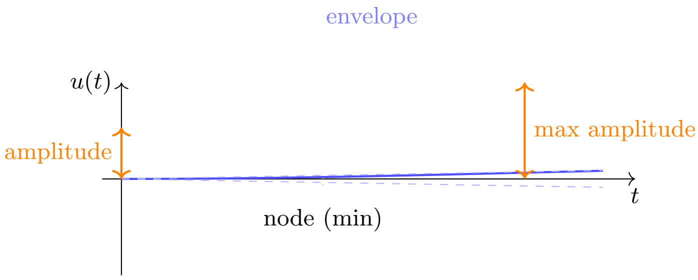


Figure 5: Beats pattern: rapid oscillations modulated by a slowly varying envelope. The amplitude reaches maxima and nodes (zeros) periodically.

Using the beats formula with $F_0 = 10$, $m = 1$, $\omega_0 = 10$, $\omega = 9$:

$$u(t) = \frac{2 \cdot 10}{1 \cdot (100 - 81)} \sin\left(\frac{10+9}{2}\,t\right) \sin\left(\frac{10-9}{2}\,t\right) = \frac{20}{19} \sin\left(\frac{19}{2}\,t\right) \sin\left(\frac{1}{2}\,t\right).$$

Description of beats:

- Fast oscillation frequency: $(\omega_0 + \omega)/2 = 19/2 = 9.5\,\mathrm{rad/s}$.
- Beat (envelope) frequency: $|\omega_0 - \omega|/2 = 1/2 = 0.5\,\mathrm{rad/s}$.
- Maximum amplitude: $20/19 \approx 1.053\,\mathrm{m}$.
- Time between successive beats (max to max): $T_{\mathrm{beat}} = 2\pi/|\omega_0 - \omega| = 2\pi/1 = 2\pi \approx 6.28\,\mathrm{s}$.
- Amplitude nodes (zeros): occur every $T_{\mathrm{beat}}/2 = \pi \approx 3.14\,\mathrm{s}$.

The mass oscillates rapidly at $9.5\,\mathrm{rad/s}$, while the amplitude slowly waxes and wanes over a period of about $6.28\,\mathrm{s}$.

10.6 RLC Circuits

Series RLC circuits (resistor, inductor, capacitor) are governed by the exact same mathematical equation as mechanical spring-mass-damper systems. This is not a coincidence — it is an instance of a deep structural analogy between mechanical and electrical systems.

Kirchhoff's Voltage Law. Consider a series circuit with an inductor of inductance L, a resistor of resistance R, and a capacitor of capacitance C, driven by a time-varying voltage source $E(t)$. Let $q(t)$ denote the charge on the capacitor at time t. The current in the circuit is $i(t) = q'(t) = \mathrm{d}q/\mathrm{d}t$.

The voltage drops across each element are:

- **Inductor:** $V_L = L\,i' = L\,q''$ (Faraday's law).
- **Resistor:** $V_R = R\,i = R\,q'$ (Ohm's law).
- **Capacitor:** $V_C = \dfrac{1}{C}\,q$ (definition of capacitance).

Kirchhoff's Voltage Law (KVL) states that the sum of voltage drops around a closed loop equals the applied voltage:

$$V_L + V_R + V_C = E(t).$$

Substituting the expressions above:

$$L\,q'' + R\,q' + \frac{1}{C}\,q = E(t). \tag{47}$$

Key Result

RLC circuit equation. The charge $q(t)$ on the capacitor in a series RLC circuit satisfies

$$L\,q''(t) + R\,q'(t) + \frac{1}{C}\,q(t) = E(t).$$

This is a second-order linear ODE, identical in form to the spring-mass-damper equation equation (41).

10.6.1 Mechanical–Electrical Analogy

The correspondence between mechanical and electrical quantities is exact:

Table 10: Mechanical–electrical analogy

Mechanical	**Electrical**	**Quantity**
m (mass)	L (inductance)	Inertia / storage of kinetic energy
c (damping coefficient)	R (resistance)	Dissipation of energy
k (spring constant)	$1/C$ (inverse capacitance)	Restoring force / storage of potential energy
$F(t)$ (external force)	$E(t)$ (voltage source)	External driving
$u(t)$ (displacement)	$q(t)$ (charge)	State variable
$u'(t)$ (velocity)	$i(t) = q'(t)$ (current)	Rate of change

The ODEs are identical under the mapping $m \leftrightarrow L$, $c \leftrightarrow R$, $k \leftrightarrow 1/C$, $F(t) \leftrightarrow E(t)$, and $u(t) \leftrightarrow q(t)$. Every result for the spring-mass system has a direct electrical counterpart.

Damping regimes in RLC circuits. The discriminant is $\Delta = R^2 - 4L/C$. The three regimes are:

- **Underdamped** ($R^2 < 4L/C$): The charge oscillates with decaying amplitude. This corresponds to an LC circuit with small resistance.
- **Critically damped** ($R^2 = 4L/C$): The charge returns to equilibrium as fast as possible without oscillation.
- **Overdamped** ($R^2 > 4L/C$): The charge decays monotonically and slowly to equilibrium.

Worked Example

A series RLC circuit has $L = 1\,\text{H}$, $R = 4\,\Omega$, and $C = \frac{1}{3}\,\text{F}$. The voltage source is $E(t) = 12\cos(2t)\,\text{V}$. At $t = 0$, the charge on the capacitor is $q(0) = 0$ and the current is $i(0) = 0$. Find $q(t)$.
Solution. *Step 1: Form the ODE.*

$$q'' + 4q' + 3q = 12\cos(2t).$$

Step 2: Homogeneous solution. The characteristic equation is

$$r^2 + 4r + 3 = 0 \implies (r+1)(r+3) = 0.$$

Roots: $r_1 = -1$, $r_2 = -3$. This is the **overdamped** case ($R^2 = 16 > 4L/C = 12$).

$$q_h(t) = c_1 e^{-t} + c_2 e^{-3t}.$$

Step 3: Particular solution. The forcing is $12\cos(2t)$. Guess:

$$q_p(t) = A\cos(2t) + B\sin(2t).$$

Differentiating:

$$q_p' = -2A\sin(2t) + 2B\cos(2t), \qquad q_p'' = -4A\cos(2t) - 4B\sin(2t).$$

Substitute into $q'' + 4q' + 3q = 12\cos(2t)$:

$$\big[-4A\cos(2t) - 4B\sin(2t)\big] + 4\big[-2A\sin(2t) + 2B\cos(2t)\big] + 3\big[A\cos(2t) + B\sin(2t)\big] = 12\cos(2t).$$

Collect coefficients:

$$(-4A + 8B + 3A)\cos(2t) + (-4B - 8A + 3B)\sin(2t) = 12\cos(2t).$$

$$\begin{cases} -A + 8B = 12, \\ -8A - B = 0 \quad \implies \quad B = -8A. \end{cases}$$

Substituting: $-A + 8(-8A) = 12 \Rightarrow -65A = 12 \Rightarrow A = -\dfrac{12}{65}$. Then $B = -8\left(-\dfrac{12}{65}\right) = \dfrac{96}{65}$.

$$q_p(t) = -\frac{12}{65}\cos(2t) + \frac{96}{65}\sin(2t).$$

Step 4: General solution.

$$q(t) = c_1 e^{-t} + c_2 e^{-3t} - \frac{12}{65}\cos(2t) + \frac{96}{65}\sin(2t).$$

Step 5: Apply initial conditions. $q(0) = 0$ and $q'(0) = i(0) = 0$.

$$q(0) = c_1 + c_2 - \frac{12}{65} = 0 \quad \Longrightarrow \quad c_1 + c_2 = \frac{12}{65}.$$

$$q'(t) = -c_1 e^{-t} - 3c_2 e^{-3t} + \frac{24}{65}\sin(2t) + \frac{192}{65}\cos(2t),$$

$$q'(0) = -c_1 - 3c_2 + \frac{192}{65} = 0 \quad \Longrightarrow \quad c_1 + 3c_2 = \frac{192}{65}.$$

Subtracting the first from the second: $2c_2 = \frac{180}{65} = \frac{36}{13}$, so $c_2 = \frac{18}{13} = \frac{90}{65}$. Then $c_1 = \frac{12}{65} - \frac{90}{65} = -\frac{78}{65} = -\frac{6}{5}$. The solution is

$$q(t) = -\frac{6}{5}\, e^{-t} + \frac{90}{65}\, e^{-3t} - \frac{12}{65}\cos(2t) + \frac{96}{65}\sin(2t).$$

The exponential terms (transient response) decay, leaving the steady-state sinusoidal response:

$$q_{\text{ss}}(t) = -\frac{12}{65}\cos(2t) + \frac{96}{65}\sin(2t) = \frac{12}{65}\sqrt{1+64}\,\cos(2t-\delta) = \frac{12\sqrt{65}}{65}\cos(2t-\delta),$$

where $\delta = \arctan(-8) \approx -1.446\,\text{rad}$.

10.7 Summary

Table 11: Damping cases for $m\,u'' + c\,u' + k\,u = 0$

Case	**Condition**	**Solution**	**Behavior**
Underdamped	$c^2 < 4mk$ $(\zeta < 1)$	$e^{-ct/(2m)}[c_1\cos(\omega_d t) + c_2\sin(\omega_d t)]$	Oscillatory decay
Critically damped	$c^2 = 4mk$ $(\zeta = 1)$	$(c_1 + c_2 t)e^{-ct/(2m)}$	Fastest non-oscillatory return
Overdamped	$c^2 > 4mk$ $(\zeta > 1)$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	Slow non-oscillatory decay

Table 12: Forced vibrations and resonance

Concept	**Key formula**
Natural frequency	$\omega_0 = \sqrt{k/m}$
Critical damping	$c_{\text{cr}} = 2\sqrt{mk} = 2m\omega_0$
Damping ratio	$\zeta = c/c_{\text{cr}} = c/(2\sqrt{mk})$
Damped frequency	$\omega_d = \sqrt{\omega_0^2 - (c/(2m))^2}$
Steady-state amplitude	$C = \dfrac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$
Steady-state phase	$\delta = \arctan\left(\dfrac{c\omega}{k - m\omega^2}\right)$
Pure resonance $(c = 0)$	$u_p(t) = \dfrac{F_0}{2m\omega_0}\, t\,\sin(\omega_0 t)$ when $\omega = \omega_0$
Practical resonance frequency	$\omega_{\text{pr}} = \sqrt{\omega_0^2 - \dfrac{c^2}{2m^2}}$ (exists when $c^2 < 2mk$)
Beats frequency	$f_{\text{beat}} = \dfrac{\lvert\omega_0 - \omega\rvert}{2\pi}$ (angular: $\lvert\omega_0 - \omega\rvert/2$)

Hint

Problem-solving workflow for mechanical/electrical applications.

1. **Model the system:** Draw a diagram, identify all forces (or voltage drops), and write the governing ODE.

2. **Classify the system:** Compute the discriminant or damping ratio to determine whether it is under-

Table 13: Mechanical–electrical analogy

Mechanical: $m\,u'' + c\,u' + k\,u = F(t)$	**Electrical:** $L\,q'' + R\,q' + \frac{1}{C}\,q = E(t)$
$m \leftrightarrow L$	Mass $\leftrightarrow$ Inductance
$c \leftrightarrow R$	Damping $\leftrightarrow$ Resistance
$k \leftrightarrow 1/C$	Spring constant $\leftrightarrow$ Inverse capacitance
$u \leftrightarrow q$	Displacement $\leftrightarrow$ Charge
$u' \leftrightarrow q' = i$	Velocity $\leftrightarrow$ Current
$F(t) \leftrightarrow E(t)$	Force $\leftrightarrow$ Voltage

damped, critically damped, or overdamped.

3. **Solve the ODE:** Apply the methods from sections 8 and 9.
4. **Interpret the solution:** Relate the mathematical result back to the physical behavior (oscillation, decay, resonance, etc.).
5. **Apply initial conditions:** Use given initial displacements/charges and velocities/currents to determine constants.

11 Laplace Transforms

The Laplace transform is one of the most powerful tools for solving linear differential equations with constant coefficients. It converts differential equations in the time domain into algebraic equations in the *frequency domain* (the "s-domain"), where they can be manipulated with elementary algebra and then inverted back. The method is particularly valuable for problems involving initial conditions, discontinuous forcing functions, and impulse inputs — situations where classical techniques become cumbersome.

11.1 Definition and Existence

Definition 11.1. Let $f(t)$ be a function defined for $t \geq 0$. The **Laplace transform** of $f(t)$ is the function

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t, \tag{48}$$

provided the integral converges.

The variable s is a complex parameter, though in most applications we treat s as real and sufficiently large so that the improper integral converges. The function $F(s)$ is the **image** of $f(t)$ under the Laplace transform.

Key Result

Existence conditions. The Laplace transform $\mathcal{L}\{f(t)\}$ exists if $f(t)$ satisfies:

1. **Piecewise continuity** on $[0, \infty)$: $f(t)$ has at most a finite number of finite jump discontinuities on any finite subinterval of $[0, \infty)$.
2. **Exponential order**: there exist constants $M > 0$, $a \in \mathbb{R}$, and $T > 0$ such that

$$|f(t)| \leq M e^{at} \quad \text{for all } t \geq T.$$

Under these conditions, the integral equation (48) converges for all $s > a$.

Intuition. The exponential factor e^{-st} in the integrand acts as a "damping" factor. As long as s is larger than the growth rate of $f(t)$, the product $e^{-st}f(t)$ decays and the integral converges.

We now compute a few Laplace transforms directly from the definition to build intuition.

Worked Example

Compute $\mathcal{L}\{1\}$.

Solution. By definition:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, \mathrm{d}t = \lim_{T\to\infty} \left[-\frac{1}{s}e^{-st}\right]_0^T = \lim_{T\to\infty} \left(-\frac{1}{s}e^{-sT} + \frac{1}{s}\right).$$

For $s > 0$, $e^{-sT} \to 0$ as $T \to \infty$, so

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

Worked Example

Compute $\mathcal{L}\{e^{at}\}$ for a constant a.
Solution.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at} \, \mathrm{d}t = \int_0^\infty e^{-(s-a)t} \, \mathrm{d}t.$$

This converges when $s > a$:

$$\mathcal{L}\{e^{at}\} = \lim_{T\to\infty} \left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^T = \frac{1}{s-a}, \qquad s > a.$$

Worked Example

Compute $\mathcal{L}\{\sin(bt)\}$ for $b > 0$.
Solution.

$$\mathcal{L}\{\sin(bt)\} = \int_0^\infty e^{-st} \sin(bt) \, \mathrm{d}t.$$

We evaluate this using integration by parts twice. Let $I = \int e^{-st} \sin(bt) \, \mathrm{d}t$. Set $u = \sin(bt)$, $\mathrm{d}v = e^{-st} \, \mathrm{d}t$:

$$I = -\frac{1}{s}e^{-st} \sin(bt) + \frac{b}{s} \int e^{-st} \cos(bt) \, \mathrm{d}t.$$

Now integrate $\int e^{-st} \cos(bt) \, \mathrm{d}t$ by parts with $u = \cos(bt)$, $\mathrm{d}v = e^{-st} \, \mathrm{d}t$:

$$\int e^{-st} \cos(bt) \, \mathrm{d}t = -\frac{1}{s}e^{-st} \cos(bt) + \frac{b}{s} \int e^{-st} \sin(bt) \, \mathrm{d}t = -\frac{1}{s}e^{-st} \cos(bt) + \frac{b}{s}I.$$

Substituting back:

$$I = -\frac{1}{s}e^{-st} \sin(bt) + \frac{b}{s}\left(-\frac{1}{s}e^{-st} \cos(bt) + \frac{b}{s}I\right).$$

Solving for I:

$$I\left(1 - \frac{b^2}{s^2}\right) = -e^{-st}\left(\frac{1}{s}\sin(bt) + \frac{b}{s^2}\cos(bt)\right),$$

$$I = -e^{-st}\frac{s\sin(bt) + b\cos(bt)}{s^2 + b^2}.$$

Evaluating the definite integral from 0 to ∞ (for $s > 0$):

$$\mathcal{L}\{\sin(bt)\} = [0] - \left[-\frac{b}{s^2 + b^2}\right] = \frac{b}{s^2 + b^2}, \qquad s > 0.$$

Direct computation from the definition is instructive but impractical for everyday use. In the next section, we compile a comprehensive table of transforms.

11.2 Laplace Transform Table

The following table collects the most commonly used Laplace transforms. Each entry includes the convergence condition on s.

Key Result

Laplace transform table.

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	**Convergence**
$\delta(t-a) \quad (a \geq 0)$	e^{-as}	all s
1	$\dfrac{1}{s}$	$s > 0$
$t^n \quad (n \in \mathbb{N}_0)$	$\dfrac{n!}{s^{n+1}}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin(bt)$	$\dfrac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\dfrac{s}{s^2+b^2}$	$s > 0$
$\sinh(bt)$	$\dfrac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh(bt)$	$\dfrac{s}{s^2-b^2}$	$s > \lvert b \rvert$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s > a$
$\sin^2(bt)$	$\dfrac{2b^2}{s(s^2+4b^2)}$	$s > 0$
$\cos^2(bt)$	$\dfrac{s^2+2b^2}{s(s^2+4b^2)}$	$s > 0$
$t\sin(bt)$	$\dfrac{2bs}{(s^2+b^2)^2}$	$s > 0$
$t\cos(bt)$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$	$s > 0$
$u_c(t)$ (Heaviside step)	$\dfrac{e^{-cs}}{s}$	$s > 0$
$J_0(bt)$ (Bessel function)	$\dfrac{1}{\sqrt{s^2+b^2}}$	$s > 0$

Hint

How to extend the table. Most additional transforms can be generated from the entries above using the properties discussed in section 11.3: linearity, shift theorems, and derivative transforms. For instance, $\mathcal{L}\{t^2e^{3t}\}$ follows from the t^ne^{at} entry with $n = 2$, $a = 3$.

11.3 Properties of the Laplace Transform

The power of the Laplace transform comes from its algebraic properties, which allow us to compute transforms without reverting to the integral definition.

11.3.1 Linearity

Key Result

Linearity. If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then for any constants $a, b \in \mathbb{R}$:

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s).$$

This follows immediately from the linearity of integration.

Worked Example

Compute $\mathcal{L}\{3e^{2t} - 4\cos(5t) + 2t\}$.
Solution. By linearity and the transform table:

$$\begin{aligned}\mathcal{L}\{3e^{2t} - 4\cos(5t) + 2t\} &= 3\,\mathcal{L}\{e^{2t}\} - 4\,\mathcal{L}\{\cos(5t)\} + 2\,\mathcal{L}\{t\} \\ &= 3 \cdot \frac{1}{s-2} - 4 \cdot \frac{s}{s^2+25} + 2 \cdot \frac{1}{s^2} \\ &= \frac{3}{s-2} - \frac{4s}{s^2+25} + \frac{2}{s^2}.\end{aligned}$$

11.3.2 First Shift Theorem (s-Shifting)

Key Result

First shift theorem. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Proof.

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s-a)t} f(t)\,\mathrm{d}t = F(s-a).$$

Worked Example

Find $\mathcal{L}\{e^{3t}\sin(2t)\}$.
Solution. From the table, $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$. By the first shift theorem with $a = 3$:

$$\mathcal{L}\{e^{3t}\sin(2t)\} = \frac{2}{(s-3)^2+4} = \frac{2}{(s-3)^2+2^2}.$$

Worked Example

Find $\mathcal{L}\{t^2 e^{-t}\}$.
Solution. From the table, $\mathcal{L}\{t^2\} = \frac{2}{s^3}$. Applying the first shift with $a = -1$:

$$\mathcal{L}\{t^2 e^{-t}\} = \frac{2}{(s+1)^3}.$$

11.3.3 Second Shift Theorem (t-Shifting)

Key Result

Second shift theorem. Let $u_c(t)$ denote the Heaviside step function (section 11.7.1). If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{u_c(t)\,f(t-c)\} = e^{-cs}F(s), \qquad c \geq 0.$$

Equivalently,

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\,f(t-c).$$

Proof.

$$\begin{aligned}\mathcal{L}\{u_c(t)\, f(t-c)\} &= \int_0^\infty e^{-st}\, u_c(t)\, f(t-c)\, \mathrm{d}t \\ &= \int_c^\infty e^{-st}\, f(t-c)\, \mathrm{d}t.\end{aligned}$$

Substitute $u = t - c$, so $\mathrm{d}u = \mathrm{d}t$, $t = u + c$:

$$\int_0^\infty e^{-s(u+c)} f(u)\, \mathrm{d}u = e^{-cs} \int_0^\infty e^{-su} f(u)\, \mathrm{d}u = e^{-cs} F(s).$$

Worked Example

Find $\mathcal{L}\{u_2(t)\,(t-2)^3\}$.

Solution. Here $c = 2$ and $f(t) = t^3$, so $F(s) = \dfrac{6}{s^4}$. By the second shift theorem:

$$\mathcal{L}\{u_2(t)\,(t-2)^3\} = e^{-2s} \cdot \frac{6}{s^4}.$$

Worked Example

Find $\mathcal{L}\{u_1(t)\,\sin(2(t-1))\}$.

Solution. Here $c = 1$ and $f(t) = \sin(2t)$, so $F(s) = \dfrac{2}{s^2+4}$. By the second shift theorem:

$$\mathcal{L}\{u_1(t)\,\sin(2(t-1))\} = e^{-s} \cdot \frac{2}{s^2+4}.$$

11.3.4 Derivative Transforms

Transforms of derivatives are the key to solving initial value problems via the Laplace transform. The initial conditions appear automatically in the transformed equation.

Key Result

Transform of the first derivative. If $f(t)$ is continuous and $f'(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Proof. Integrate by parts with $u = e^{-st}$ and $\mathrm{d}v = f'(t)\,\mathrm{d}t$:

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\, \mathrm{d}t \\ &= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\, \mathrm{d}t \\ &= \left(\lim_{T\to\infty} e^{-sT} f(T) - f(0)\right) + s \int_0^\infty e^{-st} f(t)\, \mathrm{d}t.\end{aligned}$$

Since $f(t)$ is of exponential order, the limit $\lim_{T\to\infty} e^{-sT} f(T) = 0$ for $s > a$. Thus:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Higher derivatives follow from repeated application.

Key Result

Higher derivative transforms.

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s^2 F(s) - s f(0) - f'(0), \\ \mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$

Worked Example

Solve the IVP $y'' - 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$ using Laplace transforms.
Solution. Let $Y(s) = \mathcal{L}\{y(t)\}$. Transforming each term:

$$\begin{aligned}\mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s,\\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 1,\\ \mathcal{L}\{y\} &= Y(s).\end{aligned}$$

Substituting into the ODE:

$$(s^2Y - s) - 3(sY - 1) + 2Y = 0.$$

Collect terms in $Y(s)$:

$$(s^2 - 3s + 2)Y(s) - s + 3 = 0,$$

$$Y(s) = \frac{s-3}{s^2 - 3s + 2} = \frac{s-3}{(s-1)(s-2)}.$$

Partial fraction decomposition:

$$\frac{s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Multiplying through: $s - 3 = A(s-2) + B(s-1)$. Setting $s = 1$: $-2 = -A$, so $A = 2$. Setting $s = 2$: $-1 = B$, so $B = -1$.

$$Y(s) = \frac{2}{s-1} - \frac{1}{s-2}.$$

Inverting:

$$y(t) = 2e^t - e^{2t}.$$

11.3.5 Integral Transform

Key Result

Transform of an integral. If $f(t)$ is of exponential order, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{F(s)}{s}.$$

Proof. Let $g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$. Then $g'(t) = f(t)$ and $g(0) = 0$. By the derivative transform:

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}.$$

But $\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\} = F(s)$. Hence $\mathcal{L}\{g(t)\} = F(s)/s$.

Worked Example

Find $\mathcal{L}\left\{\int_0^t e^{3\tau}\sin(2\tau)\,\mathrm{d}\tau\right\}$.
Solution. From the table (or the first shift theorem):

$$\mathcal{L}\{e^{3t}\sin(2t)\} = \frac{2}{(s-3)^2 + 4}.$$

By the integral transform:

$$\mathcal{L}\left\{\int_0^t e^{3\tau}\sin(2\tau)\,\mathrm{d}\tau\right\} = \frac{2}{s\big((s-3)^2+4\big)}.$$

11.3.6 Multiplication by t^n

Key Result

Multiplication by t^n. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s).$$

In particular, $\mathcal{L}\{tf(t)\} = -F'(s)$.

This property is useful for deriving additional table entries. For example, $\mathcal{L}\{t\sin(bt)\} = -\dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{b}{s^2+b^2}\right) = \dfrac{2bs}{(s^2+b^2)^2}$.

11.4 Inverse Laplace Transforms

The **inverse Laplace transform**, denoted $\mathcal{L}^{-1}$, recovers the time-domain function from its s-domain image:

$$\text{if } F(s) = \mathcal{L}\{f(t)\}, \text{ then } f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

By linearity, $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t)$. In practice, we compute inverse transforms using two complementary strategies: table lookup and partial fraction decomposition.

Hint

Inverse transform strategies.

1. **Table lookup:** Match $F(s)$ directly to a table entry. Look for recognizable forms: $\dfrac{1}{s-a}$, $\dfrac{b}{s^2+b^2}$, $\dfrac{s}{s^2+b^2}$, etc.
2. **Algebraic manipulation:** If $F(s)$ does not match a table entry exactly, try completing the square in the denominator, splitting into partial fractions, or using shift theorems.
3. **Second shift theorem:** If $F(s)$ contains a factor e^{-cs}, the result will involve a Heaviside step $u_c(t)$.

Completing the square. Many denominators are of the form $s^2 + 2as + (a^2 + b^2)$, which can be rewritten as $(s+a)^2 + b^2$. This allows us to use the first shift theorem in reverse.

Worked Example

Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+6s+13}\right\}$.
Solution. Complete the square in the denominator:

$$s^2 + 6s + 13 = (s+3)^2 + 4 = (s+3)^2 + 2^2.$$

Thus:

$$\frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+2^2} = \frac{1}{2}\cdot\frac{2}{(s+3)^2+2^2}.$$

From the table, $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+2^2}\right\} = \sin(2t)$. By the first shift theorem (with $a = -3$):

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{(s+3)^2+2^2}\right\} = \frac{1}{2}e^{-3t}\sin(2t).$$

11.5 Partial Fraction Decomposition

When $F(s)$ is a rational function (ratio of polynomials) and the denominator degree exceeds the numerator degree, we decompose it into simpler fractions that match table entries.

Hint

Partial fraction decomposition guide.

- **Distinct linear factors:** $\dfrac{P(s)}{(s-a)(s-b)} = \dfrac{A}{s-a} + \dfrac{B}{s-b}$.
- **Repeated linear factor:** $\dfrac{P(s)}{(s-a)^n} = \dfrac{A_1}{s-a} + \dfrac{A_2}{(s-a)^2} + \cdots + \dfrac{A_n}{(s-a)^n}$.

- **Irreducible quadratic:** $\dfrac{P(s)}{(s^2+bs+c)} = \dfrac{As+B}{s^2+bs+c}$. Complete the square in the denominator, then split into $\dfrac{A(s+b/2)}{(s+b/2)^2+(c-b^2/4)} + \dfrac{B'}{\text{denom}}$.
- **Combined:** Handle each factor type separately and combine.

Worked Example

Find $\mathcal{L}^{-1}\left\{\dfrac{3s-5}{(s-1)(s+2)}\right\}$.

Solution. Decompose:

$$\frac{3s-5}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}.$$

Multiplying both sides by $(s-1)(s+2)$:

$$3s-5 = A(s+2) + B(s-1).$$

Set $s=1$: $3(1)-5 = A(3) \Rightarrow -2 = 3A \Rightarrow A = -\dfrac{2}{3}$.

Set $s=-2$: $3(-2)-5 = B(-3) \Rightarrow -11 = -3B \Rightarrow B = \dfrac{11}{3}$.

Therefore:

$$\frac{3s-5}{(s-1)(s+2)} = -\frac{2}{3}\cdot\frac{1}{s-1} + \frac{11}{3}\cdot\frac{1}{s+2}.$$

Inverting:

$$\mathcal{L}^{-1}\left\{\frac{3s-5}{(s-1)(s+2)}\right\} = -\frac{2}{3}e^{t} + \frac{11}{3}e^{-2t}.$$

Worked Example

Find $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{(s-1)^2(s+3)}\right\}$.

Solution. The repeated factor $(s-1)^2$ requires two terms:

$$\frac{2s+1}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}.$$

Multiply through:

$$2s+1 = A(s-1)(s+3) + B(s+3) + C(s-1)^2.$$

Set $s=1$: $2(1)+1 = B(4) \Rightarrow 3 = 4B \Rightarrow B = \dfrac{3}{4}$.

Set $s=-3$: $2(-3)+1 = C(-4)^2 \Rightarrow -5 = 16C \Rightarrow C = -\dfrac{5}{16}$.

To find A, expand and equate coefficients of s^2:

$$A(s^2+2s-3) + B(s+3) + C(s^2-2s+1).$$

The s^2 coefficient is $A+C$. The left side has no s^2 term, so $A+C=0 \Rightarrow A = -C = \dfrac{5}{16}$.

Thus:

$$\frac{2s+1}{(s-1)^2(s+3)} = \frac{5/16}{s-1} + \frac{3/4}{(s-1)^2} - \frac{5/16}{s+3}.$$

Inverting each term:

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{(s-1)^2(s+3)}\right\} = \frac{5}{16}e^{t} + \frac{3}{4}t\,e^{t} - \frac{5}{16}e^{-3t}.$$

11.6 Solving Initial Value Problems

The Laplace transform method provides a systematic algorithm for solving linear IVPs with constant coefficients. Unlike classical methods (undetermined coefficients, variation of parameters), the Laplace approach incorporates initial conditions directly into the algebraic step, so there is no need to determine constants *after* finding the general solution.

Key Result

IVP algorithm via Laplace transforms.

1. **Transform both sides.** Apply $\mathcal{L}$ to the differential equation. Use the derivative transforms to express $\mathcal{L}\{y'\}$, $\mathcal{L}\{y''\}$, etc. in terms of $Y(s)$ and the initial conditions.
2. **Substitute initial conditions.** Replace $y(0)$, $y'(0)$, etc. with their given values.
3. **Solve algebraically for $Y(s)$.** Collect all $Y(s)$ terms on one side and solve for $Y(s)$.
4. **Partial fraction decomposition.** If $Y(s)$ is a rational function, decompose it into simpler fractions.
5. **Invert.** Apply $\mathcal{L}^{-1}$ to obtain $y(t)$.

Example 1: First-order IVP.

Worked Example

Solve $y' + 4y = e^{-2t}$, $y(0) = 3$.
Solution. Step 1. Transform both sides:

$$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{e^{-2t}\}.$$

Step 2. Use the derivative transform and substitute $y(0) = 3$:

$$(sY(s) - 3) + 4Y(s) = \frac{1}{s+2}.$$

Step 3. Solve for $Y(s)$:

$$(s+4)Y(s) = \frac{1}{s+2} + 3 = \frac{1 + 3(s+2)}{s+2} = \frac{3s+7}{s+2},$$
$$Y(s) = \frac{3s+7}{(s+2)(s+4)}.$$

Step 4. Partial fractions:

$$\frac{3s+7}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4}.$$

Multiply: $3s + 7 = A(s+4) + B(s+2)$. Set $s = -2$: $3(-2) + 7 = A(2) \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$. Set $s = -4$: $3(-4) + 7 = B(-2) \Rightarrow -5 = -2B \Rightarrow B = \frac{5}{2}$. **Step 5.** Invert:

$$y(t) = \frac{1}{2}e^{-2t} + \frac{5}{2}e^{-4t}.$$

Example 2: Second-order IVP.

Worked Example

Solve $y'' + 2y' + 5y = \sin(3t)$, with $y(0) = 1$ and $y'(0) = 0$.
Solution. Step 1. Transform:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{\sin(3t)\}.$$

Step 2. Use derivative transforms and substitute ICs:

$$(s^2Y - s \cdot 1 - 0) + 2(sY - 1) + 5Y = \frac{3}{s^2+9}.$$

Simplify:

$$(s^2 + 2s + 5)Y(s) - s - 2 = \frac{3}{s^2+9}.$$

Step 3. Solve for $Y(s)$:

$$Y(s) = \frac{s+2}{s^2+2s+5} + \frac{3}{(s^2+9)(s^2+2s+5)}.$$

Complete the square in the first denominator: $s^2 + 2s + 5 = (s+1)^2 + 4$. Rewrite the first term:

$$\frac{s+2}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}.$$

Inverting the first part: $\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = e^{-t}\cos(2t)$ and $\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} = \frac{1}{2}e^{-t}\sin(2t)$.

Step 4. For the second term, partial fractions on s^2:

$$\frac{3}{(s^2+9)(s^2+2s+5)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+2s+5}.$$

Multiply: $3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+9)$. Expanding and collecting powers of s:

$$3 = (A+C)s^3 + (2A+B+D)s^2 + (5A+2B+9C)s + (5B+9D).$$

Equating coefficients:

$$\begin{cases} A + C = 0 \\ 2A + B + D = 0 \\ 5A + 2B + 9C = 0 \\ 5B + 9D = 3 \end{cases}$$

From the first: $C = -A$. Substitute into the third: $5A + 2B - 9A = 0 \Rightarrow 2B = 4A \Rightarrow B = 2A$. From the second: $2A + 2A + D = 0 \Rightarrow D = -4A$. From the fourth: $5(2A) + 9(-4A) = 3 \Rightarrow 10A - 36A = 3 \Rightarrow -26A = 3 \Rightarrow A = -\frac{3}{26}$.

So $C = \frac{3}{26}$, $B = -\frac{6}{26} = -\frac{3}{13}$, $D = \frac{12}{26} = \frac{6}{13}$.

The second term becomes:

$$\frac{-\frac{3}{26}s - \frac{3}{13}}{s^2+9} + \frac{\frac{3}{26}s + \frac{6}{13}}{s^2+2s+5}.$$

Rewrite for inversion. The first fraction:

$$-\frac{3}{26}\cdot\frac{s}{s^2+9} - \frac{3}{13}\cdot\frac{1}{s^2+9} = -\frac{3}{26}\cdot\frac{s}{s^2+9} - \frac{1}{13}\cdot\frac{3}{s^2+9}.$$

Inverting: $-\frac{3}{26}\cos(3t) - \frac{1}{13}\sin(3t)$.

The second fraction: complete the square in $s^2 + 2s + 5 = (s+1)^2 + 4$:

$$\frac{\frac{3}{26}s + \frac{6}{13}}{(s+1)^2+4} = \frac{3}{26}\cdot\frac{s+1}{(s+1)^2+4} + \frac{9}{26}\cdot\frac{1}{(s+1)^2+4}.$$

Inverting: $\frac{3}{26}e^{-t}\cos(2t) + \frac{9}{52}e^{-t}\sin(2t)$.

Step 5. Combining all terms:

$$y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t) - \frac{3}{26}\cos(3t) - \frac{1}{13}\sin(3t) + \frac{3}{26}e^{-t}\cos(2t) + \frac{9}{52}e^{-t}\sin(2t).$$

Grouping like terms:

$$y(t) = \frac{29}{26}e^{-t}\cos(2t) + \frac{11}{52}e^{-t}\sin(2t) - \frac{3}{26}\cos(3t) - \frac{1}{13}\sin(3t).$$

Example 3: System of first-order IVPs.

Worked Example

Solve the system

$$\begin{cases} x' = 3x - 2y, \quad x(0) = 1, \\ y' = x + y, \quad y(0) = 0. \end{cases}$$

Solution. Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$. Transform both equations:

$$\begin{cases} sX - x(0) = 3X - 2Y, \\ sY - y(0) = X + Y. \end{cases}$$

Substitute initial conditions:

$$\begin{cases} (s-3)X + 2Y = 1, \\ -X + (s-1)Y = 0. \end{cases}$$

From the second equation: $X = (s-1)Y$. Substitute into the first:

$$(s-3)(s-1)Y + 2Y = 1,$$

$$\big((s-3)(s-1) + 2\big)Y = 1,$$

$$(s^2 - 4s + 5)Y = 1 \quad \Longrightarrow \quad Y(s) = \frac{1}{s^2 - 4s + 5}.$$

Complete the square: $s^2 - 4s + 5 = (s-2)^2 + 1$. Thus:

$$Y(s) = \frac{1}{(s-2)^2 + 1}.$$

Inverting: $y(t) = e^{2t}\sin(t)$.
Now find $X(s) = (s-1)Y(s)$:

$$X(s) = \frac{s-1}{(s-2)^2+1} = \frac{s-2+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{1}{(s-2)^2+1}.$$

Inverting:

$$x(t) = e^{2t}\cos(t) + e^{2t}\sin(t) = e^{2t}\big(\cos(t) + \sin(t)\big).$$

11.7 Step and Delta Functions

11.7.1 Heaviside Step Function

The **Heaviside step function** (or simply **unit step**) is defined by

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0.$$

When $c = 0$, we write $u(t)$ instead of $u_0(t)$.

Key Result

Laplace transform of the unit step.

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \qquad s > 0.$$

This follows directly from the definition:

$$\int_0^\infty e^{-st} u_c(t)\,\mathrm{d}t = \int_c^\infty e^{-st}\,\mathrm{d}t = \frac{e^{-cs}}{s}.$$

The step function is the building block for modeling **piecewise continuous** forcing functions. Any piecewise-defined function $g(t)$ can be expressed as a sum of step functions.

Worked Example

Express $g(t) = \begin{cases} t, & 0 \leq t < 2 \\ 2, & t \geq 2 \end{cases}$ using Heaviside step functions, and find its Laplace transform.

Solution. Write $g(t)$ as:

$$g(t) = t + (2-t)\,u_2(t).$$

To see this: for $t < 2$, $u_2(t) = 0$, so $g(t) = t$. For $t \geq 2$, $g(t) = t + 2 - t = 2$.
Now transform:

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{2u_2(t) - t\,u_2(t)\}.$$

We have $\mathcal{L}\{t\} = \dfrac{1}{s^2}$. For the second part, use the second shift theorem:

$$\mathcal{L}\{u_2(t) \cdot 2\} = \mathcal{L}\{u_2(t) \cdot 2 \cdot 1\} = 2 \cdot \frac{e^{-2s}}{s}.$$

For $\mathcal{L}\{t\,u_2(t)\}$, rewrite as $u_2(t) \cdot t = u_2(t) \cdot ((t-2) + 2)$:

$$\mathcal{L}\{u_2(t) \cdot (t-2)\} = e^{-2s} \cdot \frac{1}{s^2}, \quad \mathcal{L}\{u_2(t) \cdot 2\} = 2 \cdot \frac{e^{-2s}}{s}.$$

Therefore:

$$\mathcal{L}\{t\,u_2(t)\} = \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}.$$

Putting it all together:

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} + \frac{2e^{-2s}}{s} - \left(\frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s}\right) = \frac{1 - e^{-2s}}{s^2}.$$

11.7.2 Dirac Delta Function

The **Dirac delta function** $\delta(t-a)$ is a generalized function (distribution) characterized by its **sifting property**:

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, \mathrm{d}t = f(a).$$

Intuitively, $\delta(t-a)$ models an instantaneous impulse of unit magnitude occurring at time $t = a$.

Key Result

Laplace transform of the delta function. For $a \geq 0$:

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}.$$

Relationship to the step function. The delta function can be viewed as the "derivative" of the step function:

$$\frac{\mathrm{d}}{\mathrm{d}t} u_c(t) = \delta(t-c).$$

Consistently, differentiating $\mathcal{L}\{u_c(t)\} = e^{-cs}/s$ with respect to t should yield the transform of $\delta(t-c)$. Since $\mathcal{L}\{u_c'\} = s\,\mathcal{L}\{u_c\} - u_c(0) = s \cdot \dfrac{e^{-cs}}{s} - 0 = e^{-cs}$, we recover the correct result.

Worked Example

Solve $y'' + 4y = \delta(t-\pi)$, $y(0) = 0$, $y'(0) = 0$.
Solution. Transform both sides:

$$(s^2 Y(s) - s \cdot 0 - 0) + 4Y(s) = e^{-\pi s}.$$

Solving for $Y(s)$:

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 4} = \frac{1}{2} e^{-\pi s} \cdot \frac{2}{s^2 + 4}.$$

Invert using the second shift theorem. Since $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \sin(2t)$:

$$y(t) = \frac{1}{2}\, u_\pi(t)\, \sin\big(2(t-\pi)\big).$$

Note that $\sin(2(t-\pi)) = \sin(2t-2\pi) = \sin(2t)$. Therefore:

$$y(t) = \begin{cases} 0, & 0 \le t < \pi, \\ \dfrac{1}{2}\sin(2t), & t \ge \pi. \end{cases}$$

The system remains at rest until the impulse at $t = \pi$, after which it begins oscillating.

Worked Example

Find the impulse response of the system $y'' + 3y' + 2y = \delta(t)$, with $y(0) = 0$ and $y'(0) = 0$.
Solution. Transform:

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1,$$

$$Y(s) = \frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)}.$$

Partial fractions:

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Inverting:

$$y(t) = e^{-t} - e^{-2t}.$$

This function, often denoted $h(t)$, is the **impulse response**: the system's output when driven by a unit impulse at $t = 0$. For any forcing function $f(t)$, the solution is given by the convolution $y(t) = (h * f)(t)$, as discussed in the next section.

11.8 Convolution

Definition 11.2. The **convolution** of two functions $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau. \tag{49}$$

Convolution is commutative: $f * g = g * f$.

The fundamental theorem connecting convolution and the Laplace transform is the following.

Theorem 11.3 (Convolution Theorem). *If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then*

$$\mathcal{L}\{(f * g)(t)\} = F(s)\, G(s).$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = (f * g)(t).$$

Proof sketch. Consider $\mathcal{L}\{(f * g)(t)\}$:

$$\int_0^\infty e^{-st} \left(\int_0^t f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right) \mathrm{d}t.$$

The region of integration is $0 \le \tau \le t < \infty$. Changing the order of integration (Fubini's theorem) and substituting $u = t - \tau$:

$$\int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau)\, \mathrm{d}t \right) \mathrm{d}\tau = \int_0^\infty f(\tau) e^{-s\tau} \left(\int_0^\infty e^{-su} g(u)\, \mathrm{d}u \right) \mathrm{d}\tau.$$

The inner integral is $G(s)$, and the remaining integral is $F(s)$. Thus the result is $F(s)G(s)$.

Worked Example

Use convolution to find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}$.

Solution. Write $F(s)G(s)$ with $F(s) = \dfrac{1}{s^2}$ and $G(s) = \dfrac{1}{s^2+1}$. Then $f(t) = t$ and $g(t) = \sin(t)$. By the convolution theorem:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = (t * \sin t)(t) = \int_0^t \tau \sin(t-\tau)\,\mathrm{d}\tau.$$

Integrate by parts with $u = \tau$, $\mathrm{d}v = \sin(t-\tau)\,\mathrm{d}\tau$:

$$\begin{aligned}
\int_0^t \tau \sin(t-\tau)\,\mathrm{d}\tau &= \Big[\tau\cos(t-\tau)\Big]_0^t - \int_0^t \cos(t-\tau)\,\mathrm{d}\tau \\
&= \big(t\cos(0) - 0\big) - \Big[-\sin(t-\tau)\Big]_0^t \\
&= t - \big(-\sin(0) + \sin(t)\big) \\
&= t - \sin(t).
\end{aligned}$$

Therefore:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin(t).$$

Worked Example

Use convolution to solve $y'' + y = f(t)$, with $y(0) = 0$ and $y'(0) = 0$, where $f(t)$ is an arbitrary forcing function.

Solution. Transform:

$$s^2Y(s) + Y(s) = F(s) \quad \Longrightarrow \quad Y(s) = \frac{F(s)}{s^2+1} = F(s)\cdot\frac{1}{s^2+1}.$$

Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin(t)$, the convolution theorem gives:

$$y(t) = (\sin * f)(t) = \int_0^t \sin(\tau)\, f(t-\tau)\,\mathrm{d}\tau.$$

This formula provides the solution for *any* $f(t)$ without needing to perform partial fractions or guess particular solutions. The function $h(t) = \sin(t)$ is the impulse response of this system.

11.9 Transfer Functions

The **transfer function** provides a compact description of a linear system's input-output behavior in the frequency domain.

Definition 11.4. Consider a linear differential equation with constant coefficients:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t),$$

with all initial conditions equal to zero. The **transfer function** is the ratio

$$H(s) = \frac{Y(s)}{F(s)},$$

where $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$.

Key Result

Transfer function from ODE. Taking the Laplace transform of the ODE with zero initial conditions:

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = F(s),$$

so the transfer function is

$$H(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.$$

The output in the s-domain is simply $Y(s) = H(s)F(s)$, and in the time domain $y(t) = (h * f)(t)$ where $h(t) = \mathcal{L}^{-1}\{H(s)\}$ is the impulse response.

11.9.1 Pole Analysis and Stability

The **poles** of $H(s)$ are the roots of the denominator polynomial. They determine the long-term behavior (stability) of the system:

- All poles have negative real parts (left half-plane, LHP): the system is **stable** and the output decays to zero for bounded inputs.
- Any pole has a positive real part (right half-plane, RHP): the system is **unstable** and the output grows without bound.
- Poles on the imaginary axis (purely imaginary, no right-half-plane poles): the system is **marginally stable**; the output oscillates persistently but does not grow.

Worked Example

Find the transfer function and analyze the stability of the RLC circuit modeled by

$$L\,q'' + R\,q' + \frac{1}{C}q = E(t),$$

where $q(t)$ is the charge on the capacitor, L is the inductance, R is the resistance, C is the capacitance, and $E(t)$ is the applied voltage.
Solution. Taking the Laplace transform with zero initial conditions:

$$(Ls^2 + Rs + \tfrac{1}{C})Q(s) = E(s),$$

so the transfer function is

$$H(s) = \frac{Q(s)}{E(s)} = \frac{1}{Ls^2 + Rs + \frac{1}{C}}.$$

The poles are the roots of $Ls^2 + Rs + \dfrac{1}{C} = 0$:

$$s = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}.$$

Since $L, R, C > 0$, we have $-R < 0$. If $R^2 \geq \dfrac{4L}{C}$ (overdamped or critically damped), both roots are real and negative. If $R^2 < \dfrac{4L}{C}$ (underdamped), the roots are complex conjugates with real part $-\dfrac{R}{2L} < 0$. In all cases, the real parts are negative, so the system is stable. The physical intuition is that the resistor dissipates energy, ensuring that oscillations decay.

Worked Example

Consider the spring-mass-damper system

$$m\,x'' + c\,x' + k\,x = F(t),$$

where m is mass, c is damping, and k is spring stiffness. Find the transfer function $H(s) = X(s)/F(s)$ and analyze the stability.
Solution. With zero initial conditions:

$$(ms^2 + cs + k)X(s) = F(s),$$

so

$$H(s) = \frac{1}{ms^2 + cs + k}.$$

The poles are

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Since $m, c, k > 0$, the real parts of the poles are always negative (the same analysis as the RLC circuit above). The system is therefore stable. If $c = 0$ (no damping), the poles lie on the imaginary axis: $s = \pm i\sqrt{k/m}$, giving undamped oscillations (marginal stability).

11.9.2 Initial and Final Value Theorems

Two important theorems allow us to extract information about the time-domain behavior of $f(t)$ directly from $F(s)$ without computing the full inverse transform.

Theorem 11.5 (Initial Value Theorem)**.** *If $\mathcal{L}\{f(t)\} = F(s)$ and the limit exists, then*

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s).$$

Theorem 11.6 (Final Value Theorem)**.** *If $\mathcal{L}\{f(t)\} = F(s)$ and all poles of $sF(s)$ lie in the open left half-plane, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Hint

Caveat for the final value theorem. The condition that all poles of $sF(s)$ lie in the open LHP is essential. If $F(s)$ has poles on the imaginary axis (e.g., $\frac{1}{s^2+1}$ corresponding to $\sin t$), the limit $\lim_{t\to\infty} f(t)$ does not exist, and the final value theorem gives a misleading result. Always check pole locations before applying.

Worked Example

Verify the initial and final value theorems for $y(t) = \frac{1}{2}e^{-2t} + \frac{5}{2}e^{-4t}$ (the solution to the first-order IVP in section 11.6).

Solution. From the solution, $Y(s) = \frac{3s+7}{(s+2)(s+4)}$.

Initial value. By the theorem:

$$\lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{s(3s+7)}{(s+2)(s+4)} = \lim_{s\to\infty} \frac{3s^2+7s}{s^2+6s+8} = 3.$$

Directly: $y(0) = \frac{1}{2} + \frac{5}{2} = 3$. ✓

Final value. All poles are at $s = -2$ and $s = -4$ (both in the LHP), so the theorem applies:

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s(3s+7)}{(s+2)(s+4)} = \frac{0\cdot 7}{8} = 0.$$

Directly: $\lim_{t\to\infty} y(t) = 0$ since both exponentials decay. ✓

11.10 Summary

The Laplace transform is a powerful method for solving linear differential equations. The key ideas are:

- The transform $\mathcal{L}\{f(t)\} = F(s)$ converts functions of time into functions of frequency, turning differentiation into multiplication.
- A comprehensive transform table allows us to quickly look up common transforms (section 11.2).
- Linearity, shift theorems, and derivative transforms form the algebraic backbone of the method (section 11.3).
- Inverse transforms are obtained via table lookup and partial fraction decomposition (section 11.4, section 11.5).
- The method naturally handles initial conditions, making it ideal for IVPs (section 11.6).
- Step and delta functions allow modeling of discontinuous and impulsive inputs (section 11.7).
- The convolution theorem connects multiplication in the s-domain with convolution in the time domain (section 11.8).
- Transfer functions provide a compact input-output description and enable stability analysis via pole locations (section 11.9).

The Laplace transform method will be used throughout the remainder of this handbook, particularly in the analysis of systems of equations (section 12) and in the construction of Fourier series (section 14).

Table 14: Key Laplace transforms and properties

Concept	Key Formula/Method
Definition	$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$
Linearity	$\mathcal{L}\{af + bg\} = aF(s) + bG(s)$
First shift	$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
Second shift	$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$
First derivative	$\mathcal{L}\{f'\} = sF(s) - f(0)$
Second derivative	$\mathcal{L}\{f''\} = s^2 F(s) - sf(0) - f'(0)$
Integral	$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = F(s)/s$
Step function	$\mathcal{L}\{u_c(t)\} = e^{-cs}/s$
Delta function	$\mathcal{L}\{\delta(t-a)\} = e^{-as}$
Convolution theorem	$\mathcal{L}\{f * g\} = F(s)G(s)$
Transfer function	$H(s) = Y(s)/F(s)$ (zero ICs)
Initial value theorem	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$
Final value theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

12 Systems of Linear ODEs

Many physical systems are naturally modeled by coupled first-order differential equations rather than a single higher-order equation. The systematic study of such systems relies on linear algebra — in particular, eigenvalues and eigenvectors — to produce closed-form solutions.

12.1 Matrix Form of Linear Systems

A **system of n first-order linear ODEs with constant coefficients** can be written in compact matrix notation as

$$\mathbf{x}'(t) = A\,\mathbf{x}(t), \tag{50}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the vector of unknown functions and A is a constant $n \times n$ matrix.

Converting an nth-order ODE to a first-order system. Any linear nth-order equation

$$y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_1\,y' + a_0\,y = 0$$

can be converted to a system of n first-order equations by introducing the state variables

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \ldots, \quad x_n = y^{(n-1)}.$$

Differentiating each and using the original ODE to substitute for the highest derivative yields

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

This is the **companion matrix** form.

Worked Example

Convert the third-order equation $y''' + 4y'' + 3y' - 2y = 0$ into a 3×3 first-order system.
Solution. Define

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y''.$$

Then $x_1' = x_2$, $x_2' = x_3$, and from the ODE,

$$x_3' = y''' = -4y'' - 3y' + 2y = -4x_3 - 3x_2 + 2x_1.$$

In matrix form:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Worked Example

Convert $y'' + 3y' + 2y = 0$ to system form.
Solution. Set $x_1 = y$, $x_2 = y'$. Then

$$x_1' = x_2, \qquad x_2' = y'' = -3y' - 2y = -3x_2 - 2x_1.$$

The system is

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Notice that the characteristic polynomial of the matrix, $\det(A - \lambda I) = \lambda^2 + 3\lambda + 2$, matches the original characteristic equation exactly.

12.2 Eigenvalue Method: Real Distinct Eigenvalues

The eigenvalue method is the principal tool for solving $\mathbf{x}' = A\mathbf{x}$ when A is a constant matrix. The key idea is that eigenvectors of A generate single-mode solutions of the form $e^{\lambda t}\mathbf{v}$.

Eigenvalue-eigenvector ansatz. If λ is an eigenvalue of A and $\mathbf{v}$ is a corresponding eigenvector ($A\mathbf{v} = \lambda\mathbf{v}$), then

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$$

is a solution of $\mathbf{x}' = A\mathbf{x}$, since

$$\frac{d}{dt}\big(e^{\lambda t}\mathbf{v}\big) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(A\mathbf{v}) = A\big(e^{\lambda t}\mathbf{v}\big).$$

Key Result

Real distinct eigenvalues. If the $n \times n$ matrix A has n distinct real eigenvalues $\lambda_1, \dots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, then the general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where $c_1, \dots, c_n \in \mathbb{R}$ are arbitrary constants.

Finding eigenvalues and eigenvectors. The eigenvalues are roots of the **characteristic polynomial** $\det(A - \lambda I) = 0$. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation is $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$, or equivalently

$$\lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A) = 0.$$

For each eigenvalue λ, the eigenvectors are nonzero solutions of $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

Worked Example

Solve the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues.

$$\det\begin{pmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{pmatrix} = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0.$$

Factoring: $(\lambda - 4)(\lambda + 1) = 0$, so $\lambda_1 = 4$ and $\lambda_2 = -1$.
Step 2: Eigenvector for $\lambda_1 = 4$. Solve $(A - 4I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row gives $-3v_1 + 2v_2 = 0$, so $v_2 = \frac{3}{2}v_1$. Choosing $v_1 = 2$ gives $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Step 3: Eigenvector for $\lambda_2 = -1$. Solve $(A + I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row gives $2v_1 + 2v_2 = 0$, so $v_2 = -v_1$. Choosing $v_1 = 1$ gives $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Step 4: General solution.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In components:

$$x(t) = 2c_1 e^{4t} + c_2 e^{-t}, \qquad y(t) = 3c_1 e^{4t} - c_2 e^{-t}.$$

Worked Example

Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues.

$$\det \begin{pmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0.$$

Factoring: $(\lambda - 5)(\lambda - 2) = 0$, so $\lambda_1 = 5$ and $\lambda_2 = 2$.

Step 2: Eigenvector for $\lambda_1 = 5$.

$$(A - 5I)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the first row: $-v_1 + v_2 = 0$, so $v_2 = v_1$. Choose $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Step 3: Eigenvector for $\lambda_2 = 2$.

$$(A - 2I)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the first row: $2v_1 + v_2 = 0$, so $v_2 = -2v_1$. Choose $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Step 4: General solution.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

12.3 Eigenvalue Method: Complex Eigenvalues

When A is a real matrix, complex eigenvalues always occur in conjugate pairs $\lambda = \alpha \pm i\beta$ (with $\beta \neq 0$). The corresponding eigenvectors also come in conjugate pairs.

From complex to real-valued solutions. Let $\lambda = \alpha + i\beta$ be a complex eigenvalue of A with eigenvector $\mathbf{w} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. The complex-valued solution is

$$\mathbf{x}_c(t) = e^{(\alpha + i\beta)t}(\mathbf{a} + i\mathbf{b}).$$

By Euler's formula:

$$\begin{aligned} \mathbf{x}_c(t) &= e^{\alpha t}\big[\cos(\beta t) + i\sin(\beta t)\big](\mathbf{a} + i\mathbf{b}) \\ &= e^{\alpha t}\Big[\big(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big) + i\big(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big)\Big]. \end{aligned}$$

Since the original system has real coefficients, both the real and imaginary parts are real-valued solutions:

$$\mathbf{x}_1(t) = e^{\alpha t}\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big], \qquad \mathbf{x}_2(t) = e^{\alpha t}\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

Key Result

Complex eigenvalues. Let A be a real 2×2 matrix with eigenvalues $\lambda = \alpha \pm i\beta$ ($\beta > 0$) and corresponding eigenvector $\mathbf{w} = \mathbf{a} + i\mathbf{b}$ (where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$). Then two linearly independent real-valued solutions are

$$\mathbf{x}_1(t) = e^{\alpha t}\big[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\big],$$
$$\mathbf{x}_2(t) = e^{\alpha t}\big[\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\big].$$

The general real-valued solution is $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$.

Remark 12.1. The sign of α determines whether solutions spiral inward ($\alpha < 0$, stable spiral), spiral outward ($\alpha > 0$, unstable spiral), or trace closed orbits ($\alpha = 0$, center).

Worked Example

Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues.

$$\det\begin{pmatrix} -\lambda & -2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 + 4 = 0, \quad \text{so} \quad \lambda = \pm 2i.$$

Here $\alpha = 0$ and $\beta = 2$.
Step 2: Eigenvector for $\lambda = 2i$. Solve $(A - 2iI)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the first row: $-2i\,v_1 - 2\,v_2 = 0$, so $v_2 = -i\,v_1$. Choose $v_1 = 1$, giving

$$\mathbf{w} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \mathbf{a} + i\mathbf{b},$$

where $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.
Step 3: Real-valued solutions. Since $\alpha = 0$ and $\beta = 2$:

$$\mathbf{x}_1(t) = \mathbf{a}\cos(2t) - \mathbf{b}\sin(2t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin(2t) = \begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix},$$

$$\mathbf{x}_2(t) = \mathbf{a}\sin(2t) + \mathbf{b}\cos(2t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos(2t) = \begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}.$$

Step 4: General solution.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix} + c_2\begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}.$$

Phase plane interpretation. Eliminating t: for any nonzero constants c_1, c_2, $x(t)^2 + y(t)^2 = c_1^2 + c_2^2 =$ constant. Trajectories are circles centered at the origin, traversed counterclockwise. This is a **center** — neutrally stable closed orbits.

Worked Example

Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues.

$$\det\begin{pmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 + 4 = \lambda^2 + 2\lambda + 5 = 0.$$

The roots are $\lambda = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$. So $\alpha = -1$ and $\beta = 2$.

Step 2: Eigenvector for $\lambda = -1 + 2i$.

$$(A - (-1+2i)I)\mathbf{v} = \mathbf{0} \quad \Longrightarrow \quad \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From the first row: $-2i\,v_1 + 2\,v_2 = 0$, so $v_2 = i\,v_1$. Choose $v_1 = 1$, giving

$$\mathbf{w} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{a} + i\mathbf{b}.$$

Step 3: Real-valued solutions.

$$\mathbf{x}_1(t) = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(2t)\right] = e^{-t}\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix},$$

$$\mathbf{x}_2(t) = e^{-t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(2t)\right] = e^{-t}\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}.$$

Step 4: General solution.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\left[c_1\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}\right].$$

Phase plane interpretation. The factor e^{-t} causes all trajectories to spiral inward toward the origin. This is a **stable (sink) spiral**.

12.4 Eigenvalue Method: Repeated Eigenvalues

A repeated eigenvalue creates a subtlety: if A has only one linearly independent eigenvector for a double eigenvalue λ, the matrix is called **defective** and we cannot form two independent solutions using just $e^{\lambda t}\mathbf{v}$. Instead, we need a **generalized eigenvector**.

Generalized eigenvectors. Suppose A has a repeated eigenvalue λ with only one eigenvector $\mathbf{v}_1$ satisfying $(A - \lambda I)\mathbf{v}_1 = \mathbf{0}$. We find a **generalized eigenvector $\mathbf{v}_2$** by solving

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1. \tag{51}$$

The two linearly independent solutions are

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}, \qquad \mathbf{x}_2(t) = \big(\mathbf{v}_2 + \mathbf{v}_1\,t\big)e^{\lambda t}.$$

Key Result

Repeated eigenvalue with one eigenvector (defective case). If A has a double eigenvalue λ and a single eigenvector $\mathbf{v}_1$, let $\mathbf{v}_2$ satisfy equation (51). The general solution of $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2\big(\mathbf{v}_2 + \mathbf{v}_1\,t\big)e^{\lambda t}.$$

Remark 12.2. If A has a repeated eigenvalue but *two* linearly independent eigenvectors (which occurs only when $A = \lambda I$), the solution is the same as the distinct case: $\mathbf{x}(t) = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)e^{\lambda t}$.

Worked Example

Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues. Since A is upper triangular, the eigenvalues are the diagonal entries: $\lambda = 3$ (double root).
Step 2: Eigenvector. Solve $(A - 3I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first row gives $v_2 = 0$. The free variable v_1 gives $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Only one eigenvector, so A is defective.

Step 3: Generalized eigenvector. Solve $(A - 3I)\mathbf{v}_2 = \mathbf{v}_1$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The first row gives $u_2 = 1$. The second row gives $0 = 0$, so u_1 is free. Choose $u_1 = 0$, giving $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 4: General solution.

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t},$$

$$\mathbf{x}_2(t) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{3t} = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{3t}.$$

Therefore

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} e^{3t} = e^{3t} \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}.$$

Worked Example

Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 1: Eigenvalues.

$$\det \begin{pmatrix} -2 - \lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (-2 - \lambda)(-\lambda) + 1 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0.$$

Double eigenvalue: $\lambda = -1$.

Step 2: Eigenvector. Solve $(A + I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From $-v_1 + v_2 = 0$: choose $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Step 3: Generalized eigenvector. Solve $(A + I)\mathbf{v}_2 = \mathbf{v}_1$:

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From $-u_1 + u_2 = 1$, choose $u_1 = 0$, giving $u_2 = 1$. So $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 4: General solution.

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t},$$

$$\mathbf{x}_2(t) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] e^{-t} = \begin{pmatrix} t \\ 1 + t \end{pmatrix} e^{-t}.$$

The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} t \\ 1 + t \end{pmatrix} e^{-t}.$$

All trajectories decay to the origin as $t \to \infty$ (since $\lambda = -1 < 0$), but with the additional polynomial factor t in the second component.

12.5 Phase Plane Analysis

For 2×2 systems $\mathbf{x}' = A\mathbf{x}$, the qualitative behavior of solutions near the equilibrium $\mathbf{x} = \mathbf{0}$ is determined entirely by the eigenvalues of A. The **phase plane** — a plot of trajectories in the (x_1, x_2)-plane — reveals the system's dynamics.

Six equilibrium classifications.

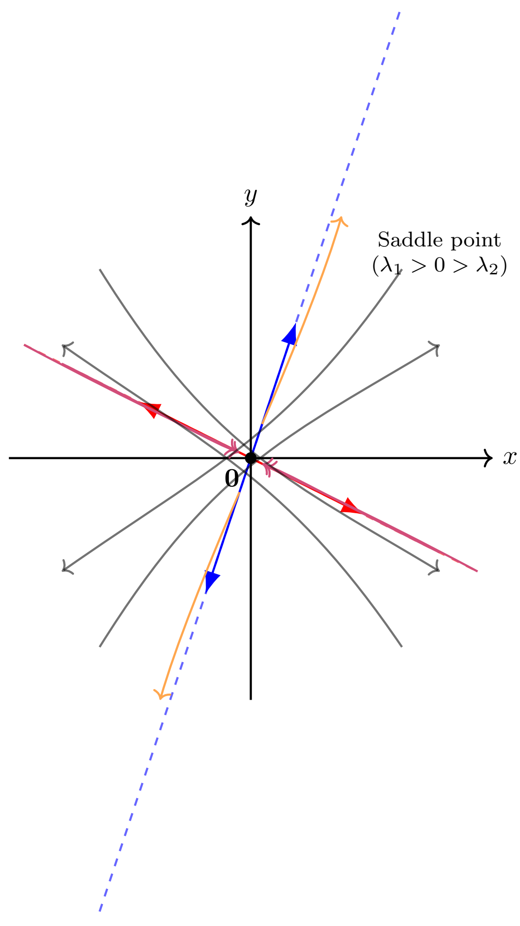
Key Result

Phase plane classification for 2×2 systems $\mathbf{x}' = A\mathbf{x}$.

Eigenvalues	**Classification**
Real, both positive	Unstable node (source) — trajectories leave $\mathbf{0}$
Real, both negative	Stable node (sink) — trajectories enter $\mathbf{0}$
Real, opposite signs	Saddle point — trajectories approach along one eigenvector, depart along the other
Purely imaginary $\pm i\beta$	Center — closed orbits (ellipses) around $\mathbf{0}$
Complex, $\Re(\lambda) > 0$	Unstable spiral — trajectories spiral outward
Complex, $\Re(\lambda) < 0$	Stable spiral — trajectories spiral inward

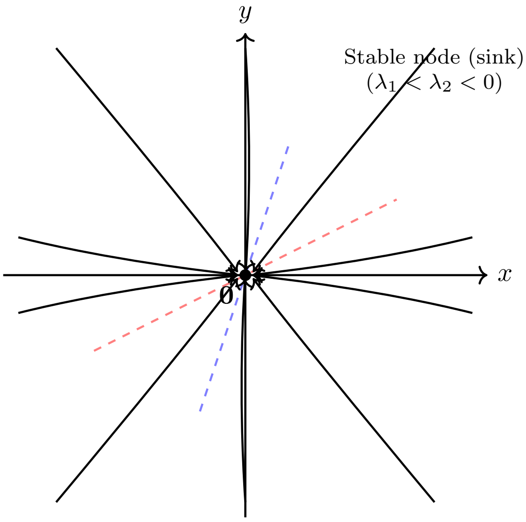
Reading trajectories from eigenvectors. For real eigenvalues, the eigenvectors define **invariant lines** through the origin. Trajectories on these lines are straight, while all other trajectories are curves that become tangent to the **slow eigenvector** (the eigenvector associated with the eigenvalue closest to zero) as $t \to \infty$.

12.5.1 Saddle Point



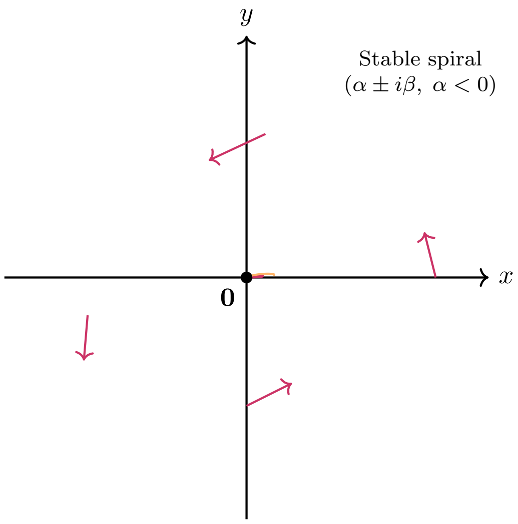
Saddle for $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. From section 12.2, $\lambda_1 = 4$ (unstable) with $\mathbf{v}_1 = (2,3)^\top$, and $\lambda_2 = -1$ (stable) with $\mathbf{v}_2 = (1,-1)^\top$. The diagram above shows the generic saddle pattern: trajectories converge along the stable manifold (dashed red, direction of $\mathbf{v}_2$) and diverge along the unstable manifold (dashed blue, direction of $\mathbf{v}_1$).

12.5.2 Stable Node (Sink)



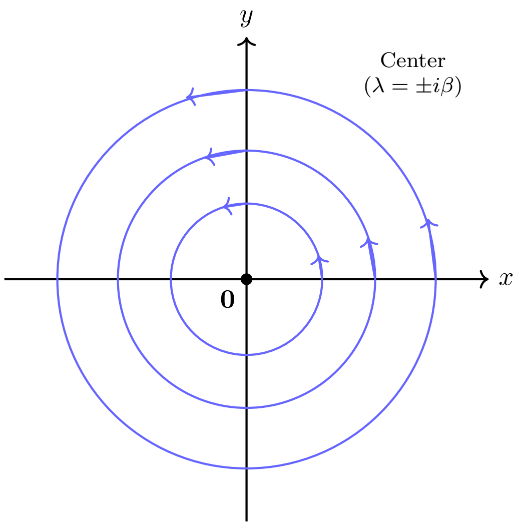
All trajectories converge to the origin. Far from the origin, trajectories approach tangent to the eigenvector associated with the eigenvalue closer to zero (the "slow" direction).

12.5.3 Stable Spiral (Sink Spiral)



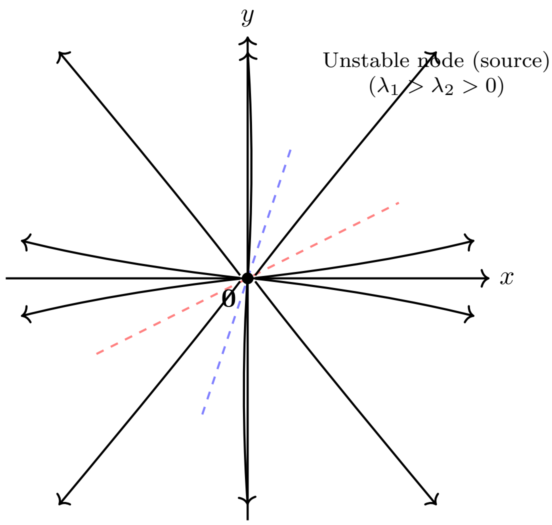
Trajectories spiral toward the origin. The number of complete rotations depends on the ratio $\beta/|\alpha|$ (frequency vs. decay rate).

12.5.4 Center



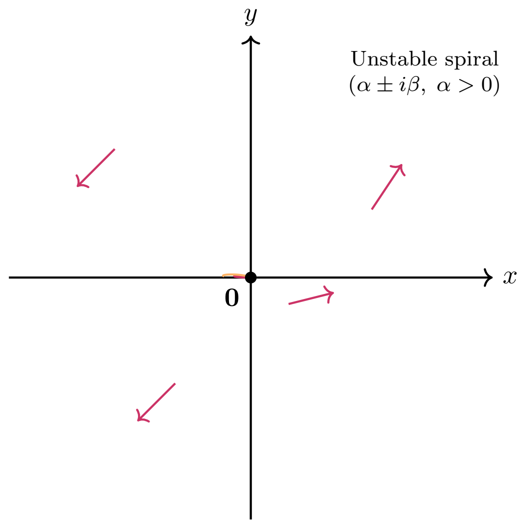
Trajectories are closed orbits (circles or ellipses). The equilibrium is **Lyapunov stable** but not asymptotically stable: nearby solutions stay nearby but do not converge to the origin.

12.5.5 Unstable Node (Source)



All trajectories diverge from the origin. Far from the origin, trajectories become tangent to the eigenvector associated with the larger eigenvalue (the "fast" direction).

12.5.6 Unstable Spiral (Source Spiral)



Trajectories spiral outward from the origin. The rate of divergence is governed by the real part α of the eigenvalues, while the frequency of rotation depends on β.

12.5.7 Classifying Equilibria: Worked Examples

Worked Example

Classify the equilibrium of $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \mathbf{x}$.

Solution. We computed the eigenvalues in section 12.2: $\lambda_1 = 4 > 0$ and $\lambda_2 = -1 < 0$. Opposite signs $\implies$ **saddle point**. The unstable manifold is along $\mathbf{v}_1 = (2, 3)^\top$ and the stable manifold along $\mathbf{v}_2 = (1, -1)^\top$.

Worked Example

Classify the equilibrium of $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} \mathbf{x}$.

Solution. Since A is upper triangular, the eigenvalues are the diagonal entries: $\lambda_1 = -2$ and $\lambda_2 = -3$.

Both are negative $\implies$ **stable node (sink)**.
Eigenvectors. For $\lambda_1 = -2$, solve $(A + 2I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving $v_2 = 0$ and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (fast eigendirection, along the x-axis).
For $\lambda_2 = -3$, solve $(A + 3I)\mathbf{v} = \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

giving $v_1 + v_2 = 0$, so $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (slow eigendirection).
Since $|\lambda_1| > |\lambda_2|$, trajectories approach the origin tangent to the slow eigenvector $\mathbf{v}_2$.

12.6 Trace-Determinant Classification

For a 2×2 matrix A, the eigenvalues are determined by two scalar quantities:

$$\tau = \operatorname{tr}(A) = \lambda_1 + \lambda_2, \qquad \Delta = \det(A) = \lambda_1 \lambda_2.$$

The characteristic equation $\lambda^2 - \tau\lambda + \Delta = 0$ has discriminant

$$D = \tau^2 - 4\Delta.$$

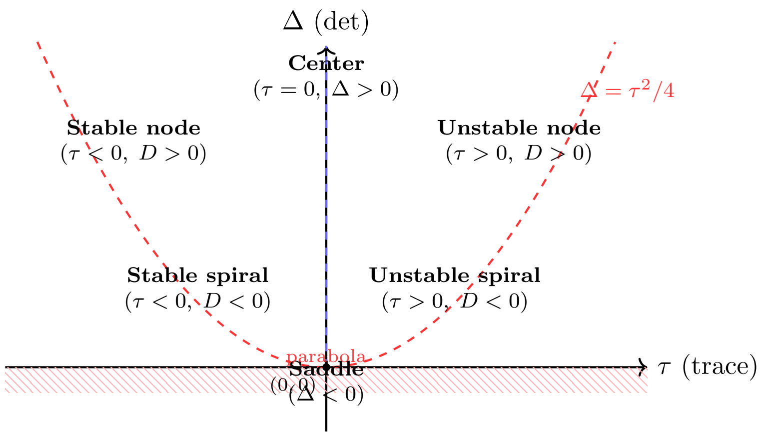
The sign of D determines whether the eigenvalues are real ($D > 0$) or complex ($D < 0$).

Key Result

Trace-determinant classification. The equilibrium $\mathbf{0}$ of $\mathbf{x}' = A\mathbf{x}$ is classified by the position of (τ, Δ) in the trace-determinant plane:

Region	Conditions	Classification
$\Delta < 0$	(below τ-axis)	Saddle
$\Delta > 0$, $\tau > 0$, $\tau^2 > 4\Delta$	(above parabola, right of Δ-axis)	Unstable node
$\Delta > 0$, $\tau < 0$, $\tau^2 > 4\Delta$	(above parabola, left of Δ-axis)	Stable node
$\Delta > 0$, $\tau > 0$, $\tau^2 < 4\Delta$	(below parabola, right of Δ-axis)	Unstable spiral
$\Delta > 0$, $\tau < 0$, $\tau^2 < 4\Delta$	(below parabola, left of Δ-axis)	Stable spiral
$\Delta > 0$, $\tau = 0$	(on positive Δ-axis)	Center

The trace-determinant plane.



The parabola $\Delta = \tau^2/4$ separates real from complex eigenvalues. The Δ-axis ($\tau = 0$) separates stable from unstable systems. The τ-axis ($\Delta = 0$) separates saddles from nodes/spirals/centers.

Worked Example

Classify the system $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \mathbf{x}$ using the trace-determinant plane.

Solution. Compute:

$$\tau = \mathrm{tr}(A) = 1 + (-2) = -1, \qquad \Delta = \det(A) = (1)(-2) - (2)(1) = -4.$$

Since $\Delta = -4 < 0$, the point $(\tau, \Delta) = (-1, -4)$ lies below the τ-axis in the saddle region. This confirms: **saddle point**.

Verification: eigenvalues satisfy $\lambda^2 + \lambda - 4 = 0$, giving $\lambda = \frac{-1 \pm \sqrt{17}}{2} \approx 1.56, -2.56$ — opposite signs, consistent with a saddle.

Worked Example

Classify $\mathbf{x}' = \begin{pmatrix} -1 & 2 \\ -3 & -1 \end{pmatrix} \mathbf{x}$.

Solution.

$$\tau = -1 + (-1) = -2, \qquad \Delta = (-1)(-1) - (2)(-3) = 1 + 6 = 5.$$

Discriminant: $D = \tau^2 - 4\Delta = 4 - 20 = -16 < 0$. Since $\tau = -2 < 0$ and $D < 0$, the point $(-2, 5)$ lies in the **stable spiral** region (left of Δ-axis, below the parabola). This is consistent with the eigenvalues $\lambda = -1 \pm 2i$ found in section 12.3.

Worked Example

Classify $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 2 & 0 \end{pmatrix} \mathbf{x}$.

Solution.

$$\tau = 0, \qquad \Delta = 0 - (-10) = 10.$$

The point $(0, 10)$ lies on the positive Δ-axis, which corresponds to **center** classification. Verification: $\lambda^2 + 10 = 0$, so $\lambda = \pm i\sqrt{10}$ — purely imaginary, confirming closed orbits.

12.7 Stability Theory

Stability analysis for linear systems follows directly from the eigenvalue structure.

Theorem 12.3 (Asymptotic stability of linear systems)**.** *The equilibrium $\mathbf{x} = \mathbf{0}$ of $\mathbf{x}' = A\mathbf{x}$ is **asymptotically stable** if and only if every eigenvalue λ of A satisfies $\Re(\lambda) < 0$.*

Proof. If A has a basis of eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, the general solution is

$$\mathbf{x}(t) = \sum_{k=1}^{n} c_k e^{\lambda_k t} \mathbf{v}_k.$$

Each term satisfies $|e^{\lambda_k t}| = e^{\Re(\lambda_k) t}$. If $\Re(\lambda_k) < 0$ for all k, then every term decays exponentially to zero as $t \to \infty$, so $\mathbf{x}(t) \to \mathbf{0}$. Conversely, if any $\Re(\lambda_k) > 0$, the corresponding term grows without bound; if any $\Re(\lambda_k) = 0$ with nontrivial Jordan block, polynomial growth occurs. Hence the criterion is both necessary and sufficient. □

Corollary 12.4 (Stability for 2×2 systems)**.** *For a 2×2 matrix A, the equilibrium $\mathbf{0}$ is asymptotically stable if and only if*

$$\mathrm{tr}(A) < 0 \quad \textit{and} \quad \det(A) > 0.$$

Proof. The eigenvalues are $\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$.

- If $\Delta < 0$, one eigenvalue is positive $\implies$ unstable.
- If $\Delta > 0$ and $\tau \geq 0$, at least one eigenvalue has nonnegative real part $\implies$ not asymptotically stable.
- If $\Delta > 0$ and $\tau < 0$, both eigenvalues have negative real parts (either both real negative, or complex with negative real part $\tau/2$) $\implies$ asymptotically stable.

□

Theorem 12.5 (Lyapunov stability)**.** *The equilibrium* $\mathbf{x} = \mathbf{0}$ *is* ***Lyapunov stable*** *(but not asymptotically stable) if and only if all eigenvalues satisfy* $\Re(\lambda) \le 0$ *and every eigenvalue with* $\Re(\lambda) = 0$ *has a trivial Jordan block (i.e. the matrix is diagonalizable over the imaginary axis).*

Remark 12.6. The most common example of Lyapunov stability without asymptotic stability is the **center** ($\lambda = \pm i\beta$). Nearby solutions stay close to the origin forever but never converge to it.

Worked Example

Show that $\mathbf{x}' = \begin{pmatrix} -2 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{x}$ is asymptotically stable.

Solution.

$$\tau = -2 + 0 = -2 < 0, \qquad \Delta = (-2)(0) - (3)(-1) = 3 > 0.$$

By Theorem 12.4, both conditions are satisfied. Eigenvalues: $\lambda = \frac{-2 \pm \sqrt{4-12}}{2} = -1 \pm i\sqrt{2}$. Real part $-1 < 0$, confirming asymptotic stability (stable spiral).

Worked Example

Show that $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$ is unstable.

Solution.

$$\tau = \operatorname{tr}(A) = 3 + 3 = 6, \qquad \Delta = \det(A) = (3)(3) - (1)(1) = 9 - 1 = 8.$$

Discriminant: $D = \tau^2 - 4\Delta = 36 - 32 = 4 > 0$. Since $\tau = 6 > 0$ and $\Delta = 8 > 0$ with $D > 0$, the point $(6, 8)$ lies in the **unstable node** region (right of Δ-axis, above the parabola).
Verification: eigenvalues satisfy $\lambda^2 - 6\lambda + 8 = 0$, giving

$$\lambda = \frac{6 \pm \sqrt{36 - 32}}{2} = \frac{6 \pm 2}{2}, \quad \text{so} \quad \lambda_1 = 4, \quad \lambda_2 = 2.$$

Both eigenvalues are positive, confirming an **unstable node (source)**. All nonzero solutions diverge from the origin as $t \to \infty$.

12.8 Matrix Exponential

The **matrix exponential** provides a compact, general formula for the solution of $\mathbf{x}' = A\mathbf{x}$, analogous to e^{at} for the scalar equation $x' = ax$.

Definition. For any square matrix A, define

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}. \tag{52}$$

This power series converges for all square matrices A and all $t \in \mathbb{R}$.

Key Result

Properties of the matrix exponential.

1. $e^{A \cdot 0} = I$.
2. $\dfrac{d}{dt} e^{At} = A\, e^{At} = e^{At} A$.
3. $e^{A(t+s)} = e^{At} e^{As}$ for all $s, t \in \mathbb{R}$.
4. e^{At} is always invertible, with $(e^{At})^{-1} = e^{-At}$.

Solution of initial value problems. If A has distinct eigenvalues and is diagonalizable as $A = PDP^{-1}$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$e^{At} = P\, e^{Dt} P^{-1} = P \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} P^{-1}. \tag{53}$$

The solution of the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is then

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0. \tag{54}$$

Worked Example

Compute e^{At} for $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$.

Step 1: Diagonalization. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$ (diagonal entries, since A is triangular).

Eigenvector for $\lambda_1 = 2$: $(A - 2I)\mathbf{v} = \mathbf{0}$ gives $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{v} = \mathbf{0}$, so $v_2 = 0$ and $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 3$: $(A - 3I)\mathbf{v} = \mathbf{0}$ gives $\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = \mathbf{0}$, so $-v_1 + v_2 = 0$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Construct:

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Step 2: Compute $e^{At} = Pe^{Dt}P^{-1}$.

$$e^{Dt} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

$$P\,e^{Dt} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{pmatrix},$$

$$P\,e^{Dt}\,P^{-1} = \begin{pmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & -e^{2t} + e^{3t} \\ 0 & e^{3t} \end{pmatrix}.$$

Therefore

$$e^{At} = \begin{pmatrix} e^{2t} & e^{3t} - e^{2t} \\ 0 & e^{3t} \end{pmatrix}.$$

Verification. Check $e^{A\cdot 0} = I$: substituting $t = 0$ gives $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. ✓

Worked Example

Solve the initial value problem $\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Step 1: Eigenvalues and eigenvectors.

$$\det\begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 9 = 0, \quad \text{so} \quad \lambda = 1 \pm 3.$$

$\lambda_1 = 4$, $\lambda_2 = -2$.

For $\lambda_1 = 4$: $\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -2$: $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\mathbf{v} = \mathbf{0} \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Step 2: e^{At}.

$$\begin{aligned} e^{At} &= P\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix}P^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{pmatrix}. \end{aligned}$$

Step 3: Solution.

$$\begin{aligned} \mathbf{x}(t) = e^{At}\begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{4t} + e^{-2t} \\ e^{4t} - e^{-2t} \end{pmatrix}. \end{aligned}$$

In components: $x(t) = e^{4t} + e^{-2t}$, $y(t) = e^{4t} - e^{-2t}$. The dominant behavior for large t is e^{4t} (consistent with the positive eigenvalue $\lambda_1 = 4$).

12.9 Summary

Table 15: Eigenvalue cases for $\mathbf{x}' = A\mathbf{x}$ (2×2 systems)

Eigenvalues	**Eigenvectors**	**General solution**
Real distinct: $\lambda_1 \neq \lambda_2$	Two independent $\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$
Complex conjugate: $\alpha \pm i\beta$	$\mathbf{a} \pm i\mathbf{b}$	$\mathbf{x}(t) = c_1 e^{\alpha t}[\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t] + c_2 e^{\alpha t}[\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t]$
Repeated, one eigenvector: λ (defective)	$\mathbf{v}_1$ (eigenvector), $\mathbf{v}_2$ (generalized)	$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\lambda t}(\mathbf{v}_2 + t\mathbf{v}_1)$

Table 16: Phase plane classifications and stability

Eigenvalues	**Phase plane**	**Trace/Det**	**Stability**
$\lambda_1 > 0 > \lambda_2$	Saddle	$\Delta < 0$	Unstable
$\lambda_1 > \lambda_2 > 0$	Unstable node	$\Delta > 0,\ \tau > 0,\ \tau^2 > 4\Delta$	Unstable
$\lambda_1 < \lambda_2 < 0$	Stable node	$\Delta > 0,\ \tau < 0,\ \tau^2 > 4\Delta$	Asymptotically stable
$\pm i\beta$	Center	$\Delta > 0,\ \tau = 0$	Lyapunov stable (not asymptotic)
$\alpha \pm i\beta,\ \alpha > 0$	Unstable spiral	$\Delta > 0,\ \tau > 0,\ \tau^2 < 4\Delta$	Unstable
$\alpha \pm i\beta,\ \alpha < 0$	Stable spiral	$\Delta > 0,\ \tau < 0,\ \tau^2 < 4\Delta$	Asymptotically stable

Table 17: Key formulas

Concept	**Formula**
Companion matrix ($y'' + a_1 y' + a_0 y = 0$)	$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$
Characteristic equation	$\det(A - \lambda I) = \lambda^2 - \tau\lambda + \Delta = 0$
Generalized eigenvector	$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$
Matrix exponential	$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$
Solution via diagonalization	$e^{At} = P\ \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\ P^{-1}$
IVP solution	$\mathbf{x}(t) = e^{At}\mathbf{x}_0$
Asymptotic stability (2×2)	$\mathrm{tr}(A) < 0$ and $\det(A) > 0$

Hint

Problem-solving workflow for $\mathbf{x}' = A\mathbf{x}$.

1. Compute $\tau = \mathrm{tr}(A)$ and $\Delta = \det(A)$.
2. Calculate the discriminant $D = \tau^2 - 4\Delta$.
3. Classify using the trace-determinant plane (section 12.6).
4. Find eigenvalues λ and eigenvectors $\mathbf{v}$ explicitly.
5. Assemble the general solution using the appropriate case from table 15.
6. If initial conditions are given, solve for the constants c_k.

13 Series Solutions

When the coefficients of a linear differential equation are not constant, the methods of section 8 and section 9 (characteristic equations, undetermined coefficients, variation of parameters) generally fail. In such cases, **series solutions** provide a powerful alternative: we represent the unknown solution as an infinite series and determine the coefficients by substituting into the differential equation.

This chapter develops three closely related methods:

1. The **power series method** for equations with analytic coefficients (section 13.1).
2. **Euler–Cauchy (equidimensional) equations**, a special class admitting closed-form solutions (section 13.2).
3. The **Frobenius method**, which extends the power series idea to equations with regular singular points (section 13.3).

All three methods share a common theme: substitute a series ansatz, reindex, collect coefficients, and solve the resulting recurrence relation.

13.1 Power Series Method

When to use. The power series method applies to the second-order linear homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{55}$$

when both $p(x)$ and $q(x)$ are **analytic** at a point x_0 (typically $x_0 = 0$). A function is analytic at x_0 if it equals its Taylor series in a neighborhood of x_0. This includes polynomials, e^x, $\sin x$, $\cos x$, and rational functions away from their poles.

If $p(x)$ or $q(x)$ has a singularity at x_0, the power series method in its basic form fails — we must instead use the Frobenius method (section 13.3).

Step-by-step algorithm.

1. **Ansatz.** Assume a power series solution about $x_0 = 0$:
$$y(x) = \sum_{n=0}^{\infty} a_n\,x^n, \qquad a_n \text{ to be determined.}$$
2. **Differentiate term-by-term:**
$$y'(x) = \sum_{n=1}^{\infty} n\,a_n\,x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2}.$$
3. **Substitute** y, y', and y'' into the ODE equation (55).
4. **Reindex** all sums so they involve the same power x^k. This typically means shifting the index: set $k = n-m$ in each sum so every sum starts at the same k and involves x^k.
5. **Collect coefficients** of each power x^k. The resulting coefficient expression must vanish for every k, producing a **recurrence relation** for the coefficients a_n.
6. **Solve the recurrence** to find the coefficients in terms of a_0 and a_1 (which are arbitrary — they correspond to the two degrees of freedom in a second-order ODE).
7. **Write the general solution** as a linear combination of the two independent series.

Radius of convergence.

Theorem 13.1 (Radius of Convergence)**.** *Consider the equation equation* (55) *with $p(x)$ and $q(x)$ analytic at $x_0 = 0$. If R_p and R_q denote the radii of convergence of the Taylor series of $p(x)$ and $q(x)$ at $x_0 = 0$, then the power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at least on the interval $(-R, R)$ where*

$$R = \min(R_p,\ R_q).$$

Equivalently, R is at least as large as the distance from x_0 to the nearest singularity of $p(x)$ or $q(x)$ in the complex plane.

In practice, most textbook problems have polynomial coefficients (so $R_p = R_q = \infty$), and the series converges for all $x \in \mathbb{R}$. When $p(x)$ or $q(x)$ is a rational function, R is the distance from x_0 to the nearest pole.

Hint

Quick radius check. If the ODE has polynomial coefficients, the power series converges for all x. If coefficients are rational, compute the distance from the expansion point to the nearest pole. For example, $y'' + \frac{1}{1-x^2}y = 0$ expanded about $x_0 = 0$ has poles at $x = \pm 1$, so $R = 1$.

Worked examples.

Worked Example

Solve $y'' - y = 0$ using the power series method and show that the solution recovers the exponential functions.
Solution. Here $p(x) = 0$ and $q(x) = -1$, both analytic everywhere. Assume

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^n.$$

Differentiate:

$$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Substitute into $y'' - y = 0$:

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - \sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Reindex the first sum by setting $k = n - 2$ (so $n = k + 2$):

$$\sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2}\, x^k - \sum_{k=0}^{\infty} a_k\, x^k = 0.$$

Both sums now involve x^k starting at $k = 0$. Collect coefficients of x^k:

$$(k+2)(k+1)\, a_{k+2} - a_k = 0 \quad \Longrightarrow \quad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

This is the **recurrence relation**. It connects every even coefficient to a_0 and every odd coefficient to a_1.
Even coefficients:

$$\begin{aligned}
a_2 &= \frac{a_0}{2 \cdot 1} = \frac{a_0}{2!}, \\
a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}, \\
a_6 &= \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!}, \\
&\ \vdots \\
a_{2m} &= \frac{a_0}{(2m)!}.
\end{aligned}$$

Odd coefficients:

$$\begin{aligned}
a_3 &= \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!}, \\
a_5 &= \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \\
a_7 &= \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!}, \\
&\ \vdots \\
a_{2m+1} &= \frac{a_1}{(2m+1)!}.
\end{aligned}$$

The general solution is

$$y(x) = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}.$$

These are the Taylor series for $\cosh x$ and $\sinh x$:

$$y(x) = a_0 \cosh x + a_1 \sinh x.$$

Equivalently, writing $c_1 = \frac{a_0+a_1}{2}$ and $c_2 = \frac{a_0-a_1}{2}$:

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

This recovers the exponential solutions found via the characteristic equation in section 8. The power series method works, but for constant-coefficient equations it is overkill — the characteristic equation is much faster. The power series method truly shines when coefficients are *not* constant.

Worked Example

Solve the Legendre equation $(1-x^2)y'' - 2xy' + 2y = 0$ using the power series method. Find a polynomial solution.

Solution. This is a special case of the **Legendre equation** $(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$ with $\ell = 1$. The coefficients $p(x) = \dfrac{-2x}{1-x^2}$ and $q(x) = \dfrac{2}{1-x^2}$ have poles at $x = \pm 1$, so by Theorem 13.1 the series converges for $|x| < 1$.

Assume $y = \sum_{n=0}^{\infty} a_n x^n$. Then

$$y' = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}.$$

Substitute into $(1-x^2)y'' - 2xy' + 2y = 0$:

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - 2x\sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + 2\sum_{n=0}^{\infty} a_n\, x^n = 0.$$

Expand $(1-x^2)y''$:

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^n - \sum_{n=1}^{\infty} 2n\, a_n\, x^n + \sum_{n=0}^{\infty} 2a_n\, x^n = 0.$$

Reindex the first sum by $k = n-2$ (so $n = k+2$):

$$\sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2}\, x^k - \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^n - \sum_{n=1}^{\infty} 2n\, a_n\, x^n + \sum_{n=0}^{\infty} 2a_n\, x^n = 0.$$

All sums now involve powers of x starting from x^0. Renaming the dummy index to k everywhere and collecting coefficients of x^k:

$$(k+2)(k+1)\, a_{k+2} - k(k-1)\, a_k - 2k\, a_k + 2a_k = 0,$$

where we understand $a_k = 0$ for $k < 0$. Simplify the a_k coefficient:

$$-k(k-1) - 2k + 2 = -k^2 + k - 2k + 2 = -k^2 - k + 2 = -(k-1)(k+2).$$

The recurrence relation is

$$(k+2)(k+1)\, a_{k+2} - (k-1)(k+2)\, a_k = 0.$$

For $k \neq -2$ (always true since $k \geq 0$), divide by $(k+2)$:

$$(k+1)\, a_{k+2} - (k-1)\, a_k = 0 \quad \Longrightarrow \quad a_{k+2} = \frac{k-1}{k+1}\, a_k.$$

Now compute coefficients:

$$a_2 = \frac{0-1}{0+1}\, a_0 = -a_0,$$
$$a_4 = \frac{2-1}{2+1}\, a_2 = \frac{1}{3}(-a_0) = -\frac{a_0}{3},$$
$$a_6 = \frac{4-1}{4+1}\, a_4 = \frac{3}{5}\left(-\frac{a_0}{3}\right) = -\frac{a_0}{5}.$$

For the odd series:

$$a_3 = \frac{1-1}{1+1}\, a_1 = 0 \cdot a_1 = 0.$$

Since $a_3 = 0$, every subsequent odd coefficient vanishes ($a_5 = \frac{3-1}{3+1}a_3 = 0$, etc.). The odd series terminates immediately.
If we choose $a_1 = 1$ and $a_0 = 0$, the solution is simply

$$y(x) = a_1 x = x.$$

Let us verify: $y = x \Rightarrow y' = 1, y'' = 0$. Substituting into the ODE:

$$(1 - x^2) \cdot 0 - 2x \cdot 1 + 2 \cdot x = -2x + 2x = 0. \quad \checkmark$$

Thus $y_1(x) = x$ is a **polynomial solution** (a Legendre polynomial of degree 1, denoted $P_1(x)$). The second solution involves the infinite even series, which is related to the Legendre function of the second kind $Q_1(x)$.
Note. Legendre polynomials $P_n(x)$ are polynomial solutions of the Legendre equation for integer $\ell = n$. They arise in physics (e.g. spherical harmonics in quantum mechanics, gravitational potential theory) and form an orthogonal basis on $[-1, 1]$.

Remark 13.2. The power series method often produces solutions that are recognized as classical **special functions**. Bessel functions (Frobenius method, section 13.3), Legendre polynomials (as in Example 2 above), Hermite polynomials, and Laguerre polynomials all arise as power series solutions of specific ODEs. These special functions have their own properties, tables, and computational implementations, making the power series approach not just theoretical but deeply practical.

13.2 Euler–Cauchy Equations

The Euler–Cauchy equation (also called the **equidimensional equation**) is a special second-order ODE with variable coefficients that nevertheless admits closed-form solutions through a power ansatz.

Form and substitution. The equation has the form

$$x^2\, y'' + \alpha\, x\, y' + \beta\, y = 0, \qquad x > 0, \tag{56}$$

where $\alpha, \beta \in \mathbb{R}$. The key observation is that every term is **equidimensional**: the k-th derivative is multiplied by x^k. This suggests the ansatz

$$y(x) = x^r,$$

where r is a constant to be determined.

Derivation of the indicial equation. Compute the derivatives of $y = x^r$:

$$y' = r\, x^{r-1}, \qquad y'' = r(r-1)\, x^{r-2}.$$

Substitute into equation (56):

$$\begin{aligned} x^2 \cdot r(r-1)\, x^{r-2} + \alpha\, x \cdot r\, x^{r-1} + \beta \cdot x^r &= 0,\\ r(r-1)\, x^r + \alpha\, r\, x^r + \beta\, x^r &= 0,\\ \big[r(r-1) + \alpha\, r + \beta\big]\, x^r &= 0. \end{aligned}$$

Since $x^r \neq 0$ for $x > 0$, we must have

$$r(r-1) + \alpha\, r + \beta = 0. \tag{57}$$

This is the **indicial equation** (also called the **characteristic equation** of the Euler–Cauchy equation). It is a quadratic in r:

$$r^2 + (\alpha - 1)\, r + \beta = 0,$$

with solutions

$$r = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}.$$

Three cases.

Key Result

Euler–Cauchy: solution forms based on the indicial equation.

Roots of equation (57)	General solution
Distinct real $r_1 \neq r_2$	$y(x) = c_1\, x^{r_1} + c_2\, x^{r_2}$
Repeated $r_1 = r_2 = r$	$y(x) = x^r\big[c_1 + c_2\, \ln(x)\big]$
Complex $\lambda \pm i\omega$	$y(x) = x^\lambda\big[c_1\cos(\omega\ln x) + c_2\sin(\omega\ln x)\big]$

The repeated-root case is analogous to the repeated-root case for constant-coefficient equations (section 8.3): the logarithmic factor plays the role of the extra x factor. The complex-root case involves trigonometric functions of $\ln x$, reflecting the self-similar (scale-invariant) nature of the Euler–Cauchy equation.

Worked examples.

Worked Example

Solve $x^2\, y'' - x\, y' - 3y = 0$ for $x > 0$.
Solution. This is an Euler–Cauchy equation with $\alpha = -1$ and $\beta = -3$. The indicial equation is

$$r(r-1) + (-1)r + (-3) = 0.$$

Simplify:

$$r^2 - r - r - 3 = r^2 - 2r - 3 = 0.$$

Factor:

$$(r-3)(r+1) = 0.$$

Distinct real roots: $r_1 = 3$ and $r_2 = -1$. The general solution is

$$y(x) = c_1\, x^3 + c_2\, x^{-1} = c_1\, x^3 + \frac{c_2}{x}.$$

Verification. Compute $y' = 3c_1x^2 - c_2x^{-2}$ and $y'' = 6c_1x + 2c_2x^{-3}$. Substitute into the ODE:

$$\begin{aligned}
&x^2(6c_1x + 2c_2x^{-3}) - x(3c_1x^2 - c_2x^{-2}) - 3(c_1x^3 + c_2x^{-1})\\
&= 6c_1x^3 + 2c_2x^{-1} - 3c_1x^3 + c_2x^{-1} - 3c_1x^3 - 3c_2x^{-1}\\
&= (6-3-3)c_1x^3 + (2+1-3)c_2x^{-1} = 0. \quad \checkmark
\end{aligned}$$

Worked Example

Solve $x^2\, y'' + 3x\, y' + y = 0$ for $x > 0$.
Solution. Here $\alpha = 3$ and $\beta = 1$. The indicial equation is

$$r(r-1) + 3r + 1 = 0.$$

Simplify:

$$r^2 - r + 3r + 1 = r^2 + 2r + 1 = (r+1)^2 = 0.$$

Repeated root: $r = -1$. The general solution is

$$y(x) = x^{-1}\big[c_1 + c_2\, \ln(x)\big] = \frac{c_1 + c_2\, \ln x}{x}.$$

Verification. Let $y = \dfrac{\ln x}{x}$. Then

$$y' = \frac{1/x \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}, \qquad y'' = \frac{(-1/x)\cdot x^2 - (1-\ln x)\cdot 2x}{x^4} = \frac{-x - 2x + 2x\ln x}{x^4} = \frac{2\ln x - 3}{x^3}.$$

Substitute:

$$x^2 \cdot \frac{2\ln x - 3}{x^3} + 3x \cdot \frac{1 - \ln x}{x^2} + \frac{\ln x}{x} = \frac{2\ln x - 3}{x} + \frac{3 - 3\ln x}{x} + \frac{\ln x}{x} = \frac{2\ln x - 3 + 3 - 3\ln x + \ln x}{x} = 0. \quad \checkmark$$

Worked Example

Solve $x^2 y'' + 2x\,y' + 10y = 0$ for $x > 0$.
Solution. Here $\alpha = 2$ and $\beta = 10$. The indicial equation is

$$r(r-1) + 2r + 10 = 0.$$

Simplify:

$$r^2 - r + 2r + 10 = r^2 + r + 10 = 0.$$

Discriminant: $\Delta = 1 - 40 = -39 < 0$. Complex roots:

$$r = \frac{-1 \pm i\sqrt{39}}{2}.$$

Here $\lambda = -\frac{1}{2}$ and $\omega = \frac{\sqrt{39}}{2}$. The general solution is

$$y(x) = x^{-1/2}\left[c_1 \cos\left(\tfrac{\sqrt{39}}{2}\ln x\right) + c_2 \sin\left(\tfrac{\sqrt{39}}{2}\ln x\right)\right].$$

Equivalently:

$$y(x) = \frac{1}{\sqrt{x}}\left[c_1 \cos\left(\tfrac{\sqrt{39}}{2}\ln x\right) + c_2 \sin\left(\tfrac{\sqrt{39}}{2}\ln x\right)\right].$$

This represents oscillations in $\ln x$ whose amplitude decays like $x^{-1/2}$.

Remark 13.3. Euler–Cauchy equations are scale-invariant: replacing x by cx (for any constant $c > 0$) transforms the equation into itself. This is why the solutions involve powers of x and trigonometric functions of $\ln x$ — both are natural under scaling.

13.3 Frobenius Method

The power series method (section 13.1) requires the coefficients $p(x)$ and $q(x)$ to be analytic at the expansion point. When this condition fails, we may still obtain series solutions if the singularity is not too severe. The Frobenius method handles precisely this situation.

Regular singular point.

Definition 13.4 (Regular Singular Point)**.** Consider the equation $y'' + p(x)y' + q(x)y = 0$. The point x_0 is a **regular singular point** if:

1. $p(x)$ or $q(x)$ (or both) is *not* analytic at x_0, but
2. both $x\,p(x)$ and $x^2\,q(x)$ *are* analytic at x_0 (after a shift to $x_0 = 0$ if necessary).

If neither $x\,p(x)$ nor $x^2\,q(x)$ is analytic at x_0, then x_0 is an **irregular singular point**, and the Frobenius method does not apply.

Hint

Quick test. For the equation $y'' + \dfrac{A}{x}y' + \dfrac{B}{x^2}y = 0$, the point $x_0 = 0$ is a regular singular point because $x\,p(x) = A$ and $x^2\,q(x) = B$ are both analytic (they are constants). This is exactly the Euler–Cauchy equation (section 13.2), which is a special case where the Frobenius series terminates into a closed-form solution.

Frobenius ansatz. When $x_0 = 0$ is a regular singular point, we assume a solution of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n\,x^n = \sum_{n=0}^{\infty} a_n\,x^{n+r}, \qquad a_0 \neq 0. \tag{58}$$

The exponent r is **not** assumed to be an integer — it is a parameter to be determined by the equation. When $r = 0$, the Frobenius ansatz reduces to the ordinary power series (section 13.1).

Indicial equation. Substituting equation (58) into the ODE and collecting the lowest power of x produces a quadratic equation for r, called the **indicial equation**. This equation determines the possible values of r and hence the form of the leading term of the series.

Three cases.

Key Result

Frobenius method: solution forms based on the indicial roots.
Let r_1 and r_2 be the roots of the indicial equation, with $r_1 \geq r_2$ (real parts if complex).

Condition on r_1, r_2	**Solution structure**
$r_1 - r_2$ is *not* a nonnegative integer	Two independent Frobenius series: $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$
$r_1 = r_2$ (repeated root)	$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$, $y_2(x) = y_1(x) \ln(x) + x^{r_1} \sum_{n=0}^{\infty} b_n x^n$
$r_1 - r_2 = N$ (positive integer)	$y_1(x)$ from the larger root r_1; $y_2(x)$ may or may not contain $\ln(x)$; requires separate analysis

Cases 2 and 3 involve the logarithmic term $\ln(x)$, analogous to the repeated-root case for constant-coefficient equations. The detailed derivations of Cases 2 and 3 are beyond the scope of this chapter (see standard texts on advanced ODE theory for full treatments). We illustrate Case 1 with Bessel's equation.

Worked example: Bessel's equation of order 0.

Worked Example

Solve Bessel's equation of order 0:

$$x^2\, y'' + x\, y' + x^2\, y = 0,$$

using the Frobenius method about $x_0 = 0$.
Solution. First write the equation in standard form:

$$y'' + \frac{1}{x} y' + y = 0.$$

Here $p(x) = \dfrac{1}{x}$ and $q(x) = 1$. Check the regular singular point condition:

$$x\, p(x) = 1 \quad (\text{analytic at } x = 0), \qquad x^2\, q(x) = x^2 \quad (\text{analytic at } x = 0).$$

So $x_0 = 0$ is a regular singular point. We proceed with the Frobenius ansatz:

$$y(x) = \sum_{n=0}^{\infty} a_n\, x^{n+r}, \qquad a_0 \neq 0.$$

Compute the derivatives:

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1}, \qquad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2}.$$

Substitute into $x^2 y'' + x y' + x^2 y = 0$:

$$x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r-2} + x \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n\, x^{n+r} = 0.$$

Simplify powers of x:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n\, x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n\, x^{n+r} + \sum_{n=0}^{\infty} a_n\, x^{n+r+2} = 0.$$

Combine the first two sums:

$$(n+r)(n+r-1) + (n+r) = (n+r)(n+r-1+1) = (n+r)^2.$$

So the equation becomes

$$\sum_{n=0}^{\infty}(n+r)^2\, a_n\, x^{n+r} + \sum_{n=0}^{\infty} a_n\, x^{n+r+2} = 0.$$

Reindex the second sum: let $k = n+2$ (so $n = k-2$). When $n = 0$, $k = 2$:

$$\sum_{k=2}^{\infty} a_{k-2}\, x^{k+r} = \sum_{n=2}^{\infty} a_{n-2}\, x^{n+r}.$$

Now combine:

$$\underbrace{r^2\, a_0\, x^r}_{n=0} + \underbrace{(1+r)^2\, a_1\, x^{1+r}}_{n=1} + \sum_{n=2}^{\infty}\Big[(n+r)^2\, a_n + a_{n-2}\Big]\, x^{n+r} = 0.$$

Indicial equation. The lowest power is x^r. Its coefficient must vanish:

$$r^2\, a_0 = 0.$$

Since $a_0 \neq 0$ by assumption, we must have $r^2 = 0$, so $r = 0$ (repeated root). This is **Case 2** ($r_1 = r_2$).
Next coefficient. The coefficient of $x^{1+r} = x^1$ is $(1+r)^2 a_1 = a_1$ (since $r = 0$). For this to vanish, $a_1 = 0$.
Recurrence relation. For $n \geq 2$:

$$(n+r)^2\, a_n + a_{n-2} = 0 \quad \Longrightarrow \quad a_n = -\frac{a_{n-2}}{(n+r)^2}.$$

With $r = 0$:

$$a_n = -\frac{a_{n-2}}{n^2}.$$

Since $a_1 = 0$, all odd coefficients vanish ($a_3 = -a_1/9 = 0$, $a_5 = -a_3/25 = 0$, etc.). For even coefficients:

$$\begin{aligned} a_2 &= -\frac{a_0}{2^2}, \\ a_4 &= -\frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2} = \frac{a_0}{(2^2)(2^2 \cdot 2^2)} = \frac{a_0}{2^4 \cdot (1! \cdot 2!)^2} \cdot 2^2, \end{aligned}$$

which simplifies more cleanly as follows. Let a_{2m} be the m-th even coefficient:

$$a_{2m} = \frac{(-1)^m}{2^{2m}\,(m!)^2}\, a_0.$$

Choosing $a_0 = 1$, the first Frobenius solution is

$$y_1(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

This is the **Bessel function of the first kind of order 0**, denoted $J_0(x)$:

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

Second solution. Since $r_1 = r_2 = 0$ is a repeated root (Case 2), the second linearly independent solution has the form

$$y_2(x) = J_0(x)\, \ln(x) + \sum_{n=1}^{\infty} b_n\, x^n.$$

This function is denoted $Y_0(x)$ (the **Bessel function of the second kind of order 0**). The coefficients b_n are determined by substituting y_2 into the ODE; the derivation is more involved and omitted here.
The general solution of Bessel's equation of order 0 is

$$y(x) = c_1\, J_0(x) + c_2\, Y_0(x).$$

Remark 13.5. Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ appear ubiquitously in physics and engineering: wave propagation in circular membranes, heat conduction in cylindrical rods, electromagnetic fields in waveguides, and quantum mechanics (schrödinger equation in cylindrical coordinates). The Frobenius method is the primary tool for constructing these special functions analytically.

Remark 13.6. For the Legendre equation discussed in section 13.1, the points $x = \pm 1$ are regular singular points. The Frobenius method can be applied about those points as well, yielding the associated Legendre functions. When the parameter ℓ is a nonnegative integer, the series terminates into the familiar Legendre polynomials $P_n(x)$.

13.4 Summary

Table 18: Series solution methods: when to use and solution forms

Method	**When to use**	**Solution form**
Power series	$p(x)$ and $q(x)$ analytic at x_0	$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$
Euler–Cauchy	$x^2 y'' + \alpha x y' + \beta y = 0$	$y = c_1 x^{r_1} + c_2 x^{r_2}$ (see section 13.2)
Frobenius	x_0 is a regular singular point; $xp(x)$ and $x^2 q(x)$ analytic at x_0	$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$

Table 19: Frobenius method: indicial roots and solution structure

Indicial roots r_1, r_2	**Solution structure**
$r_1 - r_2 \notin \{0, 1, 2, \dots\}$	Two Frobenius series (no $\ln x$)
$r_1 = r_2$	One Frobenius series + logarithmic term
$r_1 - r_2 = N \in \{1, 2, 3, \dots\}$	One Frobenius series; second solution may contain $\ln x$

Table 20: Euler–Cauchy equation: indicial roots and solution forms

Roots of $r(r-1) + \alpha r + \beta = 0$	**General solution**
Distinct real $r_1 \neq r_2$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Repeated r	$y(x) = x^r \left[c_1 + c_2 \ln(x)\right]$
Complex $\lambda \pm i\omega$	$y(x) = x^{\lambda} \left[c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)\right]$

Hint

Problem-solving workflow for series solutions.

1. **Classify the point x_0.** Are $p(x)$ and $q(x)$ analytic at x_0? If yes, use the **power series method** (section 13.1).
2. If not, check whether x_0 is a **regular singular point**: are $x\,p(x)$ and $x^2\,q(x)$ analytic at x_0? If yes, use the **Frobenius method** (section 13.3).
3. If the equation has the special form $x^2 y'' + \alpha x y' + \beta y = 0$, use the **Euler–Cauchy method** (section 13.2), which gives closed-form solutions directly.
4. After finding the series, check whether it corresponds to a known special function (Bessel, Legendre, etc.) — this often simplifies further analysis.

14 Fourier Series

Periodic functions appear throughout applied mathematics: from alternating currents in electrical circuits to sound waves in acoustics, from the vibrations of strings to the seasonal forcing in climate models. Fourier series provide the analytical framework for representing an arbitrary periodic function as a superposition of simple sine and cosine waves. This chapter develops the theory and computational machinery of Fourier series, then shows how they serve as a powerful tool for solving linear differential equations with periodic forcing — a topic that will carry over into the boundary value problems of section 15 and the PDE chapters that follow.

14.1 Fourier Coefficients

Let $f(x)$ be a function defined on the interval $[-L, L]$ that is $2L$-periodic, meaning $f(x+2L) = f(x)$ for all x. The **Fourier series** of f is the trigonometric expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \tag{59}$$

The constants a_0, a_n, and b_n are the **Fourier coefficients**. Their explicit formulas follow from the orthogonality of the sine and cosine basis functions, which we develop in section 14.2. Here we state them as a key result.

Key Result

Fourier coefficients on $[-L, L]$. For a function $f(x)$ defined on $[-L, L]$, the Fourier coefficients are

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,\mathrm{d}x, \tag{60}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n \geq 1, \tag{61}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n \geq 1. \tag{62}$$

Why the $a_0/2$ convention? Writing the constant term as $a_0/2$ rather than a_0 ensures that the formula for a_0 matches the pattern for a_n when $n = 0$. Indeed, $\cos(0) = 1$, so a_0 computed by equation (61) with $n = 0$ gives $(1/L)\int_{-L}^{L} f(x)\,\mathrm{d}x$, which is exactly equation (60). The factor $1/2$ in equation (59) then makes the average value of the series equal to $a_0/2$.

Worked examples.

Worked Example

Compute the Fourier series of the square wave

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases}$$

extended as a 2π-periodic function.

Solution. Here $L = \pi$. We compute the coefficients.

Coefficient a_0:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\,\mathrm{d}x + \int_{0}^{\pi} 1\,\mathrm{d}x \right) = \frac{1}{\pi}(-\pi + \pi) = 0.$$

Coefficients a_n:

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\cos(nx)\,\mathrm{d}x + \int_{0}^{\pi} 1 \cdot \cos(nx)\,\mathrm{d}x \right).$$

Since $f(x)$ is an odd function (symmetric about the origin), the product $f(x)\cos(nx)$ is odd $\times$ even $=$ odd. The integral of an odd function over a symmetric interval is zero, so $a_n = 0$ for all $n \geq 1$.

Coefficients b_n:

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (-1)\sin(nx)\,\mathrm{d}x + \int_{0}^{\pi} 1 \cdot \sin(nx)\,\mathrm{d}x \right).$$

Now $f(x)\sin(nx)$ is odd $\times$ odd $=$ even, so we can simplify:

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} \sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx) \right]_0^{\pi} = \frac{2}{n\pi}\big(1 - \cos(n\pi)\big).$$

Since $\cos(n\pi) = (-1)^n$, we have $1 - (-1)^n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$. Therefore:

$$b_n = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The Fourier series is

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin\big((2k+1)x\big)}{2k+1} = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right).$$

Worked Example

Compute the Fourier series of the sawtooth wave $f(x) = x$ on $(-\pi, \pi)$, extended as a 2π-periodic function.
Solution. Here $L = \pi$. The function is odd, so $a_0 = 0$ and $a_n = 0$ for all n.
Coefficients b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, \mathrm{d}x.$$

The integrand is odd $\times$ odd $=$ even, so

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx)\, \mathrm{d}x.$$

Integrate by parts with $u = x$, $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$:

$$\int_0^{\pi} x \sin(nx)\, \mathrm{d}x = \Big[-\frac{x}{n} \cos(nx) \Big]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \cos(nx)\, \mathrm{d}x.$$

The boundary term gives $-\frac{\pi}{n}\cos(n\pi) + 0 = -\frac{\pi}{n}(-1)^n = \frac{\pi}{n}(-1)^{n+1}$. The remaining integral vanishes:

$$\frac{1}{n} \int_0^{\pi} \cos(nx)\, \mathrm{d}x = \frac{1}{n} \Big[\frac{1}{n} \sin(nx) \Big]_0^{\pi} = 0.$$

Therefore:

$$b_n = \frac{2}{\pi} \cdot \frac{\pi}{n} (-1)^{n+1} = \frac{2}{n} (-1)^{n+1}.$$

The Fourier series is

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right).$$

14.2 Orthogonality

The formulas for the Fourier coefficients equation (60)–equation (62) are derived from the **orthogonality** of the trigonometric basis functions. Two functions $\phi(x)$ and $\psi(x)$ are **orthogonal** on $[-L, L]$ if their inner product vanishes:

$$\int_{-L}^{L} \phi(x)\, \psi(x)\, \mathrm{d}x = 0.$$

Theorem 14.1 (Orthogonality of Trigonometric Functions)**.** *For positive integers n, m:*

(i) $$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big) \cos\Big(\frac{m\pi x}{L}\Big)\, \mathrm{d}x = \begin{cases} 0, & n \neq m, \\ L, & n = m \geq 1, \\ 2L, & n = m = 0. \end{cases}$$

(ii) $$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big) \sin\Big(\frac{m\pi x}{L}\Big)\, \mathrm{d}x = \begin{cases} 0, & n \neq m, \\ L, & n = m. \end{cases}$$

(iii) $$\int_{-L}^{L} \sin\Big(\frac{n\pi x}{L}\Big) \cos\Big(\frac{m\pi x}{L}\Big)\, \mathrm{d}x = 0 \quad \textit{for all } n, m \geq 0.$$

Proof of (i). Use the product-to-sum identity:

$$\cos A \cos B = \frac{1}{2} \big[\cos(A - B) + \cos(A + B) \big].$$

Let $A = \dfrac{n\pi x}{L}$ and $B = \dfrac{m\pi x}{L}$. Then:

$$\int_{-L}^{L} \cos\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x = \frac{1}{2}\int_{-L}^{L}\cos\Big(\frac{(n-m)\pi x}{L}\Big)\,\mathrm{d}x + \frac{1}{2}\int_{-L}^{L}\cos\Big(\frac{(n+m)\pi x}{L}\Big)\,\mathrm{d}x.$$

If $n \neq m$, then $n - m$ is a nonzero integer, and

$$\int_{-L}^{L}\cos\Big(\frac{(n-m)\pi x}{L}\Big)\,\mathrm{d}x = \Big[\frac{L}{(n-m)\pi}\sin\Big(\frac{(n-m)\pi x}{L}\Big)\Big]_{-L}^{L} = \frac{L}{(n-m)\pi}\big[\sin((n-m)\pi) - \sin(-(n-m)\pi)\big] = 0,$$

since $\sin(k\pi) = 0$ for any integer k. Similarly, the second integral vanishes because $n + m$ is a nonzero integer.

If $n = m \geq 1$, then $n - m = 0$ and $\cos(0) = 1$:

$$\frac{1}{2}\int_{-L}^{L} 1\,\mathrm{d}x + \frac{1}{2}\int_{-L}^{L}\cos\Big(\frac{2n\pi x}{L}\Big)\,\mathrm{d}x = \frac{1}{2}(2L) + 0 = L.$$

If $n = m = 0$, then $\cos(0)\cos(0) = 1$, and the integral is $2L$.

Proof of (ii). Use $\sin A \sin B = \frac{1}{2}\big[\cos(A - B) - \cos(A + B)\big]$. The same analysis as above applies: both integrals vanish for $n \neq m$. For $n = m$:

$$\frac{1}{2}\int_{-L}^{L} 1\,\mathrm{d}x - \frac{1}{2}\int_{-L}^{L}\cos\Big(\frac{2n\pi x}{L}\Big)\,\mathrm{d}x = L - 0 = L.$$

Proof of (iii). Use $\sin A \cos B = \frac{1}{2}\big[\sin(A + B) + \sin(A - B)\big]$. Both terms integrate to zero since the sine of a multiple of π is zero at the endpoints.

How orthogonality yields the coefficients. Multiply both sides of equation (59) by $\cos(m\pi x/L)$ and integrate over $[-L, L]$. By orthogonality, every term in the infinite sum vanishes except the one with $n = m$:

$$\int_{-L}^{L} f(x)\cos\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x = a_m \int_{-L}^{L}\cos^2\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x = a_m \cdot L,$$

which immediately gives $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(m\pi x/L)\,\mathrm{d}x$. The derivation for b_n is identical, using multiplication by $\sin(m\pi x/L)$ instead. Multiplying by 1 and integrating gives a_0.

14.3 Even and Odd Functions

When $f(x)$ possesses symmetry, the Fourier series simplifies dramatically. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

Key Result

Fourier series of even and odd functions.

- If $f(x)$ is **even** on $[-L, L]$, then $b_n = 0$ for all n, and

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,\mathrm{d}x, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x.$$

  The series contains only cosine terms (a **cosine series**).

- If $f(x)$ is **odd** on $[-L, L]$, then $a_0 = 0$ and $a_n = 0$ for all n, and

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x.$$

  The series contains only sine terms (a **sine series**).

Intuition. The product of an even function with a sine (odd) is odd, and the integral of an odd function over $[-L, L]$ is zero. Hence an even function has no sine terms. Conversely, the product of an odd function with a cosine (even) is odd, so an odd function has no cosine terms.

Worked examples.

Worked Example

Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$.
Solution. Since $|-x| = |x|$, the function is even. We need only compute the cosine coefficients. With $L = \pi$:

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi.$$

For $n \geq 1$:

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)\,\mathrm{d}x.$$

Integrate by parts with $u = x$, $\mathrm{d}v = \cos(nx)\,\mathrm{d}x$:

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = \Big[\frac{x}{n}\sin(nx)\Big]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,\mathrm{d}x.$$

The boundary term vanishes (since $\sin(n\pi) = 0$), and the remaining integral is

$$-\frac{1}{n}\Big[-\frac{1}{n}\cos(nx)\Big]_0^\pi = \frac{1}{n^2}\big((-1)^n - 1\big).$$

Therefore:

$$a_n = \frac{2}{\pi}\cdot\frac{1}{n^2}\big((-1)^n - 1\big) = \begin{cases} 0, & n \text{ even}, \\ -\dfrac{4}{\pi n^2}, & n \text{ odd}. \end{cases}$$

The Fourier series is

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos\big((2k+1)x\big)}{(2k+1)^2} = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots\right).$$

Worked Example

Find the Fourier series of $f(x) = x^2$ on $[-\pi, \pi]$.
Solution. The function is even, so $b_n = 0$. With $L = \pi$:

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

For $n \geq 1$:

$$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)\,\mathrm{d}x.$$

Integrate by parts twice. First, $u = x^2$, $\mathrm{d}v = \cos(nx)\,\mathrm{d}x$:

$$\int_0^\pi x^2\cos(nx)\,\mathrm{d}x = \Big[\frac{x^2}{n}\sin(nx)\Big]_0^\pi - \frac{2}{n}\int_0^\pi x\sin(nx)\,\mathrm{d}x.$$

The boundary term vanishes. From section 14.1 (sawtooth example), $\int_0^\pi x\sin(nx)\,\mathrm{d}x = \frac{\pi}{n}(-1)^{n+1}$. So:

$$\int_0^\pi x^2\cos(nx)\,\mathrm{d}x = -\frac{2}{n}\cdot\frac{\pi}{n}(-1)^{n+1} = \frac{2\pi}{n^2}(-1)^n.$$

Therefore:

$$a_n = \frac{2}{\pi}\cdot\frac{2\pi}{n^2}(-1)^n = \frac{4(-1)^n}{n^2}.$$

The Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right).$$

14.4 Half-Range Expansions

In practice, a function $f(x)$ is often given only on the half-interval $[0, L]$. We can construct a Fourier series by **extending** f to $[-L, L]$ as either an even function (yielding a cosine series) or an odd function (yielding a sine series). These are called **half-range expansions**.

Key Result

Half-range expansions on $[0, L]$.

- **Cosine series (even extension).** Extend f to $[-L, L]$ by defining $f(-x) = f(x)$. The Fourier series contains only cosine terms:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big),$$
with $a_n = \frac{2}{L}\int_0^L f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$ for $n \geq 0$.
- **Sine series (odd extension).** Extend f to $[-L, L]$ by defining $f(-x) = -f(x)$. The Fourier series contains only sine terms:
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big),$$
with $b_n = \frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$.

Worked examples.

Worked Example

Find the half-range cosine series for $f(x) = x$ on $[0, \pi]$.
Solution. We extend $f(x) = x$ on $[0, \pi]$ to an even function on $[-\pi, \pi]$. With $L = \pi$:
$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi.$$
For $n \geq 1$:
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,\mathrm{d}x.$$
This is the same integral as in section 14.3:
$$a_n = \frac{2}{\pi}\cdot\frac{1}{n^2}\big((-1)^n - 1\big) = \begin{cases} 0, & n \text{ even},\\ -\dfrac{4}{\pi n^2}, & n \text{ odd}.\end{cases}$$
The half-range cosine series is
$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos\big((2k+1)x\big)}{(2k+1)^2} \qquad \text{for } x \in [0, \pi].$$
Note: this series actually represents the *even* extension $|x|$ on $[-\pi, \pi]$. On $[0, \pi]$, $|x| = x$, so the equality holds.

Worked Example

Find the half-range sine series for $f(x) = 1$ on $[0, \pi]$.
Solution. We extend $f(x) = 1$ on $[0, \pi]$ to an odd function on $[-\pi, \pi]$. With $L = \pi$:
$$b_n = \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\Big[-\frac{1}{n}\cos(nx)\Big]_0^{\pi} = \frac{2}{n\pi}\big(1 - (-1)^n\big).$$
This is nonzero only for odd n:
$$b_n = \begin{cases} \dfrac{4}{n\pi}, & n \text{ odd},\\ 0, & n \text{ even}.\end{cases}$$

The half-range sine series is

$$1 = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\bigl((2k+1)x\bigr)}{2k+1} = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right).$$

This series converges to 1 on $(0,\pi)$ but converges to 0 at $x = 0$ and $x = \pi$ (the endpoints where the odd extension has jump discontinuities). This behavior is explained by the Dirichlet convergence theorem (section 14.6).

14.5 Complex Fourier Series

The real Fourier series equation (59) can be written more compactly using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. The **complex Fourier series** is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L}, \tag{63}$$

where the **complex Fourier coefficients** are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\,\mathrm{d}x. \tag{64}$$

Key Result

Relationship between real and complex coefficients.

$$c_0 = \frac{a_0}{2}, \tag{65}$$

$$c_n = \frac{a_n - ib_n}{2} \quad \text{for } n > 0, \tag{66}$$

$$c_{-n} = \frac{a_n + ib_n}{2} \quad \text{for } n > 0. \tag{67}$$

Equivalently, $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$.
If $f(x)$ is real-valued, then $c_{-n} = \overline{c_n}$ (complex conjugate).

Derivation. Substitute $e^{\pm in\pi x/L} = \cos(n\pi x/L) \pm i\sin(n\pi x/L)$ into equation (63), collect real and imaginary parts, and compare with equation (59). The reverse direction uses $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$.

Worked example.

Worked Example

Find the complex Fourier series of $f(x) = e^{ax}$ on $[-\pi,\pi]$, where $a \in \mathbb{R}$.
Solution. With $L = \pi$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\, e^{-inx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\,\mathrm{d}x.$$

For $n \neq 0$ (or $a \neq 0$):

$$c_n = \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a-in)}\bigl(e^{(a-in)\pi} - e^{-(a-in)\pi}\bigr).$$

Since $e^{-in\pi} = (-1)^n$ and $e^{in\pi} = (-1)^n$:

$$c_n = \frac{(-1)^n}{2\pi(a-in)}\bigl(e^{a\pi} - e^{-a\pi}\bigr) = \frac{(-1)^n\sinh(a\pi)}{\pi(a-in)}.$$

For $n = 0$:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,\mathrm{d}x = \frac{e^{a\pi} - e^{-a\pi}}{2\pi a} = \frac{\sinh(a\pi)}{\pi a}.$$

The complex Fourier series is

$$e^{ax} = \frac{\sinh(a\pi)}{\pi}\left(\frac{1}{a} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{a - in}\, e^{inx}\right).$$

14.6 Convergence and Gibbs Phenomenon

Theorem 14.2 (Dirichlet Convergence Theorem)**.** *Let $f(x)$ be $2L$-periodic and piecewise smooth on $[-L, L]$ (i.e., f and f' are piecewise continuous). Then the Fourier series of f converges at every point x to*

$$S(x) = \frac{f(x^+) + f(x^-)}{2},$$

where $f(x^+) = \lim_{\epsilon\to 0^+} f(x+\epsilon)$ and $f(x^-) = \lim_{\epsilon\to 0^+} f(x-\epsilon)$ are the right-hand and left-hand limits. In particular, at any point where f is continuous, the series converges to $f(x)$.

Implications. If $f(x)$ is continuous on $[-L, L]$, its Fourier series converges pointwise to $f(x)$ everywhere. At a jump discontinuity where the function jumps from $f(x^-)$ to $f(x^+)$, the series converges to the *average* of the two values.

Gibbs phenomenon. Near a jump discontinuity of size $J = |f(x^+) - f(x^-)|$, the partial sums of the Fourier series exhibit a persistent overshoot: even as the number of terms $N \to \infty$, the overshoot approaches approximately

$$0.08949 \cdot J \quad \text{(about 8.95\% of the jump).}$$

This is **not** a numerical artifact; it is an inherent feature of Fourier series at discontinuities. The overshoot does *not* disappear as N increases — it merely becomes more concentrated near the discontinuity.

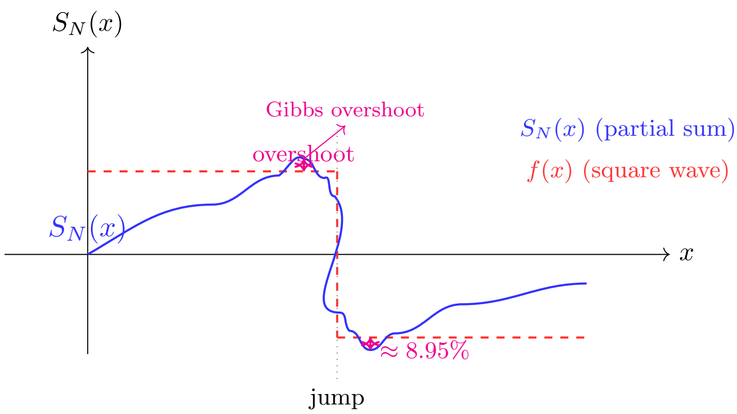


Figure 6: Gibbs phenomenon: the partial sum $S_N(x)$ of a Fourier series overshoots the function value at a jump discontinuity by approximately 8.95% of the jump size, even as $N \to \infty$. The dashed red line is the ideal square wave $f(x)$; the solid blue curve is the partial sum.

Hint

The Gibbs constant. The exact overshoot is $\frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt - 1 \approx 0.08949$ times the jump size. The integral $\int_0^{\pi} \frac{\sin t}{t}\,dt$ is related to the *sinc integral* $\mathrm{Si}(\pi) \approx 1.85194$.

14.7 Parseval's Identity

Theorem 14.3 (Parseval's Identity)**.** *Let $f(x)$ have the Fourier series*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right].$$

Then

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2). \tag{68}$$

In words: the average energy (mean square) of the function equals the sum of the squared magnitudes of its Fourier coefficients.

Proof sketch. Square the Fourier series and integrate over $[-L, L]$. By orthogonality (Theorem 14.1), all cross-terms vanish:

$$\int_{-L}^{L}\cos\Bigl(\frac{n\pi x}{L}\Bigr)\cos\Bigl(\frac{m\pi x}{L}\Bigr)\,\mathrm{d}x = 0 \text{ for } n \neq m,$$

and similarly for the sine–sine and sine–cosine products. The surviving terms are:

$$\int_{-L}^{L}\frac{a_0^2}{4}\,\mathrm{d}x = \frac{a_0^2}{4}\cdot 2L = \frac{a_0^2}{2}\cdot L,$$

$$\int_{-L}^{L}a_n^2\cos^2\Bigl(\frac{n\pi x}{L}\Bigr)\,\mathrm{d}x = a_n^2\cdot L, \qquad \int_{-L}^{L}b_n^2\sin^2\Bigl(\frac{n\pi x}{L}\Bigr)\,\mathrm{d}x = b_n^2\cdot L.$$

Dividing the total by L gives equation (68).

Key Result

Parseval's identity in the complex form. For the complex Fourier coefficients c_n of equation (64):

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2\,\mathrm{d}x = \sum_{n=-\infty}^{\infty}|c_n|^2.$$

Application: computing infinite series.

Worked Example

Use Parseval's identity to evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4}$.

Solution. From section 14.3, the Fourier series of x^2 on $[-\pi, \pi]$ is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

Here $a_0 = \dfrac{2\pi^2}{3}$ and $a_n = \dfrac{4(-1)^n}{n^2}$, with $b_n = 0$. Parseval's identity (equation (68)) with $L = \pi$ gives:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^4\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2.$$

The left side:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}x^4\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi}x^4\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi^5}{5} = \frac{2\pi^4}{5}.$$

The right side:

$$\frac{a_0^2}{2} = \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 = \frac{2\pi^4}{9},$$

$$\sum_{n=1}^{\infty}a_n^2 = \sum_{n=1}^{\infty}\frac{16}{n^4} = 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Equating both sides:

$$\frac{2\pi^4}{5} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Solving:

$$16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} = 2\pi^4\left(\frac{1}{5} - \frac{1}{9}\right) = 2\pi^4\cdot\frac{4}{45} = \frac{8\pi^4}{45}.$$

Therefore:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

This is the famous result $\zeta(4) = \dfrac{\pi^4}{90}$.

14.8 Applications to ODEs

Fourier series provide a powerful method for solving linear ODEs with **periodic forcing**. The basic idea is to expand the periodic forcing function as a Fourier series, then solve the ODE for each harmonic component — exploiting the superposition principle for linear equations.

General method. Consider the forced harmonic oscillator

$$x'' + \omega_0^2 x = f(t), \tag{69}$$

where $f(t)$ is 2π-periodic. Expand $f(t)$ as a Fourier series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(nt) + b_n \sin(nt)\big].$$

By linearity, we can solve the ODE for each term separately:

1. For the constant term $a_0/2$: the particular solution is $x_p = \dfrac{a_0}{2\omega_0^2}$.
2. For each $\cos(nt)$ term: solve $x'' + \omega_0^2 x = a_n \cos(nt)$. If $n \neq \omega_0$, the particular solution is

$$x_n^{(c)}(t) = \frac{a_n}{\omega_0^2 - n^2} \cos(nt).$$

3. For each $\sin(nt)$ term: solve $x'' + \omega_0^2 x = b_n \sin(nt)$. If $n \neq \omega_0$,

$$x_n^{(s)}(t) = \frac{b_n}{\omega_0^2 - n^2} \sin(nt).$$

The full particular solution is the sum of all these mode-by-mode solutions.

Remark 14.4. **Resonance warning.** If $n = \omega_0$ for some harmonic present in the forcing, the denominator $\omega_0^2 - n^2$ vanishes and the method breaks down. This is the phenomenon of **resonance**: the forced response grows without bound (secular growth). In this case, the particular solution takes the form

$$x_p(t) = \frac{a_n}{2\omega_0} t \sin(\omega_0 t) \quad \text{or} \quad \frac{b_n}{2\omega_0} t \cos(\omega_0 t),$$

producing a linearly growing amplitude. This is consistent with the theory of undetermined coefficients developed in section 9.

Worked Example

Solve the initial value problem

$$x'' + 4x = f(t), \qquad x(0) = 0, \quad x'(0) = 0,$$

where $f(t)$ is the 2π-periodic square wave:

$$f(t) = \begin{cases} 1, & 0 < t < \pi, \\ -1, & \pi < t < 2\pi. \end{cases}$$

Solution. From the first example in section 14.1, the Fourier series of this square wave is

$$f(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin\big((2k+1)t\big)}{2k+1}.$$

Let $n_k = 2k+1$. The ODE for the k-th mode is

$$x_k'' + 4x_k = \frac{4}{\pi n_k}\sin(n_k t).$$

Since $\omega_0^2 = 4$, we have $\omega_0 = 2$. Check for resonance: $n_k = 1, 3, 5, \ldots$, and $n_k \neq 2$ for any k, so no resonance occurs. The particular solution for the k-th mode is

$$x_k(t) = \frac{4/(\pi n_k)}{4 - n_k^2}\sin(n_k t) = \frac{4}{\pi n_k(4 - n_k^2)}\sin(n_k t).$$

The complete particular solution is the sum:

$$x_p(t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\big((2k+1)t\big)}{(2k+1)\big(4 - (2k+1)^2\big)}.$$

The general solution is $x(t) = x_h(t) + x_p(t)$, where the homogeneous solution is

$$x_h(t) = C_1\cos(2t) + C_2\sin(2t).$$

Apply initial conditions: $x(0) = 0$ implies $C_1 + x_p(0) = 0$. Since $x_p(0) = 0$ (all sine terms vanish at $t = 0$), we get $C_1 = 0$. For $x'(0) = 0$:

$$x'(t) = 2C_2\cos(2t) + x_p'(t),$$

and $x_p'(0) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{4 - (2k+1)^2}$. Using the closed-form sum, one finds $x_p'(0) = -\frac{1}{2}$. Setting $x'(0) = 2C_2 - \frac{1}{2} = 0$ gives $C_2 = \frac{1}{4}$.
The final solution is

$$x(t) = \frac{1}{4}\sin(2t) + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\big((2k+1)t\big)}{(2k+1)\big(4 - (2k+1)^2\big)}.$$

Hint

Physical intuition for Fourier forcing. Each harmonic component of the forcing drives the system at its own frequency. The system's response at frequency n is amplified by the factor $\frac{1}{|\omega_0^2 - n^2|}$: forcing frequencies far from the natural frequency produce a small response, while forcing frequencies close to ω_0 produce a large response. This frequency-selective amplification is the fundamental principle behind filters in signal processing and is why resonance is both useful (e.g., tuning a radio) and dangerous (e.g., the Tacoma Narrows Bridge collapse).

14.9 Summary

Fourier series transform the study of periodic functions into the algebra of their spectral coefficients. By decomposing a periodic function into sine and cosine harmonics, we gain the ability to:

- Analyze and synthesize periodic signals in engineering and physics.
- Solve linear ODEs and PDEs with periodic forcing or boundary conditions.
- Evaluate infinite series via Parseval's identity.
- Understand the frequency content of complex waveforms.

The Fourier series framework developed here will be directly applied in the study of boundary value problems (section 15), the heat equation (section 16), and the wave and Laplace equations (section 17), where separation of variables naturally produces Fourier expansions.

Table 21: Fourier series: key formulas and concepts

Concept	Key Formula/Method
Fourier series on $[-L, L]$	$f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty}\big[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\big]$
Cosine coefficient a_n	$a_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,\mathrm{d}x$
Sine coefficient b_n	$b_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,\mathrm{d}x$
Even function	$b_n = 0,\ a_n = \dfrac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})\,\mathrm{d}x$
Odd function	$a_0 = a_n = 0,\ b_n = \dfrac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\,\mathrm{d}x$
Complex Fourier series	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \quad c_n = \dfrac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,\mathrm{d}x$
Dirichlet convergence	Series converges to $\dfrac{f(x^+) + f(x^-)}{2}$ at every x
Gibbs overshoot	$\approx 8.95\%$ of jump at discontinuity, persists as $N \to \infty$
Parseval's identity	$\dfrac{1}{L}\int_{-L}^{L} f(x)^2\,\mathrm{d}x = \dfrac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$
Periodic forcing of $x'' + \omega_0^2 x = f(t)$	Solve mode-by-mode: $x_n = \dfrac{a_n}{\omega_0^2 - n^2}\cos(nt) + \dfrac{b_n}{\omega_0^2 - n^2}\sin(nt)$
Resonance condition	If $n = \omega_0$, solution grows linearly: $x_p \propto t\sin(\omega_0 t)$

15 Boundary Value Problems

15.1 BVP vs IVP

In the preceding chapters we have focused almost exclusively on **initial value problems (IVPs)**: a differential equation together with conditions specified at a *single* point. For example,

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(x_0) = y_0,\ y'(x_0) = y_0'.$$

An IVP asks: given the state of a system at one instant, what will it do in the future? This is the natural framework for time-evolution problems.

A **boundary value problem (BVP)**, by contrast, prescribes conditions at *two different points* of the independent variable:

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(a) = \alpha,\ y(b) = \beta,$$

where $a \neq b$. Instead of "initial" conditions at a starting time, we impose **boundary conditions** at the endpoints of an interval. The independent variable x typically represents spatial position rather than time.

When BVPs arise. Boundary value problems appear whenever a physical quantity is constrained at the boundaries of a spatial domain. Classic examples include:

- The displacement $y(x)$ of a string fixed at both ends: $y(0) = 0$, $y(L) = 0$.
- The temperature $u(x)$ in a rod whose ends are held at prescribed temperatures.
- The electric potential between two conducting plates.

In all of these, the differential equation describes the internal physics, while the boundary conditions encode the geometry or external constraints.

Key differences from IVPs. Unlike IVPs, BVPs do *not* always have a solution, and when a solution exists it may not be unique. The three possibilities are:

1. **Unique solution**: exactly one function satisfies both the ODE and the boundary conditions.
2. **No solution**: the boundary conditions are incompatible with any solution of the ODE.

3. **Infinitely many solutions**: the boundary conditions are satisfied by a whole family of solutions. This occurs precisely when the associated **homogeneous BVP** admits nontrivial solutions—a phenomenon we explore in the next subsection.

Worked Example

Solve the BVP $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$.
Solution. The general solution of the ODE is

$$y(x) = c_1 \cos x + c_2 \sin x.$$

Apply the first boundary condition:

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1 = 0,$$

so $c_1 = 0$ and $y(x) = c_2 \sin x$.
Apply the second boundary condition:

$$y(\pi) = c_2 \sin \pi = c_2 \cdot 0 = 0.$$

This condition is satisfied for *any* c_2. Hence we have infinitely many solutions:

$$y(x) = c_2 \sin x, \qquad c_2 \in \mathbb{R}.$$

The homogeneous BVP admits nontrivial solutions; this is the hallmark of an **eigenvalue problem**.

Worked Example

Solve the BVP $y'' + y = 0$, $y(0) = 0$, $y(\pi/2) = 1$.
Solution. Again the general solution is $y(x) = c_1 \cos x + c_2 \sin x$.
Apply $y(0) = 0$: we obtain $c_1 = 0$, so $y(x) = c_2 \sin x$.
Apply $y(\pi/2) = 1$:

$$y(\pi/2) = c_2 \sin(\pi/2) = c_2 \cdot 1 = 1 \quad \Longrightarrow \quad c_2 = 1.$$

The solution is unique: $y(x) = \sin x$.

15.2 Eigenvalue Problems

The phenomenon of infinitely many solutions leads us to one of the most important structures in applied mathematics: the **eigenvalue problem**.

Consider the second-order equation

$$y'' + \lambda y = 0, \qquad 0 < x < L, \tag{70}$$

with Dirichlet boundary conditions

$$y(0) = 0, \qquad y(L) = 0. \tag{71}$$

Here λ is a parameter. For most values of λ, the only solution is the trivial one $y \equiv 0$. But for certain special values of λ—called **eigenvalues**—there exist nontrivial solutions, called **eigenfunctions**.

Case analysis. We treat three cases for the sign of λ.

Case 1: $\lambda < 0$. Let $\lambda = -\mu^2$ with $\mu > 0$. The general solution is

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} = A \cosh(\mu x) + B \sinh(\mu x).$$

Apply $y(0) = 0$: $A = 0$, so $y(x) = B \sinh(\mu x)$.

Apply $y(L) = 0$: $B \sinh(\mu L) = 0$. Since $\mu > 0$ and $L > 0$, we have $\sinh(\mu L) > 0$, so $B = 0$. Hence $y \equiv 0$. There are *no* eigenvalues with $\lambda < 0$.

Case 2: $\lambda = 0$. The equation becomes $y'' = 0$, with general solution

$$y(x) = c_1 x + c_2.$$

Apply $y(0) = 0$: $c_2 = 0$. Apply $y(L) = 0$: $c_1 L = 0$, so $c_1 = 0$. Again $y \equiv 0$. The value $\lambda = 0$ is *not* an eigenvalue.

Case 3: $\lambda > 0$. Let $\lambda = \mu^2$ with $\mu > 0$. The general solution is

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x).$$

Apply $y(0) = 0$: $c_1 = 0$, so $y(x) = c_2 \sin(\mu x)$.

Apply $y(L) = 0$: $c_2 \sin(\mu L) = 0$. For a nontrivial solution we need $c_2 \neq 0$, which requires

$$\sin(\mu L) = 0 \quad \Longrightarrow \quad \mu L = n\pi, \qquad n = 1, 2, 3, \ldots$$

Thus $\mu_n = \dfrac{n\pi}{L}$ and $\lambda_n = \mu_n^2 = \left(\dfrac{n\pi}{L}\right)^2$.

Key Result

Dirichlet eigenvalue problem. For

$$y'' + \lambda y = 0, \qquad y(0) = 0, \;\; y(L) = 0,$$

the eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

Any positive constant multiple of $y_n(x)$ is also an eigenfunction.

Neumann boundary conditions. If instead we impose

$$y'(0) = 0, \qquad y'(L) = 0,$$

the analysis changes slightly. With $\lambda = \mu^2 > 0$, we have $y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$ and $y'(x) = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x)$.

Apply $y'(0) = 0$: $c_2 \mu = 0$, so $c_2 = 0$. Then $y(x) = c_1 \cos(\mu x)$ and $y'(x) = -c_1 \mu \sin(\mu x)$.

Apply $y'(L) = 0$: $-c_1 \mu \sin(\mu L) = 0$. For nontrivial solutions ($c_1 \neq 0$), we need $\sin(\mu L) = 0$, so $\mu L = n\pi$ with $n = 0, 1, 2, \ldots$.

Note that $n = 0$ gives $\mu = 0$, $\lambda_0 = 0$, and $y_0(x) = \cos(0) = 1$ (a nonzero constant). This means $\lambda_0 = 0$ *is* an eigenvalue for Neumann conditions.

Key Result

Neumann eigenvalue problem. For

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \;\; y'(L) = 0,$$

the eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad y_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 0, 1, 2, \ldots$$

Note that $\lambda_0 = 0$ with eigenfunction $y_0(x) = 1$.

Worked Example

Find all eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \qquad y(0) = 0, \;\; y(2) = 0.$$

Solution. Here $L = 2$. From equation (70), the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 = \frac{n^2\pi^2}{4}, \qquad n = 1, 2, 3, \ldots$$

and the eigenfunctions are

$$y_n(x) = \sin\left(\frac{n\pi x}{2}\right).$$

Listing the first few:

$$\lambda_1 = \frac{\pi^2}{4}, \qquad y_1(x) = \sin\left(\frac{\pi x}{2}\right),$$
$$\lambda_2 = \pi^2, \qquad y_2(x) = \sin(\pi x),$$
$$\lambda_3 = \frac{9\pi^2}{4}, \qquad y_3(x) = \sin\left(\frac{3\pi x}{2}\right).$$

Worked Example

Find all eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \;\; y'(3) = 0.$$

Solution. Here $L = 3$ with Neumann boundary conditions. The eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 = \frac{n^2\pi^2}{9}, \qquad n = 0, 1, 2, \ldots$$

and the eigenfunctions are

$$y_n(x) = \cos\left(\frac{n\pi x}{3}\right).$$

The first few are:

$$\lambda_0 = 0, \qquad y_0(x) = 1,$$
$$\lambda_1 = \frac{\pi^2}{9}, \qquad y_1(x) = \cos\left(\frac{\pi x}{3}\right),$$
$$\lambda_2 = \frac{4\pi^2}{9}, \qquad y_2(x) = \cos\left(\frac{2\pi x}{3}\right).$$

Note that $y_0(x) = 1$ corresponds to the constant equilibrium state.

15.3 Sturm–Liouville Form

The eigenvalue problems we just studied are special cases of a broad and powerful framework: the **Sturm–Liouville (SL) problem**. This theory unifies eigenvalue problems, orthogonality, and series expansions.

General SL form. A second-order linear ODE is in **Sturm–Liouville form** when it can be written as

$$\big(p(x)\,y'\big)' + \big(q(x) + \lambda\,r(x)\big)y = 0, \qquad a \le x \le b, \tag{72}$$

where $p(x)$, $q(x)$, and $r(x)$ are given coefficient functions and λ is the eigenvalue parameter. The function $r(x)$ is called the **weight function**.

Converting to SL form. Any equation of the form

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y + \lambda\,S(x)\,y = 0 \tag{73}$$

can be converted to SL form by dividing by $P(x)$ and multiplying by an **integrating factor**. First divide by $P(x)$:

$$y'' + \frac{Q(x)}{P(x)}\,y' + \frac{R(x)}{P(x)}\,y + \lambda\,\frac{S(x)}{P(x)}\,y = 0.$$

Multiply by the integrating factor

$$\mu(x) = \exp\left(\int \frac{Q(x)}{P(x)}\,\mathrm{d}x\right).$$

Then the first two terms combine as a derivative:

$$\mu(x)\,y'' + \mu(x)\,\frac{Q(x)}{P(x)}\,y' = \big(\mu(x)\,y'\big)',$$

which is exactly the $(py')'$ structure of the SL form with $p(x) = \mu(x)$, $q(x) = \mu(x)\,\dfrac{R(x)}{P(x)}$, and $r(x) = \mu(x)\,\dfrac{S(x)}{P(x)}$.

Regular SL problem. We restrict attention to the **regular** case:

Definition 15.1 (Regular Sturm–Liouville Problem)**.** A Sturm–Liouville problem is **regular** on $[a, b]$ if:

1. $p(x)$, $p'(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$.
2. $p(x) > 0$ and $r(x) > 0$ on $[a, b]$.
3. The boundary conditions are of the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0,$$

   where $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$.

Theorem 15.2 (SL Existence Theorem)**.** *A regular Sturm–Liouville problem has:*

1. *An infinite sequence of real eigenvalues*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty.$$

2. *Each eigenvalue λ_n has exactly one corresponding eigenfunction $\phi_n(x)$ (up to a constant multiple).*
3. *The n-th eigenfunction $\phi_n(x)$ has exactly $n - 1$ zeros in the open interval (a, b).*

This theorem guarantees that eigenvalue problems arising from well-behaved physical systems always have a rich structure of eigenvalues and eigenfunctions.

Worked Example

Convert the equation $x\, y'' + 2y' + \lambda\, x\, y = 0$ to Sturm–Liouville form on $[0, 1]$, and identify $p(x)$, $q(x)$, and $r(x)$.
Solution. Here $P(x) = x$, $Q(x) = 2$, $R(x) = 0$, and $S(x) = x$.
Divide by $P(x) = x$:

$$y'' + \frac{2}{x}\, y' + \lambda\, y = 0.$$

The integrating factor is

$$\mu(x) = \exp\left(\int \frac{2}{x}\, \mathrm{d}x \right) = \exp(2 \ln x) = x^2.$$

Multiply the equation by x^2:

$$x^2\, y'' + 2x\, y' + \lambda\, x^2\, y = 0.$$

The first two terms combine as

$$x^2 y'' + 2xy' = (x^2 y')'.$$

Hence the SL form is

$$(x^2 y')' + \lambda\, x^2\, y = 0.$$

Reading off the coefficients:

$$p(x) = x^2, \qquad q(x) = 0, \qquad r(x) = x^2.$$

On $(0, 1]$, we have $p(x) = x^2 > 0$ and $r(x) = x^2 > 0$. Note that $p(0) = 0$, so strictly speaking this is a *singular* SL problem (not regular) at $x = 0$, since the regularity condition requires $p > 0$ on the *closed* interval. Singular SL problems require additional care at the singular endpoint.

15.4 Orthogonality Theorem

The most powerful consequence of the Sturm–Liouville framework is the **orthogonality** of eigenfunctions belonging to distinct eigenvalues.

Theorem 15.3 (SL Orthogonality Theorem)**.** *Let $\phi_n(x)$ and $\phi_m(x)$ be eigenfunctions corresponding to distinct eigenvalues $\lambda_n \neq \lambda_m$ of a regular Sturm–Liouville problem on $[a, b]$. Then ϕ_n and ϕ_m are* ***orthogonal with respect to the weight function*** *$r(x)$:*

$$\int_a^b \phi_n(x)\, \phi_m(x)\, r(x)\, \mathrm{d}x = 0, \qquad n \neq m.$$

Proof. The eigenfunctions satisfy the SL equations

$$\begin{cases} \big(p\,\phi_n'\big)' + \big(q + \lambda_n\, r\big)\phi_n = 0, \\ \big(p\,\phi_m'\big)' + \big(q + \lambda_m\, r\big)\phi_m = 0. \end{cases}$$

Multiply the first equation by $\phi_m(x)$ and the second by $\phi_n(x)$:

$$\begin{cases} \phi_m\,(p\,\phi_n')' + \phi_m\, q\,\phi_n + \lambda_n\,\phi_m\, r\,\phi_n = 0, \\ \phi_n\,(p\,\phi_m')' + \phi_n\, q\,\phi_m + \lambda_m\,\phi_n\, r\,\phi_m = 0. \end{cases}$$

Subtract the second from the first:

$$\phi_m\,(p\,\phi_n')' - \phi_n\,(p\,\phi_m')' + (\lambda_n - \lambda_m)\,\phi_n\,\phi_m\, r = 0.$$

The terms involving $q(x)$ cancel. Rearrange:

$$(\lambda_n - \lambda_m)\,\phi_n(x)\,\phi_m(x)\,r(x) = \phi_n\,(p\,\phi_m')' - \phi_m\,(p\,\phi_n')'.$$

The right-hand side can be written as a total derivative:

$$\phi_n\,(p\,\phi_m')' - \phi_m\,(p\,\phi_n')' = \frac{\mathrm{d}}{\mathrm{d}x}\Big[p\big(\phi_n\,\phi_m' - \phi_m\,\phi_n'\big)\Big].$$

To verify this, differentiate the expression inside the brackets:

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[p(\phi_n\phi_m' - \phi_m\phi_n')\Big] = p'(\phi_n\phi_m' - \phi_m\phi_n') + p(\phi_n'\phi_m' + \phi_n\phi_m'' - \phi_m'\phi_n' - \phi_m\phi_n'') = \phi_n(p\phi_m')' - \phi_m(p\phi_n')',$$

as claimed. Now integrate both sides from a to b:

$$(\lambda_n - \lambda_m)\int_a^b \phi_n(x)\,\phi_m(x)\,r(x)\,\mathrm{d}x = \Big[p(x)\big(\phi_n(x)\,\phi_m'(x) - \phi_m(x)\,\phi_n'(x)\big)\Big]_{x=a}^{x=b}.$$

The right-hand side consists of **boundary terms**. For a regular SL problem, the boundary conditions are of the form

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

Both ϕ_n and ϕ_m satisfy these boundary conditions. It is a standard verification that under any such separated boundary conditions, the boundary expression

$$p\big(\phi_n\phi_m' - \phi_m\phi_n'\big)$$

vanishes at both $x = a$ and $x = b$. (For example, if Dirichlet conditions $y(a) = 0$ apply, then $\phi_n(a) = \phi_m(a) = 0$ makes the expression zero immediately. For mixed conditions, a short algebraic argument shows cancellation.)

Therefore the right-hand side is zero. Since $\lambda_n \neq \lambda_m$, we divide by $(\lambda_n - \lambda_m)$ to obtain

$$\int_a^b \phi_n(x)\,\phi_m(x)\,r(x)\,\mathrm{d}x = 0,$$

completing the proof. □

Orthogonality in action. The orthogonality of $\{\sin(n\pi x/L)\}$ on $[0, L]$ is a direct consequence of Theorem 15.3. For the Dirichlet problem $y'' + \lambda y = 0$ on $[0, L]$, we have $p = 1$, $q = 0$, $r = 1$, and $\phi_n(x) = \sin(n\pi x/L)$. The theorem guarantees

$$\int_0^L \sin\Big(\frac{n\pi x}{L}\Big)\sin\Big(\frac{m\pi x}{L}\Big)\,\mathrm{d}x = 0, \qquad n \neq m,$$

which is precisely the sine orthogonality relation used throughout Fourier analysis (section 14).

Worked Example

Verify the orthogonality of $\sin(\pi x)$ and $\sin(2\pi x)$ on $[0, 1]$ by direct computation.
Solution. We compute

$$I = \int_0^1 \sin(\pi x)\,\sin(2\pi x)\,\mathrm{d}x.$$

Use the product-to-sum identity $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$:

$$\sin(\pi x)\sin(2\pi x) = \frac{1}{2}\big[\cos(-\pi x) - \cos(3\pi x)\big] = \frac{1}{2}\big[\cos(\pi x) - \cos(3\pi x)\big].$$

Integrate:

$$I = \frac{1}{2}\left[\frac{\sin(\pi x)}{\pi} - \frac{\sin(3\pi x)}{3\pi}\right]_0^1 = \frac{1}{2}\left[\frac{0}{\pi} - \frac{0}{3\pi} - 0\right] = 0.$$

The integral vanishes, confirming orthogonality.

Worked Example

Show that the eigenfunctions of $y'' + \lambda y = 0$ with Neumann conditions $y'(0) = 0$, $y'(1) = 0$ on $[0,1]$ are orthogonal with respect to the weight $r(x) = 1$.

Solution. The eigenfunctions are $\phi_n(x) = \cos(n\pi x)$ for $n = 0, 1, 2, \ldots$. We need to verify

$$\int_0^1 \cos(n\pi x)\,\cos(m\pi x)\,\mathrm{d}x = 0, \qquad n \neq m.$$

Use $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$:

$$\cos(n\pi x)\cos(m\pi x) = \frac{1}{2}\big[\cos\big((n-m)\pi x\big) + \cos\big((n+m)\pi x\big)\big].$$

Integrate:

$$\int_0^1 \cos(n\pi x)\cos(m\pi x)\,\mathrm{d}x = \frac{1}{2}\left[\frac{\sin((n-m)\pi x)}{(n-m)\pi} + \frac{\sin((n+m)\pi x)}{(n+m)\pi}\right]_0^1.$$

Since $n \neq m$ are integers, both $(n-m)\pi$ and $(n+m)\pi$ are nonzero multiples of π, so $\sin(k\pi) = 0$ for any integer k. The integral evaluates to 0, confirming orthogonality.

15.5 Eigenfunction Expansions

The orthogonality of eigenfunctions allows us to expand an arbitrary function as a series, much like a Taylor series or a Fourier series.

Key Result

Eigenfunction expansion. Let $\{\phi_n(x)\}_{n=1}^{\infty}$ be the eigenfunctions of a regular SL problem on $[a,b]$ with weight $r(x)$. Any piecewise smooth function $f(x)$ on $[a,b]$ can be expanded as

$$f(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x),$$

where the **expansion coefficients** are

$$c_n = \frac{\displaystyle\int_a^b f(x)\,\phi_n(x)\,r(x)\,\mathrm{d}x}{\displaystyle\int_a^b \phi_n(x)^2\,r(x)\,\mathrm{d}x}. \tag{74}$$

The series converges to $f(x)$ at points of continuity, and to the average $\frac{1}{2}[f(x^+) + f(x^-)]$ at points of discontinuity.

Derivation of the coefficient formula. Multiply the expansion $f(x) = \sum_{k=1}^{\infty} c_k\phi_k(x)$ by $\phi_n(x)\,r(x)$ and integrate over $[a,b]$:

$$\int_a^b f(x)\,\phi_n(x)\,r(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} c_k \int_a^b \phi_k(x)\,\phi_n(x)\,r(x)\,\mathrm{d}x.$$

By orthogonality, all terms in the sum vanish except $k = n$:

$$\int_a^b f(x)\,\phi_n(x)\,r(x)\,\mathrm{d}x = c_n \int_a^b \phi_n(x)^2\,r(x)\,\mathrm{d}x.$$

Solving for c_n gives equation (74). The denominator

$$\|\phi_n\|^2 = \int_a^b \phi_n(x)^2\, r(x)\, \mathrm{d}x$$

is the squared **norm** of the eigenfunction with respect to the weight $r(x)$.

Connection to Fourier series. The Fourier sine series is a special case of eigenfunction expansion. Consider the SL problem with $p = 1$, $q = 0$, $r = 1$ on $[0, L]$ and Dirichlet boundary conditions. The eigenfunctions are $\phi_n(x) = \sin(n\pi x/L)$, and

$$\|\phi_n\|^2 = \int_0^L \sin^2\!\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}.$$

The expansion formula then gives the familiar Fourier sine coefficients:

$$c_n = \frac{2}{L} \int_0^L f(x)\, \sin\!\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Similarly, the Fourier cosine series arises from the Neumann SL problem on $[0, L]$.

Worked Example

Expand $f(x) = x$ on $[0, 1]$ as a series of eigenfunctions of $y'' + \lambda y = 0$ with Dirichlet boundary conditions $y(0) = 0$, $y(1) = 0$.

Solution. The eigenfunctions are $\phi_n(x) = \sin(n\pi x)$ with weight $r(x) = 1$. We need

$$c_n = \frac{\displaystyle\int_0^1 x\, \sin(n\pi x)\, \mathrm{d}x}{\displaystyle\int_0^1 \sin^2(n\pi x)\, \mathrm{d}x}.$$

The denominator is

$$\int_0^1 \sin^2(n\pi x)\, \mathrm{d}x = \frac{1}{2}.$$

For the numerator, use integration by parts with $u = x$ and $\mathrm{d}v = \sin(n\pi x)\, \mathrm{d}x$:

$$\int_0^1 x\, \sin(n\pi x)\, \mathrm{d}x = \left[-\frac{x}{n\pi}\cos(n\pi x)\right]_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\, \mathrm{d}x.$$

The boundary term gives

$$-\frac{1}{n\pi}\cos(n\pi) + 0 = -\frac{(-1)^n}{n\pi} = \frac{(-1)^{n+1}}{n\pi}.$$

The remaining integral is

$$\frac{1}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 = 0.$$

So the numerator is $\dfrac{(-1)^{n+1}}{n\pi}$.

Therefore

$$c_n = \frac{(-1)^{n+1}/(n\pi)}{1/2} = \frac{2(-1)^{n+1}}{n\pi}.$$

The eigenfunction expansion is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x), \qquad 0 < x < 1.$$

Writing out the first few terms:

$$x \approx \frac{2}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{2}{3\pi}\sin(3\pi x) - \cdots$$

Worked Example

Expand $f(x) = 1$ on $[0, 1]$ as a series of eigenfunctions of $y'' + \lambda y = 0$ with Neumann boundary conditions $y'(0) = 0$, $y'(1) = 0$.
Solution. The eigenfunctions are $\phi_n(x) = \cos(n\pi x)$ for $n = 0, 1, 2, \ldots$, with weight $r(x) = 1$.
For $n = 0$: $\phi_0(x) = 1$. The norm is

$$\|\phi_0\|^2 = \int_0^1 1^2 \,dx = 1.$$

The coefficient is

$$c_0 = \frac{\int_0^1 1 \cdot 1 \,dx}{1} = 1.$$

For $n \geq 1$:

$$c_n = \frac{\displaystyle\int_0^1 1 \cdot \cos(n\pi x)\,dx}{\displaystyle\int_0^1 \cos^2(n\pi x)\,dx} = \frac{\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1}{1/2} = \frac{0}{1/2} = 0.$$

The expansion is simply

$$1 = 1,$$

i.e., only the $n = 0$ term survives. This makes sense: the constant function is itself the $n = 0$ eigenfunction.
As a more instructive exercise, expand $f(x) = x$ with Neumann conditions. Then for $n = 0$:

$$c_0 = \frac{\int_0^1 x \cdot 1 \,dx}{1} = \frac{1}{2}.$$

For $n \geq 1$:

$$c_n = \frac{\int_0^1 x\,\cos(n\pi x)\,dx}{1/2}.$$

Integrate by parts with $u = x$, $dv = \cos(n\pi x)\,dx$:

$$\int_0^1 x\cos(n\pi x)\,dx = \left[\frac{x}{n\pi}\sin(n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx.$$

The boundary term vanishes ($\sin(n\pi) = 0$). The remaining integral is

$$-\frac{1}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1 = \frac{1}{(n\pi)^2}\big(\cos(n\pi) - \cos 0\big) = \frac{(-1)^n - 1}{(n\pi)^2}.$$

This is zero for even n and $-\dfrac{2}{(n\pi)^2}$ for odd n. Therefore

$$c_n = \begin{cases} 0, & n \text{ even}, \\ -\dfrac{4}{(n\pi)^2}, & n \text{ odd}. \end{cases}$$

The expansion is

$$x = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi x) - \frac{4}{9\pi^2}\cos(3\pi x) - \frac{4}{25\pi^2}\cos(5\pi x) - \cdots$$

This converges to x on $[0, 1]$.

15.6 Applications

Eigenfunction expansions provide a powerful method for solving **nonhomogeneous** boundary value problems.

The nonhomogeneous SL problem. Consider

$$\big(p(x)\,y'\big)' + q(x)\,y + \lambda\,r(x)\,y = f(x), \qquad a < x < b, \tag{75}$$

subject to homogeneous boundary conditions at $x = a$ and $x = b$. Suppose we already know the eigenfunctions $\{\phi_n(x)\}$ and eigenvalues $\{\lambda_n\}$ of the associated homogeneous problem

$$\big(p(x)\,\phi_n'\big)' + q(x)\,\phi_n + \lambda_n\,r(x)\,\phi_n = 0.$$

Eigenfunction expansion method. Expand both the solution $y(x)$ and the source term $f(x)$ in the eigenfunction basis:

$$y(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x), \qquad f(x) = \sum_{n=1}^{\infty} f_n\,\phi_n(x).$$

Substitute the series for $y(x)$ into equation (75):

$$\sum_{n=1}^{\infty} c_n\left[\big(p\,\phi_n'\big)' + q\,\phi_n + \lambda\,r\,\phi_n\right] = f(x).$$

Using the homogeneous eigenvalue equation $\big(p\,\phi_n'\big)' + q\,\phi_n = -\lambda_n\,r\,\phi_n$:

$$\sum_{n=1}^{\infty} c_n\left(-\lambda_n\,r\,\phi_n + \lambda\,r\,\phi_n\right) = f(x).$$

$$\sum_{n=1}^{\infty} c_n\,(\lambda - \lambda_n)\,\phi_n(x)\,r(x) = f(x).$$

Multiply by $\phi_m(x)$ and integrate, using orthogonality:

$$c_m\,(\lambda - \lambda_m)\,\|\phi_m\|^2 = f_m.$$

Therefore, provided $\lambda \neq \lambda_n$ for any n:

$$c_n = \frac{f_n}{(\lambda - \lambda_n)\,\|\phi_n\|^2} = \frac{\displaystyle\int_a^b f(x)\,\phi_n(x)\,r(x)\,\mathrm{d}x}{\displaystyle(\lambda - \lambda_n)\int_a^b \phi_n(x)^2\,r(x)\,\mathrm{d}x}. \tag{76}$$

Resonance. If $\lambda = \lambda_n$ for some n and the corresponding $f_n \neq 0$, there is no solution—this is the phenomenon of **resonance**. If $f_n = 0$ for that particular n, the coefficient c_n is undetermined (infinite solutions), as discussed in the context of homogeneous BVPs.

Worked Example

Solve the nonhomogeneous BVP

$$y'' + \pi^2\,y = x, \qquad 0 < x < 1, \qquad y(0) = 0, \;\; y(1) = 0.$$

Solution. The associated homogeneous problem $y'' + \lambda y = 0$ with Dirichlet BCs has eigenfunctions $\phi_n(x) = \sin(n\pi x)$ and eigenvalues $\lambda_n = n^2\pi^2$.

Here $\lambda = \pi^2 = \lambda_1$—we are at the first eigenvalue. This raises the question of whether a solution exists. We must check the solvability condition.

Expand the source term $f(x) = x$ in the eigenfunction basis. From the previous worked example:

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\,\sin(n\pi x).$$

So $f_n = \dfrac{2(-1)^{n+1}}{n\pi}$.

For $n = 1$: $\lambda = \lambda_1 = \pi^2$, so the denominator $(\lambda - \lambda_1) = 0$. The coefficient $f_1 = \dfrac{2(-1)^2}{\pi} = \dfrac{2}{\pi} \neq 0$.

Since $f_1 \neq 0$ and $\lambda = \lambda_1$, the BVP has **no solution**. The source term $f(x) = x$ has a component along the first eigenfunction, and at resonance the system cannot respond.

To see this directly: suppose a solution $y(x)$ exists. Multiply the ODE by $\phi_1(x) = \sin(\pi x)$ and integrate:

$$\int_0^1 (y'' + \pi^2 y)\sin(\pi x)\,\mathrm{d}x = \int_0^1 x\sin(\pi x)\,\mathrm{d}x.$$

The right-hand side is $\frac{2}{\pi} \neq 0$. But integrating the left side by parts twice and using the boundary conditions yields zero (this is the Fredholm alternative). Contradiction: no solution exists.

Worked Example

Solve the nonhomogeneous BVP

$$y'' + 2\pi^2\, y = \sin(\pi x), \qquad 0 < x < 1, \qquad y(0) = 0, \;\; y(1) = 0.$$

Solution. Eigenvalues: $\lambda_n = n^2\pi^2$. Here $\lambda = 2\pi^2$, which is *not* an eigenvalue ($\sqrt{2}\pi$ is not a multiple of π). Expand $f(x) = \sin(\pi x)$ in eigenfunctions. Since $\sin(\pi x) = \phi_1(x)$, the expansion is simply

$$f(x) = \sin(\pi x) = 1 \cdot \phi_1(x).$$

So $f_1 = \|\phi_1\|^2 = \frac{1}{2}$ and $f_n = 0$ for $n \neq 1$.
The coefficients are

$$c_n = \frac{f_n}{(\lambda - \lambda_n)\, \|\phi_n\|^2}.$$

For $n = 1$:

$$c_1 = \frac{1/2}{(2\pi^2 - \pi^2) \cdot (1/2)} = \frac{1}{\pi^2}.$$

For $n \neq 1$, $f_n = 0$ so $c_n = 0$.
The solution is

$$y(x) = \frac{1}{\pi^2}\, \sin(\pi x).$$

Verification. Compute $y'' = -\sin(\pi x)$. Then

$$y'' + 2\pi^2 y = -\sin(\pi x) + 2\pi^2 \cdot \frac{1}{\pi^2} \sin(\pi x) = -\sin(\pi x) + 2\sin(\pi x) = \sin(\pi x).$$

✓ The solution satisfies the ODE and the boundary conditions.

15.7 Summary

Table 22: Eigenvalue problems and Sturm–Liouville theory

Concept	**Key formula/result**
BVP vs IVP	BVP: conditions at $x = a$ and $x = b$; IVP: conditions at one point
Dirichlet eigenvalues	$\lambda_n = (n\pi/L)^2$, $y_n(x) = \sin(n\pi x/L)$, $n = 1, 2, \ldots$
Neumann eigenvalues	$\lambda_n = (n\pi/L)^2$, $y_n(x) = \cos(n\pi x/L)$, $n = 0, 1, 2, \ldots$
SL form	$(py')' + (q + \lambda r)y = 0$
Integrating factor	$\mu(x) = \exp\left(\int \frac{Q(x)}{P(x)}\, \mathrm{d}x\right)$
SL existence	Infinite real eigenvalues $\lambda_1 < \lambda_2 < \cdots \to \infty$
Orthogonality	$\int_a^b \phi_n \phi_m\, r\, \mathrm{d}x = 0$ for $n \neq m$
Expansion coefficients	$c_n = \dfrac{\int_a^b f\phi_n\, r\, \mathrm{d}x}{\int_a^b \phi_n^2\, r\, \mathrm{d}x}$
Nonhomogeneous solution	$c_n = \dfrac{f_n}{(\lambda - \lambda_n)\, \|\phi_n\|^2}$ (if $\lambda \neq \lambda_n$)
Resonance	If $\lambda = \lambda_n$ and $f_n \neq 0$, no solution exists

Hint

Problem-solving checklist for BVPs.

1. Identify the boundary conditions and classify (Dirichlet, Neumann, mixed).
2. Determine the eigenvalues and eigenfunctions of the associated homogeneous problem.

3. Verify orthogonality of eigenfunctions with respect to the weight $r(x)$.
4. For nonhomogeneous problems, expand $f(x)$ in the eigenfunction basis.
5. Check whether λ coincides with any eigenvalue (resonance).
6. If $\lambda \neq \lambda_n$ for all n, compute coefficients using equation (76).
7. If $\lambda = \lambda_n$, check the solvability condition ($f_n = 0$).

16 Heat Equation

16.1 Physical Derivation

The heat equation is the prototypical parabolic partial differential equation (PDE). It describes how temperature diffuses through a material over time, and it serves as a mathematical model for many other diffusion processes — from the spread of a pollutant in a river to the flow of electrical charge through a semiconductor.

Fourier's law of heat conduction. In 1822, Joseph Fourier established the fundamental law governing heat conduction. Consider a thin rod aligned along the x-axis. Let $u(x,t)$ denote the temperature at position x and time t. Fourier observed that heat flows from hot regions to cold regions, and that the **heat flux** J (amount of heat energy flowing per unit area per unit time) is proportional to the temperature gradient:

$$J(x,t) = -\kappa \frac{\partial u}{\partial x}(x,t). \tag{77}$$

Here $\kappa > 0$ is the **thermal conductivity** of the material (units: W/(m·K)). The minus sign is essential: heat flows in the direction of decreasing temperature, i.e., opposite to the temperature gradient.

Energy conservation in a rod element. Now consider a small segment of the rod from x to $x + \Delta x$. The amount of heat energy E contained in this segment is

$$E = \rho\, c\, A\, \Delta x \cdot u(x,t),$$

where ρ is the mass density of the material, c is the specific heat capacity (energy per unit mass per degree), and A is the cross-sectional area of the rod.

The rate of change of the energy in this segment must equal the net heat flux into the segment (energy conservation):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = A\big[J(x,t) - J(x+\Delta x,t)\big].$$

Substituting the expressions:

$$\rho\, c\, A\, \Delta x\, \frac{\partial u}{\partial t}(x,t) = A\big[J(x,t) - J(x+\Delta x,t)\big].$$

Divide by $A\,\Delta x$:

$$\rho\, c\, \frac{\partial u}{\partial t}(x,t) = -\frac{J(x+\Delta x,t) - J(x,t)}{\Delta x}.$$

Taking the limit $\Delta x \to 0$, the right side becomes $-\frac{\partial J}{\partial x}$:

$$\rho\, c\, \frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}.$$

Derivation of the heat equation. Substitute Fourier's law equation (77) into the energy conservation equation:

$$\rho\, c\, \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big(-\kappa \frac{\partial u}{\partial x}\Big) = \kappa \frac{\partial^2 u}{\partial x^2}.$$

Assuming the material is homogeneous (κ, ρ, and c are constant), we divide by $\rho\, c$ to obtain the **one-dimensional heat equation**:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0, \tag{78}$$

where

$$\alpha = \frac{\kappa}{\rho\, c} \tag{79}$$

is the **thermal diffusivity** (units: m^2/s).

Key Result

Physical interpretation of α. The thermal diffusivity α governs the rate at which temperature disturbances propagate through a material. A large α means heat diffuses quickly (the material is a good conductor relative to its heat capacity). A small α means temperature changes propagate slowly. In the units of the heat equation, the characteristic diffusion time across a distance L is $t_{\text{diff}} \sim L^2/\alpha$.

Higher dimensions. In three dimensions, Fourier's law becomes $\boldsymbol{J} = -\kappa \nabla u$ and energy conservation gives

$$\frac{\partial u}{\partial t} = \alpha\, \nabla^2 u = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

Throughout this chapter, we focus on the one-dimensional case equation (78), which captures the essential mathematical structure.

16.2 Separation of Variables

We now solve the heat equation equation (78) on a finite rod $[0, L]$ with homogeneous boundary conditions. The primary method is **separation of variables**.

The method. We seek solutions of the form

$$u(x,t) = X(x)\, T(t),$$

where $X(x)$ depends only on space and $T(t)$ depends only on time. Substitute this ansatz into equation (78):

$$X(x)\, T'(t) = \alpha\, X''(x)\, T(t).$$

Assuming neither factor vanishes identically, divide by $\alpha\, X(x)\, T(t)$:

$$\frac{T'(t)}{\alpha\, T(t)} = \frac{X''(x)}{X(x)}. \tag{80}$$

The left side depends only on t, while the right side depends only on x. For this equality to hold for all x and t, both sides must equal the same **separation constant**, which we denote by $-\lambda$.

Theorem 16.1 (Separation of Variables for the Heat Equation)**.** *Assuming $u(x,t) = X(x)T(t)$, the heat equation equation (78) separates into two ordinary differential equations:*

$$T'(t) + \alpha\lambda\, T(t) = 0, \tag{81}$$

$$X''(x) + \lambda\, X(x) = 0, \tag{82}$$

where λ is the separation constant. The choice $\lambda > 0$ is required by the homogeneous boundary conditions and the physical requirement of decay.

Justification for the negative sign. Why do we write the separation constant as $-\lambda$ rather than $+\lambda$? There are three complementary reasons:

1. **Physical reasoning:** Temperature disturbances should decay over time, not grow. If we used $+\lambda > 0$, the time equation $T' = \alpha\lambda T$ would yield $T(t) = e^{\alpha\lambda t}$, an exponentially growing solution, which contradicts the second law of thermodynamics.

2. **Boundary conditions:** With homogeneous Dirichlet conditions $X(0) = 0$ and $X(L) = 0$, the spatial equation $X'' + \lambda X = 0$ admits nontrivial solutions only for $\lambda > 0$ (as established in the eigenvalue analysis of section 15). If $\lambda \leq 0$, only the trivial solution $X \equiv 0$ satisfies both boundary conditions.

3. **Consistency:** Using $-\lambda$ gives the time equation $T' = -\alpha\lambda T$, yielding $T(t) = e^{-\alpha\lambda t}$, which decays for $\lambda > 0$.

Solving the separated equations. The time equation equation (81) is a simple first-order linear ODE:

$$T'(t) = -\alpha\lambda\, T(t) \quad \Longrightarrow \quad T(t) = A\, e^{-\alpha\lambda t},$$

where A is an arbitrary constant.

The space equation equation (82) is exactly the eigenvalue problem studied in section 15.2. The specific eigenvalues and eigenfunctions depend on the boundary conditions, as we develop in the next subsections.

16.3 Dirichlet Boundary Conditions

Consider a rod of length L whose ends are held at zero temperature:

$$u(0,t) = 0, \qquad u(L,t) = 0, \qquad t > 0.$$

These are **homogeneous Dirichlet boundary conditions**. Together with an initial temperature distribution

$$u(x,0) = f(x), \qquad 0 < x < L,$$

we have the initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, & 0 < x < L, \;\; t > 0, \\ u(0,t) = 0, \;\; u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < L. \end{cases} \tag{83}$$

Eigenvalue problem for $X(x)$. The boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ imply $X(0) = 0$ and $X(L) = 0$. The spatial ODE is

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \;\; X(L) = 0.$$

From section 15.2, the eigenvalues and eigenfunctions are:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

Product solutions. For each eigenvalue λ_n, the corresponding time factor is

$$T_n(t) = e^{-\alpha\lambda_n t} = \exp\left[-\alpha \left(\frac{n\pi}{L}\right)^2 t\right].$$

The product solutions are

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\alpha \left(\frac{n\pi}{L}\right)^2 t\right], \qquad n = 1, 2, 3, \ldots$$

Each $u_n(x,t)$ satisfies the PDE and the homogeneous boundary conditions.

General solution. By linearity, any linear combination of product solutions is also a solution. We form the **infinite series** (Fourier sine series in space):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\alpha \left(\frac{n\pi}{L}\right)^2 t\right]. \tag{84}$$

This series satisfies the PDE and boundary conditions for any choice of coefficients $\{b_n\}$. To determine the coefficients, we apply the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

This is a Fourier sine series for $f(x)$ on $[0, L]$. Using the orthogonality of the sine functions (section 14.2), the coefficients are

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx.$$

Key Result

Heat equation with homogeneous Dirichlet BCs. For the problem equation (83), the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha(n\pi/L)^2 t},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\!\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Physical interpretation of the series solution. Each term in the series corresponds to a **mode** of the temperature distribution. The $n = 1$ mode (the fundamental mode) has the lowest decay rate and dominates the long-time behavior:

$$u(x,t) \sim b_1\,\sin\!\left(\frac{\pi x}{L}\right)\,e^{-\alpha(\pi/L)^2 t} \quad \text{as } t \to \infty.$$

Higher modes ($n \geq 2$) decay much faster because their decay rates scale as n^2. After sufficient time, the temperature profile approaches the shape of the fundamental mode.

Worked examples.

Worked Example

Solve the heat equation on a rod of length $L = \pi$ with zero-temperature ends and initial temperature $f(x) = \sin(2x)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha\,\frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi,\quad t > 0, \\ u(0,t) = 0,\quad u(\pi,t) = 0, & t > 0, \\ u(x,0) = \sin(2x), & 0 < x < \pi. \end{cases}$$

Solution. Here $L = \pi$, so the eigenvalues are $\lambda_n = n^2$ and the eigenfunctions are $X_n(x) = \sin(nx)$. The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n\,\sin(nx)\,e^{-\alpha n^2 t}.$$

Apply the initial condition:

$$u(x,0) = \sum_{n=1}^{\infty} b_n\,\sin(nx) = \sin(2x).$$

By orthogonality, $b_n = 0$ for $n \neq 2$ and $b_2 = 1$. Alternatively, compute explicitly:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(2x)\,\sin(nx)\,\mathrm{d}x.$$

For $n \neq 2$, the integral vanishes by the orthogonality of sines (Theorem 14.1). For $n = 2$:

$$b_2 = \frac{2}{\pi}\int_0^{\pi} \sin^2(2x)\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

The solution is

$$u(x,t) = e^{-4\alpha t}\,\sin(2x).$$

This is a single-mode solution: the temperature profile retains its shape (a half-wave of a sine) and simply decays exponentially in amplitude. The decay rate 4α corresponds to the second mode.

Worked Example

Solve the heat equation on a rod of length $L = \pi$ with zero-temperature ends and a triangular initial temperature distribution:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha\,\frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi,\quad t > 0, \\ u(0,t) = 0,\quad u(\pi,t) = 0, & t > 0, \\ u(x,0) = x(\pi - x), & 0 < x < \pi. \end{cases}$$

Solution. With $L = \pi$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n\,\sin(nx)\,e^{-\alpha n^2 t},$$

where

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\,\sin(nx)\,\mathrm{d}x.$$

Expand the integrand:

$$b_n = \frac{2}{\pi}\left[\pi\int_0^\pi x\sin(nx)\,\mathrm{d}x - \int_0^\pi x^2\sin(nx)\,\mathrm{d}x\right].$$

First integral. Integrate by parts with $u = x$, $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$:

$$\int_0^\pi x\sin(nx)\,\mathrm{d}x = \Big[-\frac{x}{n}\cos(nx)\Big]_0^\pi + \frac{1}{n}\int_0^\pi \cos(nx)\,\mathrm{d}x = -\frac{\pi}{n}(-1)^n + 0 = \frac{\pi}{n}(-1)^{n+1}.$$

So the first contribution is

$$\pi\cdot\frac{\pi}{n}(-1)^{n+1} = \frac{\pi^2}{n}(-1)^{n+1}.$$

Second integral. Integrate by parts with $u = x^2$, $\mathrm{d}v = \sin(nx)\,\mathrm{d}x$:

$$\int_0^\pi x^2\sin(nx)\,\mathrm{d}x = \Big[-\frac{x^2}{n}\cos(nx)\Big]_0^\pi + \frac{2}{n}\int_0^\pi x\cos(nx)\,\mathrm{d}x.$$

The boundary term gives $-\frac{\pi^2}{n}(-1)^n = \frac{\pi^2}{n}(-1)^{n+1}$. For the remaining integral, use integration by parts again with $u = x$, $\mathrm{d}v = \cos(nx)\,\mathrm{d}x$:

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = \Big[\frac{x}{n}\sin(nx)\Big]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)\,\mathrm{d}x = 0 - \frac{1}{n}\Big[-\frac{1}{n}\cos(nx)\Big]_0^\pi = \frac{1}{n^2}\big((-1)^n - 1\big).$$

So the second integral is

$$\int_0^\pi x^2\sin(nx)\,\mathrm{d}x = \frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n}\cdot\frac{(-1)^n - 1}{n^2} = \frac{\pi^2}{n}(-1)^{n+1} + \frac{2}{n^3}\big((-1)^n - 1\big).$$

Combine.

$$b_n = \frac{2}{\pi}\left[\frac{\pi^2}{n}(-1)^{n+1} - \frac{\pi^2}{n}(-1)^{n+1} - \frac{2}{n^3}\big((-1)^n - 1\big)\right] = \frac{2}{\pi}\cdot\frac{-2}{n^3}\big((-1)^n - 1\big) = \frac{4}{\pi n^3}\big(1 - (-1)^n\big).$$

Since $1 - (-1)^n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$:

$$b_n = \begin{cases} \dfrac{8}{\pi n^3}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

The solution is

$$u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin\big((2k+1)x\big)}{(2k+1)^3}\exp\big[-\alpha(2k+1)^2 t\big].$$

Writing out the first few terms:

$$u(x,t) = \frac{8}{\pi}\Big[\sin(x)\,e^{-\alpha t} + \frac{\sin(3x)}{27}\,e^{-9\alpha t} + \frac{\sin(5x)}{125}\,e^{-25\alpha t} + \cdots\Big].$$

Notice the rapid decay of higher modes: the $n = 3$ term decays $9\times$ faster than the fundamental, and the $n = 5$ term decays $25\times$ faster.

16.4 Neumann Boundary Conditions

Now consider a rod whose ends are **insulated**, meaning no heat can flow through the endpoints. The boundary conditions are

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0, \qquad t > 0.$$

These are **homogeneous Neumann boundary conditions**. Physically, $\frac{\partial u}{\partial x} = 0$ at an endpoint means the temperature gradient vanishes there, so there is no heat flux ($J = -\kappa\,\frac{\partial u}{\partial x} = 0$).

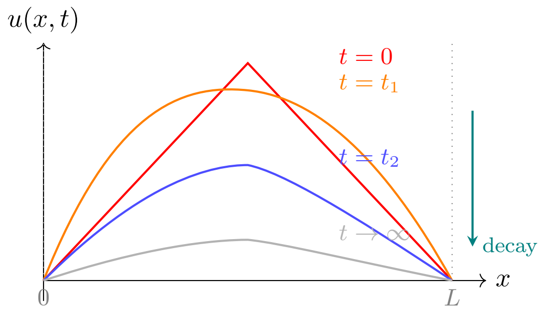


Figure 7: Temperature profile evolution for the Dirichlet heat equation. The initial triangular distribution (red) smooths out over time as higher modes decay faster. Eventually all temperature dissipates through the zero-temperature boundaries.

Eigenvalue analysis. The spatial ODE with Neumann conditions is

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \;\; X'(L) = 0.$$

As derived in section 15.2, the eigenvalues and eigenfunctions are:

$$\lambda_0 = 0, \;\; X_0(x) = 1, \qquad \text{and} \qquad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \;\; X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

The $\lambda = 0$ mode. The eigenvalue $\lambda_0 = 0$ deserves special attention. The time equation for this mode is

$$T_0'(t) + \alpha \cdot 0 \cdot T_0(t) = 0 \implies T_0'(t) = 0 \implies T_0(t) = \text{constant}.$$

This means the $n = 0$ mode is a **steady-state component** that does not decay. Physically, it represents the average temperature of the rod, which is conserved because no heat can escape through the insulated ends.

General solution.

$$u(x,t) = b_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right) \exp\left[-\alpha \left(\frac{n\pi}{L}\right)^2 t\right]. \tag{85}$$

Key Result

Heat equation with homogeneous Neumann BCs. For the problem

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad u_x(0,t) = 0, \; u_x(L,t) = 0, \qquad u(x,0) = f(x),$$

the solution is equation (85) with

$$b_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \qquad b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n \ge 1.$$

As $t \to \infty$, the exponentially decaying terms vanish and

$$\lim_{t\to\infty} u(x,t) = b_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x,$$

i.e., the temperature approaches the **average** of the initial distribution.

Physical interpretation. With insulated ends, the total heat energy in the rod is conserved. The temperature distribution smooths out as higher-frequency modes decay, but the overall average temperature remains fixed. In the long run, the rod reaches a uniform temperature equal to the initial average.

Worked example.

Worked Example

Solve the heat equation on a rod of length $L = \pi$ with insulated ends and initial temperature $f(x) = \cos(2x) + 3$:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \quad t > 0, \\ \frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi,t) = 0, & t > 0, \\ u(x,0) = \cos(2x) + 3, & 0 < x < \pi. \end{cases}$$

Solution. With $L = \pi$, the solution has the form

$$u(x,t) = b_0 + \sum_{n=1}^{\infty} b_n \cos(nx)\, e^{-\alpha n^2 t}.$$

Compute the coefficients. For $n = 0$:

$$b_0 = \frac{1}{\pi}\int_0^{\pi} \big(\cos(2x) + 3\big)\,\mathrm{d}x = \frac{1}{\pi}\left[\frac{\sin(2x)}{2} + 3x\right]_0^{\pi} = \frac{1}{\pi}\big(0 + 3\pi\big) = 3.$$

For $n \geq 1$:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \big(\cos(2x) + 3\big)\,\cos(nx)\,\mathrm{d}x.$$

Split the integral:

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(2x)\cos(nx)\,\mathrm{d}x + \frac{2}{\pi}\int_0^{\pi} 3\cos(nx)\,\mathrm{d}x.$$

The second integral vanishes: $\int_0^{\pi} \cos(nx)\,\mathrm{d}x = 0$ for $n \geq 1$. For the first integral, orthogonality of cosines gives zero for $n \neq 2$ and:

$$b_2 = \frac{2}{\pi}\int_0^{\pi} \cos^2(2x)\,\mathrm{d}x = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1.$$

The solution is

$$u(x,t) = 3 + e^{-4\alpha t}\cos(2x).$$

Check. At $t = 0$: $u(x,0) = 3 + \cos(2x)$. ✓

As $t \to \infty$: $u(x,t) \to 3$, which equals $b_0 = \frac{1}{\pi}\int_0^{\pi}(\cos(2x) + 3)\,\mathrm{d}x = 3$. The constant background temperature 3 is preserved, while the spatially varying $\cos(2x)$ component decays.

16.5 Steady-State Solution

A **steady-state solution** is a solution that does not change in time: $\frac{\partial u}{\partial t} = 0$. In the context of the heat equation, this represents the temperature distribution that the system approaches after an infinite amount of time (if such a limit exists).

Derivation. Setting $\frac{\partial u}{\partial t} = 0$ in equation (78) gives

$$0 = \alpha\frac{\partial^2 u}{\partial x^2} \quad \Longrightarrow \quad \frac{\partial^2 u}{\partial x^2} = 0.$$

Integrating twice:

$$u_{\mathrm{ss}}(x) = A\,x + B, \tag{86}$$

where A and B are constants determined by the boundary conditions.

Dirichlet BCs with nonzero temperatures. Suppose the ends of the rod are held at fixed (possibly nonzero) temperatures:

$$u(0,t) = T_1, \qquad u(L,t) = T_2.$$

The steady-state solution satisfies these boundary conditions:

$$u_{\mathrm{ss}}(0) = B = T_1, \qquad u_{\mathrm{ss}}(L) = A\,L + T_1 = T_2.$$

Solving for A:

$$A = \frac{T_2 - T_1}{L}.$$

Therefore:

$$u_{\text{ss}}(x) = T_1 + \frac{T_2 - T_1}{L}\,x. \tag{87}$$

Key Result

Linear steady-state profile. For the heat equation with fixed end temperatures $u(0) = T_1$ and $u(L) = T_2$, the steady-state temperature distribution is a **linear function**:

$$u_{\text{ss}}(x) = T_1 + \frac{T_2 - T_1}{L}\,x.$$

This represents a uniform temperature gradient from one end to the other. Heat flows from the hotter end to the colder end at a constant rate.

Physical interpretation. The linear steady state reflects a balance between the fixed boundary temperatures and the diffusion process. Once the gradient is established, heat flows at a constant rate through the rod (constant flux $J = -\kappa(T_2 - T_1)/L$), and the temperature profile no longer changes.

Relation to Neumann BCs. For insulated ends ($u_x(0) = 0$, $u_x(L) = 0$), the steady state satisfies $A = 0$, giving $u_{\text{ss}}(x) = B$. The constant B equals the average of the initial temperature (as discussed in section 16.4).

16.6 Nonhomogeneous Boundary Conditions

When the boundary conditions are nonhomogeneous — for example, one end held at a nonzero constant temperature — we cannot directly apply the separation of variables method, which requires homogeneous BCs. The standard approach is a **shifting technique**: we decompose the solution into a steady-state part that satisfies the nonhomogeneous boundary conditions and a transient part that satisfies homogeneous boundary conditions.

The shifting technique. Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha\,\frac{\partial^2 u}{\partial x^2}, & 0 < x < L,\ \ t > 0, \\ u(0,t) = T_1,\ \ u(L,t) = T_2, & t > 0, \\ u(x,0) = f(x), & 0 < x < L. \end{cases}$$

We seek a solution of the form

$$u(x,t) = v(x,t) + \phi(x),$$

where $\phi(x)$ is a time-independent function chosen to satisfy the boundary conditions:

$$\phi(0) = T_1, \qquad \phi(L) = T_2.$$

The simplest choice is the linear steady-state profile equation (87):

$$\phi(x) = T_1 + \frac{T_2 - T_1}{L}\,x.$$

Substituting $u = v + \phi$ into the heat equation:

$$\frac{\partial v}{\partial t} = \alpha\,\frac{\partial^2 v}{\partial x^2} + \alpha\,\phi''(x).$$

Since $\phi(x)$ is linear, $\phi''(x) = 0$, so v satisfies the same homogeneous heat equation:

$$\frac{\partial v}{\partial t} = \alpha\,\frac{\partial^2 v}{\partial x^2}.$$

The boundary conditions for v are homogeneous:

$$v(0,t) = u(0,t) - \phi(0) = T_1 - T_1 = 0, \qquad v(L,t) = u(L,t) - \phi(L) = T_2 - T_2 = 0.$$

The initial condition for v is

$$v(x,0) = u(x,0) - \phi(x) = f(x) - \phi(x).$$

Complete method.

1. Find $\phi(x)$ satisfying the nonhomogeneous BCs (usually the linear steady state).
2. Set $u(x,t) = v(x,t) + \phi(x)$.
3. Solve the heat equation for $v(x,t)$ with homogeneous Dirichlet BCs and initial condition $v(x,0) = f(x) - \phi(x)$, using the method of section 16.3.
4. Recover $u(x,t) = v(x,t) + \phi(x)$.

Worked example.

Worked Example

A metal rod of length $L = \pi$ has one end held at $0°$C and the other at $100°$C. Initially the rod is at a uniform temperature of $50°$C. Find the temperature $u(x,t)$ for $t > 0$.
Solution. The problem is

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 100, & t > 0, \\ u(x,0) = 50, & 0 < x < \pi. \end{cases}$$

Step 1: Find $\phi(x)$. The steady-state solution satisfying $u(0) = 0$ and $u(\pi) = 100$ is

$$\phi(x) = 0 + \frac{100 - 0}{\pi}\, x = \frac{100}{\pi}\, x.$$

Step 2: Define $v(x,t) = u(x,t) - \phi(x)$. Then v satisfies

$$\begin{cases} \frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, & 0 < x < \pi, \\ v(0,t) = 0, \quad v(\pi,t) = 0, & \\ v(x,0) = 50 - \frac{100}{\pi}\, x. & \end{cases}$$

Step 3: Solve for $v(x,t)$. Using the Dirichlet solution from section 16.3:

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)\, e^{-\alpha n^2 t},$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(50 - \frac{100}{\pi} x\right) \sin(nx)\, \mathrm{d}x.$$

Split the integral:

$$b_n = \frac{2}{\pi} \cdot 50 \int_0^{\pi} \sin(nx)\, \mathrm{d}x - \frac{200}{\pi^2} \int_0^{\pi} x \sin(nx)\, \mathrm{d}x.$$

The first integral:

$$\int_0^{\pi} \sin(nx)\, \mathrm{d}x = \left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{1 - (-1)^n}{n}.$$

The second integral (from the triangular example above):

$$\int_0^{\pi} x \sin(nx)\, \mathrm{d}x = \frac{\pi}{n}(-1)^{n+1}.$$

Therefore:

$$b_n = \frac{100}{\pi} \cdot \frac{1 - (-1)^n}{n} - \frac{200}{\pi^2} \cdot \frac{\pi}{n}(-1)^{n+1} = \frac{100}{\pi n}\left[1 - (-1)^n - 2(-1)^{n+1}\right].$$

Simplify the bracket: $1 - (-1)^n + 2(-1)^n = 1 + (-1)^n$. So:

$$b_n = \frac{100}{\pi n}\left(1 + (-1)^n\right).$$

This is nonzero only for **even** n: if n is odd, $1 + (-1)^n = 0$; if n is even, $1 + (-1)^n = 2$. Let $n = 2k$:

$$b_{2k} = \frac{200}{\pi(2k)} = \frac{100}{\pi k}, \qquad k = 1, 2, 3, \ldots$$

Step 4: Recover $u(x,t)$.

$$u(x,t) = \frac{100}{\pi}\,x + \frac{100}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k}\,e^{-4\alpha k^2 t}.$$

As $t \to \infty$, the transient part vanishes and

$$u_{\mathrm{ss}}(x) = \frac{100}{\pi}\,x,$$

a linear gradient from 0°C at $x = 0$ to 100°C at $x = \pi$.

Check at $t = 0$:

$$u(x,0) = \frac{100}{\pi}\,x + \frac{100}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2kx)}{k}.$$

The series $\sum_{k=1}^{\infty}\dfrac{\sin(2kx)}{k}$ is a known Fourier series that equals $\dfrac{\pi - 2x}{2}$ on $(0,\pi)$. Substituting:

$$u(x,0) = \frac{100}{\pi}\,x + \frac{100}{\pi}\cdot\frac{\pi - 2x}{2} = \frac{100x}{\pi} + 50 - \frac{100x}{\pi} = 50.$$

✓ The initial condition is satisfied.

16.7 Source Terms

The basic heat equation equation (78) assumes no internal heat generation. In many practical situations, there is a **source term** $Q(x,t)$ representing internal heat production (e.g., electrical heating, chemical reactions, nuclear decay):

$$\frac{\partial u}{\partial t} = \alpha\,\frac{\partial^2 u}{\partial x^2} + Q(x,t), \qquad 0 < x < L,\;\; t > 0. \tag{88}$$

Eigenfunction expansion method. We assume the solution can be expanded in the eigenfunctions of the spatial problem:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\,\sin\!\left(\frac{n\pi x}{L}\right),$$

and similarly expand the source term:

$$Q(x,t) = \sum_{n=1}^{\infty} Q_n(t)\,\sin\!\left(\frac{n\pi x}{L}\right), \qquad Q_n(t) = \frac{2}{L}\int_0^L Q(x,t)\,\sin\!\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Substituting into equation (88):

$$\sum_{n=1}^{\infty} T_n'(t)\,\sin\!\left(\frac{n\pi x}{L}\right) = \alpha\sum_{n=1}^{\infty} T_n(t)\left(-\frac{n^2\pi^2}{L^2}\right)\,\sin\!\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} Q_n(t)\,\sin\!\left(\frac{n\pi x}{L}\right).$$

By orthogonality, equate coefficients of each eigenfunction:

$$T_n'(t) + \alpha\left(\frac{n\pi}{L}\right)^2 T_n(t) = Q_n(t). \tag{89}$$

This is a first-order linear ODE for each mode $T_n(t)$, solvable by the integrating factor method.

Key Result

Solution of the modal ODE. The solution to equation (89) is

$$T_n(t) = e^{-\alpha\lambda_n t}\left[T_n(0) + \int_0^t Q_n(\tau)\,e^{\alpha\lambda_n\tau}\,\mathrm{d}\tau\right],$$

where $\lambda_n = (n\pi/L)^2$ and $T_n(0)$ is determined by the initial condition.

Worked example.

Worked Example

Solve the heat equation with a spatially uniform source on a rod of length $L = \pi$ with zero-temperature ends and initially zero temperature:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + Q_0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 < x < \pi, \end{cases}$$

where Q_0 is a positive constant representing uniform internal heat generation.
Solution. With $L = \pi$, the eigenfunctions are $\sin(nx)$ and $\lambda_n = n^2$.
Expand the source. For a constant source $Q(x,t) = Q_0$:

$$Q_n(t) = \frac{2}{\pi} \int_0^\pi Q_0 \sin(nx)\,\mathrm{d}x = \frac{2Q_0}{\pi} \Big[-\frac{\cos(nx)}{n} \Big]_0^\pi = \frac{2Q_0}{n\pi} \big(1 - (-1)^n\big).$$

This is nonzero only for odd n:

$$Q_n = \begin{cases} \dfrac{4Q_0}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Solve the modal ODE. With zero initial temperature, $T_n(0) = 0$. For odd n:

$$T_n'(t) + \alpha n^2 T_n(t) = \frac{4Q_0}{n\pi}.$$

Using the integrating factor $e^{\alpha n^2 t}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big[e^{\alpha n^2 t} T_n(t) \Big] = \frac{4Q_0}{n\pi}\, e^{\alpha n^2 t}.$$

Integrate from 0 to t:

$$e^{\alpha n^2 t} T_n(t) - T_n(0) = \frac{4Q_0}{n\pi} \int_0^t e^{\alpha n^2 \tau}\,\mathrm{d}\tau = \frac{4Q_0}{n\pi} \cdot \frac{e^{\alpha n^2 t} - 1}{\alpha n^2}.$$

Since $T_n(0) = 0$:

$$T_n(t) = \frac{4Q_0}{n^3 \pi \alpha} \Big(1 - e^{-\alpha n^2 t}\Big).$$

Reconstruct the solution. Letting $n = 2k+1$ for odd modes:

$$u(x,t) = \frac{4Q_0}{\pi\alpha} \sum_{k=0}^{\infty} \frac{1 - e^{-\alpha(2k+1)^2 t}}{(2k+1)^3} \sin\big((2k+1)x\big).$$

Long-time behavior. As $t \to \infty$, the exponential terms vanish:

$$u_{\mathrm{ss}}(x) = \frac{4Q_0}{\pi\alpha} \sum_{k=0}^{\infty} \frac{\sin\big((2k+1)x\big)}{(2k+1)^3}.$$

This is the Fourier sine series of a quadratic function. In fact, this series equals $\dfrac{Q_0}{2\alpha}\, x(\pi - x)$, which is the parabolic steady-state profile. This makes physical sense: with uniform heat generation and zero-temperature ends, the temperature reaches a parabolic profile — hottest in the middle, zero at the ends — with $\dfrac{\mathrm{d}^2 u_{\mathrm{ss}}}{\mathrm{d}x^2} = -\dfrac{Q_0}{\alpha}$, consistent with the steady-state equation $\alpha u'' + Q_0 = 0$.
The full solution is

$$u(x,t) = \frac{Q_0}{2\alpha}\, x(\pi - x) - \frac{4Q_0}{\pi\alpha} \sum_{k=0}^{\infty} \frac{e^{-\alpha(2k+1)^2 t}}{(2k+1)^3} \sin\big((2k+1)x\big),$$

showing the approach to the parabolic steady state from an initially cold rod.

16.8 Summary

The heat equation unifies several concepts developed in section 14 and section 15: separation of variables, eigenvalue problems, orthogonality, and Fourier series expansion. The choice of boundary conditions determines the eigenfunction basis and the qualitative behavior of the solution.

Table 23: Heat equation solution methods by boundary condition type

BC type	**Eigenfunctions**	**Key features**
Dirichlet:	$\sin(n\pi x/L)$, $n = 1, 2, \dots$	Temperature forced to zero at both ends; all modes decay; $u \to 0$ as $t \to \infty$
Neumann:	1, $\cos(n\pi x/L)$, $n = 0, 1, 2, \dots$	Insulated ends; $\lambda_0 = 0$ mode persists; $u \to$ average of $f(x)$ as $t \to \infty$
Mixed (one Dirichlet, one Neumann):	$\sin\big((n + \frac{1}{2})\pi x/L\big)$, $n = 0, 1, 2, \dots$	Eigenvalues $\lambda_n = \big((n + \frac{1}{2})\pi/L\big)^2$; all modes decay
Nonhomogeneous Dirichlet:	Shifting: $u = v + \phi$	$\phi(x)$ = steady-state profile; v solved with homogeneous BCs
Source term $Q(x,t)$:	Eigenfunction expansion	Mode-by-mode ODEs: $T_n' + \alpha\lambda_n T_n = Q_n(t)$; solved via integrating factor
Steady state:	$u_{\text{ss}}(x) = Ax + B$	Linear profile for Dirichlet BCs; constant for Neumann BCs

Hint

Problem-solving checklist for the heat equation.

1. Write down the PDE, boundary conditions, and initial condition.
2. Classify the boundary conditions (homogeneous/nonhomogeneous, Dirichlet/Neumann).
3. If BCs are nonhomogeneous, use the shifting technique to homogenize them.
4. Identify the eigenfunctions and eigenvalues for the spatial problem.
5. Write the general series solution with undetermined coefficients.
6. Compute the coefficients from the initial condition using orthogonality.
7. If a source term is present, expand it in eigenfunctions and solve the modal ODEs.
8. Interpret the long-time behavior of the solution.

17 Wave and Laplace Equations

This chapter covers two fundamental second-order PDEs: the **wave equation**, which governs vibrations and wave propagation, and **Laplace's equation**, which describes steady-state phenomena in electrostatics, fluid flow, and heat conduction. Both arise naturally from the separation of variables method developed in section 16, but they exhibit qualitatively different behavior — oscillatory versus decaying.

17.1 Wave Equation Derivation

We derive the one-dimensional wave equation from the physics of a vibrating string.

Physical setup. Consider a taut, flexible string stretched along the x-axis, fixed at its endpoints. Let $y(x,t)$ denote the transverse displacement of the string at position x and time t. We assume:

- The string is perfectly flexible (no resistance to bending).
- The motion is purely transverse (horizontal displacements are negligible).
- The tension T in the string is uniform and remains constant during vibration.
- The string has a uniform linear mass density ρ (mass per unit length).
- Displacements are small, so the angle θ between the string and the horizontal satisfies $\tan\theta \approx \sin\theta \approx \theta$.

Newton's second law for a string element. Consider a small segment of the string from x to $x+\Delta x$. The mass of this segment is $m = \rho\,\Delta x$. The forces acting on it are the tensions at the two endpoints, directed tangent to the string. Let θ and $\theta + \Delta\theta$ denote the angles the string makes with the horizontal at the left and right endpoints of the segment, respectively.

The net transverse force on the segment is

$$F_y = T\sin(\theta + \Delta\theta) - T\sin(\theta).$$

Applying Newton's second law in the transverse direction:

$$\rho\,\Delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\sin(\theta + \Delta\theta) - T\sin(\theta).$$

Small-angle approximation. For small displacements, the slope of the string at any point is the tangent of the angle:

$$\tan\theta \approx \sin\theta = \frac{\partial y}{\partial x}(x,t).$$

Similarly, at the right endpoint:

$$\sin(\theta + \Delta\theta) \approx \frac{\partial y}{\partial x}(x+\Delta x, t).$$

Substituting into the force equation:

$$\rho\,\Delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left[\frac{\partial y}{\partial x}(x+\Delta x, t) - \frac{\partial y}{\partial x}(x,t)\right].$$

Divide both sides by $\rho\,\Delta x$:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\cdot\frac{\frac{\partial y}{\partial x}(x+\Delta x, t) - \frac{\partial y}{\partial x}(x,t)}{\Delta x}.$$

Taking the limit $\Delta x \to 0$, the difference quotient becomes a derivative:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.$$

The wave equation. Define the wave speed

$$c = \sqrt{\frac{T}{\rho}},$$

which has units of velocity. The one-dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad -\infty < x < \infty, \quad t > 0. \tag{90}$$

Key Result

Wave speed. The speed $c = \sqrt{T/\rho}$ depends only on the physical properties of the string. Increasing the tension T increases the wave speed (tighter strings transmit vibrations faster). Increasing the linear density ρ decreases the wave speed (heavier strings respond more sluggishly).

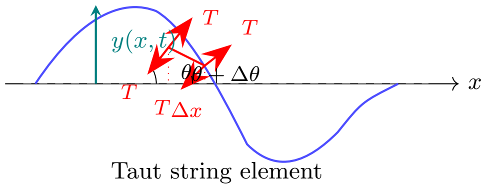


Figure 8: Derivation of the wave equation. A small segment of the string of length Δx is subject to tension forces at its endpoints. The net transverse force drives the vertical acceleration, leading to the wave equation.

Initial conditions. To obtain a unique solution, the wave equation requires two initial conditions: the initial displacement and the initial velocity:

$$y(x,0) = f(x), \qquad \frac{\partial y}{\partial t}(x,0) = g(x), \tag{91}$$

where $f(x)$ and $g(x)$ are given functions. This is natural for a second-order equation in time: we need both the position and velocity at $t = 0$.

17.2 d'Alembert's Solution

For the wave equation on an infinite string ($-\infty < x < \infty$), there is a closed-form solution known as d'Alembert's formula. The key insight is that the wave equation naturally factors into two independent propagation directions.

Characteristic coordinates. Define new variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

These are the **characteristic coordinates**. The coordinate ξ is constant along lines moving to the right with speed c, while η is constant along lines moving to the left with speed c.

We express the partial derivatives in terms of ξ and η. By the chain rule:

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta},$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\,\frac{\partial y}{\partial \xi} + c\,\frac{\partial y}{\partial \eta}.$$

Computing the second derivatives:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right) = \frac{\partial^2 y}{\partial \xi^2} + 2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta} + \frac{\partial^2 y}{\partial \eta^2},$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(-c\,\frac{\partial y}{\partial \xi} + c\,\frac{\partial y}{\partial \eta}\right) = -c\left(-c\,\frac{\partial^2 y}{\partial \xi^2} + c\,\frac{\partial^2 y}{\partial \xi\,\partial \eta}\right) + c\left(-c\,\frac{\partial^2 y}{\partial \eta\,\partial \xi} + c\,\frac{\partial^2 y}{\partial \eta^2}\right)$$

$$= c^2\,\frac{\partial^2 y}{\partial \xi^2} - 2c^2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta} + c^2\,\frac{\partial^2 y}{\partial \eta^2}.$$

Substitute into the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2\,\dfrac{\partial^2 y}{\partial x^2}$:

$$c^2\Big(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}\Big) = c^2\Big(y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta}\Big).$$

Cancel c^2 and the matching terms $y_{\xi\xi}$ and $y_{\eta\eta}$:

$$-2y_{\xi\eta} = 2y_{\xi\eta} \quad\Longrightarrow\quad 4y_{\xi\eta} = 0 \quad\Longrightarrow\quad \frac{\partial^2 y}{\partial \xi\,\partial \eta} = 0.$$

Integration. The reduced equation $y_{\xi\eta} = 0$ is very simple. Integrate with respect to η:

$$\frac{\partial y}{\partial \xi} = F'(\xi),$$

where $F'(\xi)$ is an arbitrary function of ξ alone. Now integrate with respect to ξ:

$$y(\xi, \eta) = F(\xi) + G(\eta),$$

where $G(\eta)$ is another arbitrary function (the "constant of integration" when integrating with respect to ξ). Converting back to x and t:

$$y(x, t) = F(x - ct) + G(x + ct).$$

Key Result

General solution of the wave equation. The general solution of equation (90) on an infinite string is

$$y(x, t) = F(x - ct) + G(x + ct),$$

where F and G are arbitrary twice-differentiable functions. The term $F(x - ct)$ represents a **right-traveling wave** (shape F propagating to the right at speed c), and $G(x + ct)$ represents a **left-traveling wave** (shape G propagating to the left at speed c).

Applying the initial conditions. We now determine F and G from equation (91):

$$y(x, 0) = F(x) + G(x) = f(x), \tag{92}$$

$$\frac{\partial y}{\partial t}(x, 0) = -c\,F'(x) + c\,G'(x) = g(x). \tag{93}$$

Integrate equation (93) from some fixed point x_0 to x:

$$-c\big[F(x) - F(x_0)\big] + c\big[G(x) - G(x_0)\big] = \int_{x_0}^{x} g(s)\,\mathrm{d}s.$$

Divide by c:

$$-F(x) + G(x) + \big[F(x_0) - G(x_0)\big] = \frac{1}{c}\int_{x_0}^{x} g(s)\,\mathrm{d}s.$$

Let $C = F(x_0) - G(x_0)$ be a constant. Now we have a system of two equations:

$$\begin{cases} F(x) + G(x) = f(x), \\ -F(x) + G(x) = \dfrac{1}{c}\displaystyle\int_{x_0}^{x} g(s)\,\mathrm{d}s - C. \end{cases}$$

Adding and subtracting:

$$G(x) = \frac{1}{2}\,f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s - \frac{C}{2}, \qquad F(x) = \frac{1}{2}\,f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,\mathrm{d}s + \frac{C}{2}.$$

Substitute into $y(x, t) = F(x - ct) + G(x + ct)$:

$$y(x, t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(s)\,\mathrm{d}s - \int_{x_0}^{x-ct} g(s)\,\mathrm{d}s\right].$$

The two integrals combine:

$$\int_{x_0}^{x+ct} g(s)\,\mathrm{d}s - \int_{x_0}^{x-ct} g(s)\,\mathrm{d}s = \int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

Key Result

d'Alembert's formula. For the initial value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2\,\frac{\partial^2 y}{\partial x^2}, \qquad y(x, 0) = f(x), \quad \frac{\partial y}{\partial t}(x, 0) = g(x),$$

the solution is

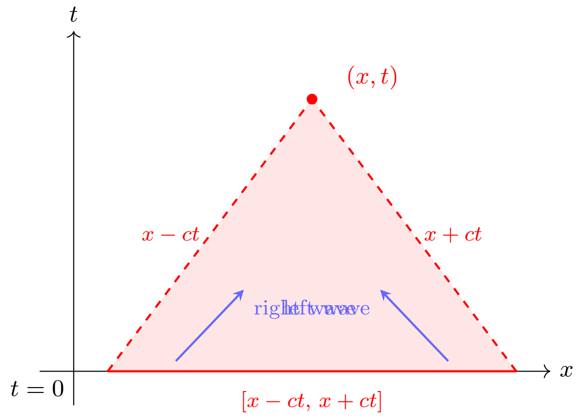
$$y(x, t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{94}$$

Interpretation of the formula. The first term, $\frac{1}{2}[f(x-ct)+f(x+ct)]$, splits the initial displacement into two copies, each traveling in opposite directions at speed c. The second term accounts for the initial velocity: it averages the initial velocity over the interval $[x-ct, x+ct]$ and distributes its effect to point x.

Theorem 17.1 (Domain of Dependence)**.** *The value of the solution $y(x,t)$ at any point (x,t) depends only on the initial data $f(s)$ and $g(s)$ in the interval*

$$[x-ct,\ x+ct].$$

This interval is called the ***domain of dependence*** *of the point (x,t). Information travels at the finite speed c; disturbances outside this interval have no influence on the solution at (x,t).*



Characteristic cone and domain of dependence

Figure 9: Domain of dependence for d'Alembert's solution. The solution at point (x,t) depends only on initial data in the interval $[x-ct,\ x+ct]$ on the x-axis. The dashed red lines are the characteristic lines $x-ct = \text{const}$ and $x+ct = \text{const}$.

Worked examples.

Worked Example

A string is initially at rest in the shape $f(x) = e^{-x^2}$. Find the solution for $t > 0$.
Solution. Here $f(x) = e^{-x^2}$ and $g(x) = 0$ (the string is initially at rest). By d'Alembert's formula:

$$y(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right].$$

Interpretation. The initial Gaussian bump splits into two identical bumps of half the original amplitude. One travels to the right at speed c, the other to the left. As t increases, the two bumps separate and the amplitude at any fixed point decays to zero (since the bumps move away).

Worked Example

A string initially flat ($f(x) = 0$) receives an impulse: the initial velocity is

$$g(x) = \begin{cases} v_0, & 0 < x < L, \\ 0, & \text{otherwise}, \end{cases}$$

where v_0 is a constant. Find $y(x,t)$ using d'Alembert's formula.
Solution. Since $f(x) = 0$, only the second term survives:

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

The integral picks up contributions only where $s \in (0, L)$. The intersection of $[x-ct,\ x+ct]$ with $[0, L]$ depends on the position of the interval relative to $[0, L]$. Consider the case where the entire interval $[x-ct,\ x+ct]$ lies within $[0, L]$, i.e., $0 \le x-ct$ and $x+ct \le L$. Then:

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} v_0\,\mathrm{d}s = \frac{v_0}{2c}\cdot 2ct = v_0\,t.$$

In this region, the displacement grows linearly with time.
For the case where $x - ct < 0$ but $x + ct < L$ (left edge of the wave is outside the impulse region):

$$y(x,t) = \frac{1}{2c}\int_0^{x+ct} v_0\,\mathrm{d}s = \frac{v_0}{2c}(x+ct).$$

The full piecewise solution depends on the relative positions of the interval endpoints. The key takeaway: the impulse generates a trapezoidal wave profile that spreads out at speed c in both directions.

17.3 Finite String and Standing Waves

When the string has finite length L and is fixed at both ends, waves cannot propagate to infinity. Instead, they reflect at the boundaries, and the superposition of right- and left-traveling waves produces **standing waves** (normal modes).

Problem statement. Consider a string of length L fixed at both ends:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = c^2\,\dfrac{\partial^2 y}{\partial x^2}, & 0 < x < L,\;\; t > 0, \\ y(0,t) = 0,\;\; y(L,t) = 0, & t > 0, \\ y(x,0) = f(x),\;\; \frac{\partial y}{\partial t}(x,0) = g(x), & 0 < x < L. \end{cases} \tag{95}$$

Separation of variables. We seek product solutions $y(x,t) = X(x)\,T(t)$. Substituting into the wave equation:

$$X(x)\,T''(t) = c^2\,X''(x)\,T(t).$$

Divide by $c^2 X(x)\,T(t)$:

$$\frac{T''(t)}{c^2\,T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This separates into two ODEs:

$$X''(x) + \lambda\,X(x) = 0, \tag{96}$$

$$T''(t) + c^2\lambda\,T(t) = 0. \tag{97}$$

The boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$ imply $X(0) = 0$ and $X(L) = 0$. The spatial equation equation (96) with these boundary conditions is exactly the Dirichlet eigenvalue problem studied in section 15.2. The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\!\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, 3, \ldots$$

For each λ_n, the temporal equation equation (97) becomes

$$T_n''(t) + \omega_n^2\,T_n(t) = 0, \qquad \text{where } \omega_n = c\sqrt{\lambda_n} = \frac{n\pi c}{L}.$$

The general solution is

$$T_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t).$$

General solution. The product solutions $y_n(x,t) = X_n(x)\,T_n(t)$ are

$$y_n(x,t) = \sin\!\left(\frac{n\pi x}{L}\right)\left[A_n\cos\!\left(\frac{n\pi ct}{L}\right) + B_n\sin\!\left(\frac{n\pi ct}{L}\right)\right].$$

By linearity, the general solution is the infinite sum:

$$y(x,t) = \sum_{n=1}^{\infty}\sin\!\left(\frac{n\pi x}{L}\right)\left[A_n\cos\!\left(\frac{n\pi ct}{L}\right) + B_n\sin\!\left(\frac{n\pi ct}{L}\right)\right]. \tag{98}$$

Determining the coefficients. The initial conditions determine A_n and B_n. At $t = 0$:

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x).$$

This is a Fourier sine series for $f(x)$ on $[0, L]$. By orthogonality:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{99}$$

For the velocity, differentiate equation (98) with respect to t:

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[-A_n \frac{n\pi c}{L} \sin\left(\frac{n\pi ct}{L}\right) + B_n \frac{n\pi c}{L} \cos\left(\frac{n\pi ct}{L}\right)\right].$$

At $t = 0$:

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = g(x).$$

By orthogonality:

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x. \tag{100}$$

Key Result

Finite string with fixed ends. The solution to equation (95) is equation (98) with coefficients given by equation (99) and equation (100).

Normal modes and harmonics. Each term n in the series equation (98) is a **normal mode**:

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right].$$

Key properties:

- The spatial shape $\sin(n\pi x/L)$ is fixed; the amplitude oscillates in time with angular frequency $\omega_n = n\pi c/L$.
- The nodes (points of zero displacement) are at $x = 0, L/n, 2L/n, \ldots, L$. The n-th mode has exactly $n - 1$ interior nodes.
- The **fundamental frequency** (first harmonic) is $\omega_1 = \pi c/L$. The n-th harmonic has frequency $\omega_n = n\,\omega_1$. This integer relationship is responsible for the musical harmonics we hear from strings.

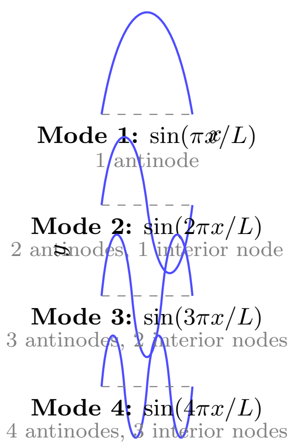


Figure 10: Normal mode shapes for a string of length L fixed at both ends. The n-th mode has the shape $\sin(n\pi x/L)$ with n antinodes and $n - 1$ interior nodes. Higher modes oscillate at higher frequencies $\omega_n = n\omega_1$.

Worked examples.

Worked Example

A guitar string of length L is plucked into a triangular shape:

$$f(x) = \begin{cases} \dfrac{2h}{L}\,x, & 0 \le x \le \dfrac{L}{2}, \\[2mm] \dfrac{2h}{L}\,(L-x), & \dfrac{L}{2} \le x \le L, \end{cases}$$

and released from rest ($g(x) = 0$). Find the solution $y(x,t)$.
Solution. Since $g(x) = 0$, all $B_n = 0$. The solution reduces to

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\!\left(\frac{n\pi x}{L}\right) \cos\!\left(\frac{n\pi c t}{L}\right),$$

with

$$A_n = \frac{2}{L}\int_0^L f(x)\,\sin\!\left(\frac{n\pi x}{L}\right)\,dx.$$

Split the integral at $x = L/2$:

$$A_n = \frac{2}{L}\left[\frac{2h}{L}\int_0^{L/2} x\,\sin\!\left(\frac{n\pi x}{L}\right)\,dx + \frac{2h}{L}\int_{L/2}^{L} (L-x)\,\sin\!\left(\frac{n\pi x}{L}\right)\,dx\right].$$

Let $k = n\pi/L$ for brevity. Compute the first integral by parts ($u = x$, $dv = \sin(kx)\,dx$):

$$\int_0^{L/2} x\sin(kx)\,dx = \left[-\frac{x}{k}\cos(kx)\right]_0^{L/2} + \frac{1}{k}\int_0^{L/2}\cos(kx)\,dx = -\frac{L}{2k}\cos\!\left(\frac{n\pi}{2}\right) + \frac{1}{k^2}\sin\!\left(\frac{n\pi}{2}\right).$$

For the second integral, substitute $u = L - x$ ($du = -dx$):

$$\int_{L/2}^{L} (L-x)\sin(kx)\,dx = \int_0^{L/2} u\sin(k(L-u))\,du.$$

Since $\sin(k(L-u)) = \sin(n\pi - ku) = \sin(n\pi)\cos(ku) - \cos(n\pi)\sin(ku) = (-1)^{n+1}\sin(ku)$, the second integral is $(-1)^{n+1}$ times the first. For a symmetric triangular pluck, the result simplifies to:

$$A_n = \frac{8h}{n^2\pi^2}\sin\!\left(\frac{n\pi}{2}\right).$$

This is nonzero only for **odd** n (since $\sin(n\pi/2) = 0$ for even n). For n odd, $\sin(n\pi/2) = (-1)^{(n-1)/2}$. Therefore:

$$A_n = \begin{cases} \dfrac{8h}{n^2\pi^2}\,(-1)^{(n-1)/2}, & n \text{ odd}, \\[2mm] 0, & n \text{ even}. \end{cases}$$

The final solution is

$$y(x,t) = \frac{8h}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\sin\!\left(\frac{(2k+1)\pi x}{L}\right)\cos\!\left(\frac{(2k+1)\pi c t}{L}\right).$$

Notice that only odd harmonics are present. This is a general property: symmetric initial conditions excite only odd modes. The amplitude of the n-th mode decays as $1/n^2$, so the fundamental dominates strongly.

Worked Example

A string of length $L = \pi$ is fixed at both ends. At $t = 0$ it is at its equilibrium position but given an initial velocity $g(x) = \sin(3x)$. Find $y(x,t)$.
Solution. Here $f(x) = 0$ and $g(x) = \sin(3x)$, so $A_n = 0$ for all n. We only need B_n:

$$B_n = \frac{2}{cn\pi}\int_0^{\pi} \sin(3x)\,\sin(nx)\,dx.$$

By the orthogonality of sines, this integral is zero unless $n = 3$. For $n = 3$:

$$B_3 = \frac{2}{3c\pi} \int_0^{\pi} \sin^2(3x)\,dx = \frac{2}{3c\pi} \cdot \frac{\pi}{2} = \frac{1}{3c}.$$

The solution is a single mode:

$$y(x,t) = \frac{1}{3c} \sin(3x) \sin(3ct).$$

Interpretation. The initial velocity profile already has the shape of the third normal mode, so only the third mode is excited. The string oscillates at the third harmonic frequency $\omega_3 = 3c$ with amplitude $1/(3c)$. The two nodes at $x = 0$ and $x = \pi$ are fixed, and there is one additional node at $x = \pi/3$ and $x = 2\pi/3$.

17.4 Two-Dimensional Wave Equation

The wave equation generalizes to two spatial dimensions to describe the transverse vibration of a membrane (e.g., a drumhead). Let $u(x,y,t)$ be the displacement of the membrane at position (x,y) and time t.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = c^2\,\nabla^2 u, \tag{101}$$

where ∇^2 is the two-dimensional Laplacian.

Separation on a rectangular membrane. Consider a rectangular membrane $0 < x < a$, $0 < y < b$ with fixed edges:

$$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0.$$

We seek product solutions $u(x,y,t) = X(x)\,Y(y)\,T(t)$. Substituting into equation (101):

$$X\,Y\,T'' = c^2\,(X''\,Y\,T + X\,Y''\,T).$$

Divide by $c^2 XYT$:

$$\frac{T''}{c^2\,T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Since the left side depends only on t and the right side depends only on x and y, both must equal a separation constant, which we take as $-\lambda$:

$$T'' + c^2 \lambda\,T = 0, \qquad \frac{X''}{X} + \frac{Y''}{Y} = -\lambda.$$

The spatial equation further separates:

$$\frac{X''}{X} = -\mu, \qquad \frac{Y''}{Y} = -\nu, \qquad \mu + \nu = \lambda.$$

With fixed edges, we have:

$$X'' + \mu X = 0, \;\; X(0) = X(a) = 0 \quad\Longrightarrow\quad \mu_m = \left(\frac{m\pi}{a}\right)^2, \;\; X_m(x) = \sin\!\left(\frac{m\pi x}{a}\right),$$

$$Y'' + \nu Y = 0, \;\; Y(0) = Y(b) = 0 \quad\Longrightarrow\quad \nu_n = \left(\frac{n\pi}{b}\right)^2, \;\; Y_n(y) = \sin\!\left(\frac{n\pi y}{b}\right),$$

for $m, n = 1, 2, 3, \ldots$.

The eigenvalues are

$$\lambda_{mn} = \mu_m + \nu_n = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right),$$

and the angular frequencies are

$$\omega_{mn} = c\sqrt{\lambda_{mn}} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Key Result

Rectangular membrane. The general solution is

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\!\left(\frac{m\pi x}{a}\right) \sin\!\left(\frac{n\pi y}{b}\right) \Big[A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t) \Big],$$

where $\omega_{mn} = c\pi\sqrt{m^2/a^2 + n^2/b^2}$ and the coefficients are determined by the initial displacement and velocity.

Worked Example

A rectangular membrane of size $a \times b$ is fixed on all edges. The initial displacement is $u(x, y, 0) = \sin(\pi x/a)\sin(\pi y/b)$ and the initial velocity is $u_t(x, y, 0) = 0$. Find the solution $u(x, y, t)$.
Solution. The general solution for a rectangular membrane with fixed edges is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\Big[A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\Big],$$

where $\omega_{mn} = c\pi\sqrt{m^2/a^2 + n^2/b^2}$.
The initial velocity condition gives

$$u_t(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = 0,$$

so by the orthogonality of sine products, $B_{mn} = 0$ for all m, n.
The initial displacement condition gives

$$u(x, y, 0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

Again by orthogonality, $A_{11} = 1$ and $A_{mn} = 0$ for all $(m, n) \neq (1, 1)$.
The solution is a single mode:

$$u(x, y, t) = \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\cos\left(c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}\, t\right).$$

Interpretation. The initial displacement already has the shape of the fundamental mode $(m, n) = (1, 1)$, so only the fundamental mode is excited. The membrane oscillates at its lowest eigenfrequency $\omega_{11} = c\pi\sqrt{1/a^2 + 1/b^2}$ with unit amplitude. If the initial displacement had contained higher modes (e.g., a sum of several products of sines), each mode would oscillate independently at its own frequency.

Discussion. Unlike the one-dimensional string, the frequencies of a rectangular membrane are *not* integer multiples of a fundamental. The ratio $\omega_{mn}/\omega_{11} = \sqrt{m^2a^2 + n^2b^2}\,/\,\sqrt{a^2 + b^2}$ is generally irrational. This is why a drum produces a sound with no clear fundamental pitch — a **non-harmonic** spectrum.

17.5 Laplace's Equation in Rectangles

We now turn to **Laplace's equation**, the prototypical elliptic PDE:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{102}$$

Solutions of Laplace's equation are called **harmonic functions**. They describe steady-state temperature distributions (no time dependence in the heat equation), electrostatic potentials (no charges), and incompressible, irrotational fluid flow.

Rectangular domain. Consider Laplace's equation on a rectangle $0 < x < a$, $0 < y < b$. We need four boundary conditions, one on each edge. A common and pedagogically useful case has three edges held at zero and one edge given by a prescribed function:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & 0 < x < a,\ \ 0 < y < b, \\ u(0, y) = 0,\ \ u(a, y) = 0, & 0 < y < b, \\ u(x, 0) = 0, & 0 < x < a, \\ u(x, b) = f(x), & 0 < x < a. \end{cases} \tag{103}$$

Separation of variables. Seek $u(x,y) = X(x)\,Y(y)$. Substituting into Laplace's equation:

$$X''\,Y + X\,Y'' = 0 \quad \Longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

The spatial ODEs are:

$$X'' + \lambda X = 0, \qquad X(0) = 0, \;\; X(a) = 0, \tag{104}$$

$$Y'' - \lambda Y = 0, \qquad Y(0) = 0. \tag{105}$$

Equation equation (104) is the familiar Dirichlet eigenvalue problem on $[0, a]$. The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2, \qquad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n = 1, 2, 3, \ldots$$

For each λ_n, equation equation (105) becomes

$$Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0.$$

The general solution is

$$Y_n(y) = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right).$$

The boundary condition $Y(0) = 0$ forces $A_n = 0$, so

$$Y_n(y) = B_n \sinh\left(\frac{n\pi y}{a}\right).$$

General solution. The product solutions are $u_n(x,y) = \sin(n\pi x/a)\,\sinh(n\pi y/a)$. By linearity:

$$u(x,y) = \sum_{n=1}^{\infty} A_n\,\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right). \tag{106}$$

Apply the remaining boundary condition $u(x,b) = f(x)$:

$$f(x) = u(x,b) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{a}\right)\,\sin\left(\frac{n\pi x}{a}\right).$$

This is a Fourier sine series for $f(x)$ on $[0, a]$. Define $C_n = A_n \sinh(n\pi b/a)$:

$$C_n = \frac{2}{a}\int_0^a f(x)\,\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x.$$

Therefore:

$$A_n = \frac{2}{a\,\sinh(n\pi b/a)}\int_0^a f(x)\,\sin\left(\frac{n\pi x}{a}\right)\,\mathrm{d}x. \tag{107}$$

Key Result

Laplace's equation on a rectangle. For the problem equation (103), the solution is equation (106) with coefficients given by equation (107).

Worked example.

Worked Example

Solve Laplace's equation on the square $0 < x < \pi$, $0 < y < \pi$, with

$$u(0,y) = u(\pi,y) = u(x,0) = 0, \qquad u(x,\pi) = \sin(2x).$$

Solution. Here $a = \pi$, $b = \pi$, and $f(x) = \sin(2x)$. The solution is

$$u(x,y) = \sum_{n=1}^{\infty} A_n\,\sin(nx)\sinh(ny).$$

The coefficients are

$$A_n = \frac{2}{\pi\,\sinh(n\pi)}\int_0^{\pi} \sin(2x)\,\sin(nx)\,\mathrm{d}x.$$

By orthogonality of sines, the integral vanishes for $n \neq 2$. For $n = 2$:

$$\int_0^{\pi} \sin^2(2x)\, dx = \frac{\pi}{2}.$$

So $A_2 = \dfrac{2}{\pi \sinh(2\pi)} \cdot \dfrac{\pi}{2} = \dfrac{1}{\sinh(2\pi)}$, and $A_n = 0$ for $n \neq 2$. The solution is

$$u(x, y) = \frac{\sinh(2y)}{\sinh(2\pi)} \sin(2x).$$

Check. At $y = \pi$: $u(x, \pi) = \dfrac{\sinh(2\pi)}{\sinh(2\pi)} \sin(2x) = \sin(2x)$. ✓
At $y = 0$: $u(x, 0) = \dfrac{\sinh(0)}{\sinh(2\pi)} \sin(2x) = 0$. ✓
At $x = 0$ and $x = \pi$: $\sin(2x) = 0$, so $u = 0$. ✓

17.6 Laplace's Equation in Polar Coordinates

When the domain has circular geometry (e.g., a disk), polar coordinates are the natural choice. The two-dimensional Laplacian in polar coordinates (r, θ) is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{108}$$

Separation of variables. Assume $u(r, \theta) = R(r)\,\Theta(\theta)$. Substitute into equation (108):

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\,\Theta'' = 0.$$

Divide by $R\Theta/r^2$:

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

This gives two ODEs:

$$\Theta'' + \lambda\,\Theta = 0, \tag{109}$$

$$r^2 R'' + rR' - \lambda R = 0. \tag{110}$$

Angular equation. The solution $\Theta(\theta)$ must be **periodic** with period 2π (since θ and $\theta + 2\pi$ represent the same physical point). This periodicity condition restricts λ to be a non-negative integer square.

Case 1: $\lambda = n^2$ with $n = 1, 2, 3, \ldots$. The general solution is

$$\Theta_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta).$$

Periodicity is automatically satisfied since $\cos(n(\theta + 2\pi)) = \cos(n\theta)$ and similarly for sine.

Case 2: $\lambda = 0$ ($n = 0$). The angular equation becomes $\Theta'' = 0$, with solution

$$\Theta_0(\theta) = C_0 + D_0\,\theta.$$

Periodicity $\Theta_0(\theta + 2\pi) = \Theta_0(\theta)$ requires $D_0 = 0$, so $\Theta_0(\theta) = C_0$ (a constant).

Radial equation. Equation equation (110) is an **Euler–Cauchy equation** (also called an equidimensional equation). The substitution $R(r) = r^k$ yields:

$$r^2\, k(k-1) r^{k-2} + r\, k\, r^{k-1} - \lambda\, r^k = 0 \quad \Longrightarrow \quad k^2 - \lambda = 0 \quad \Longrightarrow \quad k = \pm\sqrt{\lambda}.$$

Case 1: $\lambda = n^2$ with $n \geq 1$. The roots are $k = \pm n$, so

$$R_n(r) = C_n\, r^n + D_n\, r^{-n}.$$

If the domain includes the origin ($r = 0$), we must require that the solution remains **bounded** there. Since $r^{-n} \to \infty$ as $r \to 0$, we set $D_n = 0$. Thus

$$R_n(r) = C_n\, r^n, \qquad n = 1, 2, 3, \ldots$$

Case 2: $\lambda = 0$ ($n = 0$). The radial equation becomes

$$r^2 R'' + rR' = 0.$$

Let $S = R'$, then $r^2 S' + rS = 0$, which is separable:

$$\frac{S'}{S} = -\frac{1}{r} \quad \Longrightarrow \quad \ln|S| = -\ln r + \text{const} \quad \Longrightarrow \quad S = \frac{K}{r}.$$

Integrating once more:

$$R_0(r) = C_0 + D_0 \ln r.$$

Boundedness at $r = 0$ requires $D_0 = 0$, so $R_0(r) = C_0$ (a constant).

General bounded solution on a disk. Combining the angular and radial parts:

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n \Big[A_n \cos(n\theta) + B_n \sin(n\theta)\Big]. \tag{111}$$

The factor $A_0/2$ (rather than A_0) is a notational convention that makes the $n = 0$ coefficient formula consistent with the Fourier series formula.

Dirichlet problem on a disk. Suppose the boundary condition on a disk of radius a is

$$u(a, \theta) = f(\theta), \qquad 0 \le \theta < 2\pi,$$

where $f(\theta)$ is a given 2π-periodic function. Then:

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n \Big[A_n \cos(n\theta) + B_n \sin(n\theta)\Big].$$

This is precisely the Fourier series of $f(\theta)$ on $[0, 2\pi]$. The coefficients are:

$$A_0 = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, \mathrm{d}\theta, \tag{112}$$

$$A_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta)\, \mathrm{d}\theta, \qquad n \ge 1, \tag{113}$$

$$B_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin(n\theta)\, \mathrm{d}\theta, \qquad n \ge 1. \tag{114}$$

Key Result

Dirichlet problem on a disk. For Laplace's equation $\nabla^2 u = 0$ on the disk $r < a$ with boundary condition $u(a, \theta) = f(\theta)$, the bounded solution is

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \Big[a^n A_n \cos(n\theta) + a^n B_n \sin(n\theta)\Big],$$

where $a^n A_n$ and $a^n B_n$ are the standard Fourier coefficients of $f(\theta)$ on $[0, 2\pi]$. Equivalently,

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, \mathrm{d}\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \int_0^{2\pi} f(\phi) \cos\big(n(\phi - \theta)\big)\, \mathrm{d}\phi.$$

Poisson kernel. The solution can be written as a single integral using the **Poisson kernel**:

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - \phi)\, f(\phi)\, \mathrm{d}\phi, \tag{115}$$

where the Poisson kernel is

$$P(r, \psi) = \frac{a^2 - r^2}{a^2 - 2ar\cos\psi + r^2}. \tag{116}$$

The Poisson kernel has several important properties: $P(a, \psi) = 0$ for $\psi \ne 0$ (the boundary data is sharply localized), and $P(0, \psi) = 1$ (the value at the center is the average of the boundary data).

Worked example.

Worked Example

Solve Laplace's equation on the disk $r < 2$ with boundary condition

$$u(2,\theta) = 3 + 5\cos(2\theta) - 4\sin(3\theta).$$

Solution. Here $a = 2$ and $f(\theta) = 3 + 5\cos(2\theta) - 4\sin(3\theta)$. This is already in the form of a Fourier series, so we can read off the coefficients directly.
Compare with the general solution on the boundary:

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} 2^n \Big[A_n \cos(n\theta) + B_n \sin(n\theta)\Big].$$

From the constant term: $A_0/2 = 3$, so $A_0 = 6$.
From the $\cos(2\theta)$ term: $2^2 A_2 = 5$, so $A_2 = 5/4$.
From the $\sin(3\theta)$ term: $2^3 B_3 = -4$, so $B_3 = -4/8 = -1/2$.
All other coefficients vanish. The solution is

$$u(r,\theta) = 3 + \frac{5}{4} r^2 \cos(2\theta) - \frac{1}{2} r^3 \sin(3\theta).$$

Check. At $r = 2$:

$$u(2,\theta) = 3 + \frac{5}{4} \cdot 4\, \cos(2\theta) - \frac{1}{2} \cdot 8\, \sin(3\theta) = 3 + 5\cos(2\theta) - 4\sin(3\theta).$$

✓ The boundary condition is satisfied.
Physical interpretation. The solution is a superposition of a constant (average value 3) and two harmonic modes. The r^2 and r^3 factors cause higher modes to be suppressed near the center — the temperature (or potential) is smoothest at the origin and picks up spatial variation as you move outward toward the boundary.

17.7 Summary

This chapter has covered two fundamental second-order PDEs: the wave equation (hyperbolic) and Laplace's equation (elliptic). Together with the heat equation (parabolic) from section 16, these form the three classical types of PDEs.

Hint

Problem-solving checklist.

1. **Wave equation, infinite domain:** Use d'Alembert's formula directly.
2. **Wave equation, finite domain:** Use separation of variables to get the standing wave series. Compute Fourier sine coefficients from initial data.
3. **Wave equation, 2D rectangle:** Double separation; the eigenfrequencies ω_{mn} are generally non-harmonic.
4. **Laplace's equation, rectangle:** Separate in the direction with homogeneous BCs on both ends (gives trig functions); the other direction gives sinh/cosh. Match the nonhomogeneous boundary with a Fourier series.
5. **Laplace's equation, disk:** Use polar coordinates. Angular equation gives integer n from periodicity. Radial equation is Euler–Cauchy; enforce boundedness at $r = 0$. Match boundary data with a Fourier series.
6. **Poisson's equation ($\nabla^2 u = f$):** Use eigenfunction expansion (as with the heat equation with a source), expanding both u and f in the appropriate eigenfunction basis.

18 Nonlinear Systems

Linear systems admit a powerful theory built on superposition and eigenvalue analysis (section 12). When the equations become nonlinear, the principle of superposition breaks down: the sum of two solutions is generally

Table 24: Chapter summary: wave equation and Laplace's equation

Concept	Key formula/method
Wave equation (1D)	$\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}, \quad c = \sqrt{T/\rho}$
d'Alembert's formula	$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$
Domain of dependence	Solution at (x,t) depends only on data in $[x-ct,\, x+ct]$
Finite string (fixed ends)	$y(x,t) = \sum_{n=1}^{\infty} \sin\Big(\dfrac{n\pi x}{L}\Big)\big[A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\big]$
Normal mode frequencies	$\omega_n = \dfrac{n\pi c}{L} = n\,\omega_1$ (harmonic series)
2D wave equation	$u_{tt} = c^2(u_{xx} + u_{yy})$; frequencies $\omega_{mn} = c\pi\sqrt{m^2/a^2 + n^2/b^2}$
Laplace's equation	$\nabla^2 u = 0$; solutions are harmonic functions
Laplace on rectangle	$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\Big(\dfrac{n\pi x}{a}\Big)\sinh\Big(\dfrac{n\pi y}{a}\Big)$
Laplace in polar	$u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta} = 0$; Euler–Cauchy radial equation
Dirichlet on disk	$u(r,\theta) = \dfrac{A_0}{2} + \sum_{n=1}^{\infty} r^n[A_n \cos(n\theta) + B_n \sin(n\theta)]$
Poisson kernel	$P(r,\psi) = \dfrac{a^2 - r^2}{a^2 - 2ar\cos\psi + r^2}$

not a solution. No universal closed-form solution formula exists for nonlinear systems. Instead, we rely on **qualitative methods** — studying the structure of solutions without finding explicit formulas.

18.1 Nonlinear Autonomous Systems

A **nonlinear autonomous system** in two variables takes the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x, y), \tag{117}$$

where f and g are nonlinear functions. The system is *autonomous* because f and g do not depend explicitly on t.

Why analytical methods fail. For linear systems $\mathbf{x}' = A\mathbf{x}$, we can construct the general solution from eigenvalues and eigenvectors. For nonlinear systems:

- **No superposition:** If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions, $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is *not* generally a solution.
- **No general formula:** There is no algorithm that produces a closed-form solution for an arbitrary nonlinear system.
- **Rich behavior:** Nonlinear systems exhibit phenomena absent in linear systems, including multiple equilibria, limit cycles, bifurcations, and chaos.

The strategy is to understand the system's **phase portrait** — a qualitative map of all possible solution trajectories in the (x, y)-plane — by combining local linear analysis near equilibria with global theorems.

Equilibrium points. As in section 7.1 and section 12.5, equilibrium (singular) points are constant solutions where the system comes to rest.

Definition 18.1 (Equilibrium point)**.** An **equilibrium point** (or **fixed point**, or **critical point**) of equation (117) is a point (x^*, y^*) satisfying

$$f(x^*, y^*) = 0 \quad \text{and} \quad g(x^*, y^*) = 0.$$

At an equilibrium, both derivatives vanish and the solution is stationary: $x(t) \equiv x^*$, $y(t) \equiv y^*$.

Finding equilibria is an algebraic problem: solve the system of two nonlinear equations $f = 0$, $g = 0$. Unlike the one-dimensional case (section 7), there may be zero, one, or many equilibrium points.

Worked Example

Find all equilibrium points of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^2 - xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y - y^2 - xy.$$

Solution. Set both equations to zero simultaneously:

$$x(1 - x - y) = 0, \tag{118}$$
$$y(1 - x - y) = 0. \tag{119}$$

From equation (118), either $x = 0$ or $1 - x - y = 0$.
Case 1: $x = 0$. Substituting into equation (119): $y(1 - 0 - y) = 0$, so $y = 0$ or $y = 1$. This gives equilibria $(0, 0)$ and $(0, 1)$.
Case 2: $1 - x - y = 0$, i.e. $y = 1 - x$. Substituting into equation (119): $y(0) = 0$, which is satisfied for any y. But we must also have $x = 1 - y$ from the same relation. Setting $x = 0$ gives $y = 1$ (already found), and setting $y = 0$ gives $x = 1$, yielding the equilibrium $(1, 0)$.
Verification: Check $(1, 0)$: $f(1, 0) = 1 - 1 - 0 = 0$, $g(1, 0) = 0 - 0 - 0 = 0$. ✓
The system has three equilibrium points: $(0, 0)$, $(0, 1)$, and $(1, 0)$.

18.2 Jacobian Linearization

The central technique for analyzing nonlinear systems near an equilibrium is **linearization**. Just as in section 7.2, we approximate the nonlinear functions by their first-order Taylor expansions near the equilibrium. This reduces the nonlinear system to a linear one, whose behavior we understand completely from section 12.

Derivation via Taylor expansion. Let (x^*, y^*) be an equilibrium. Introduce deviation variables $u = x - x^*$ and $v = y - y^*$. Expanding f and g in a two-variable Taylor series about (x^*, y^*):

$$f(x,y) = f(x^*,y^*) + \frac{\partial f}{\partial x}\Big|_{(x^*,y^*)}(x-x^*) + \frac{\partial f}{\partial y}\Big|_{(x^*,y^*)}(y-y^*) + O\big(|u|^2+|v|^2\big),$$
$$g(x,y) = g(x^*,y^*) + \frac{\partial g}{\partial x}\Big|_{(x^*,y^*)}(x-x^*) + \frac{\partial g}{\partial y}\Big|_{(x^*,y^*)}(y-y^*) + O\big(|u|^2+|v|^2\big).$$

Since (x^*, y^*) is an equilibrium, $f(x^*,y^*) = g(x^*,y^*) = 0$. Discarding the higher-order terms gives the **linearized system**

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x}\big|_{(x^*,y^*)} & \frac{\partial f}{\partial y}\big|_{(x^*,y^*)} \\ \frac{\partial g}{\partial x}\big|_{(x^*,y^*)} & \frac{\partial g}{\partial y}\big|_{(x^*,y^*)} \end{pmatrix}}_{=\,J} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{120}$$

Key Result

Jacobian matrix and linearization. The **Jacobian matrix** of the system $x' = f(x,y)$, $y' = g(x,y)$ is

$$J(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

Evaluated at an equilibrium (x^*, y^*), the linearized system is $\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x^*,y^*)\begin{pmatrix} u \\ v \end{pmatrix}$, where $u = x - x^*$, $v = y - y^*$. Classify (x^*, y^*) by the eigenvalues of $J(x^*, y^*)$ using the phase plane classification from section 12.

Key Result

Eigenvalue classification at an equilibrium of a nonlinear system.
Let λ_1, λ_2 be the eigenvalues of $J(x^*, y^*)$.

Eigenvalues	**Type**	**Stability**
Real, both > 0	Unstable node (source)	Unstable
Real, both < 0	Stable node (sink)	Asymptotically stable
Real, opposite signs	Saddle point	Unstable
Complex $\alpha \pm i\beta$, $\alpha > 0$	Unstable spiral	Unstable
Complex $\alpha \pm i\beta$, $\alpha < 0$	Stable spiral	Asymptotically stable
Purely imaginary $\pm i\beta$	Center (inconclusive)	*Linearization inconclusive*

The center case is the only one where the linearization does *not* determine the true nonlinear behavior. See section 18.4 for details.

Worked Example

Linearize and classify the equilibria of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x - x^2.$$

(This is the undamped nonlinear pendulum in phase-plane form.)
Step 1: Find equilibria. Set $y = 0$ and $x - x^2 = 0 \implies x(1-x) = 0$.

$$\text{Equilibria: } (0,0) \text{ and } (1,0).$$

Step 2: Jacobian. With $f(x,y) = y$ and $g(x,y) = x - x^2$:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 1, \quad \frac{\partial g}{\partial x} = 1 - 2x, \quad \frac{\partial g}{\partial y} = 0.$$

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}.$$

Step 3: Classify $(0, 0)$.

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Characteristic equation: $\lambda^2 - 1 = 0 \implies \lambda = \pm 1$. Opposite signs $\implies$ **saddle point** (unstable).
Step 4: Classify $(1, 0)$.

$$J(1, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Characteristic equation: $\lambda^2 + 1 = 0 \implies \lambda = \pm i$. Purely imaginary eigenvalues $\implies$ **center by linearization**. The linearization predicts closed orbits, but the nonlinear behavior requires further analysis (confirmed below to be a true center).

Worked Example

Linearize and classify the equilibria of the competing species model

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x - x^2 - xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y - y^2 - xy.$$

Step 1: Find equilibria.

$$x(2 - x - y) = 0, \tag{121}$$
$$y(1 - x - y) = 0. \tag{122}$$

From equation (121), either $x = 0$ or $x + y = 2$.
Case 1: $x = 0$. From equation (122): $y(1 - y) = 0$, giving $y = 0$ or $y = 1$. Equilibria: $(0, 0)$ and $(0, 1)$.
Case 2: $x + y = 2$, so $y = 2 - x$. From equation (122): $y(1 - x - y) = y(1 - 2) = -y = 0$, so $y = 0$, which gives $x = 2$. Equilibrium: $(2, 0)$.
The three equilibria are $(0, 0)$, $(0, 1)$, and $(2, 0)$.
Step 2: Jacobian. With $f(x, y) = 2x - x^2 - xy$ and $g(x, y) = y - y^2 - xy$:

$$J(x, y) = \begin{pmatrix} 2 - 2x - y & -x \\ -y & 1 - 2y - x \end{pmatrix}.$$

Step 3: Classify $(0, 0)$.

$$J(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda_1 = 2, \ \lambda_2 = 1.$$

Both positive $\implies$ **unstable node (source)**. Both species die out is unstable: a small introduction of either population grows.
Step 4: Classify $(0, 1)$.

$$J(0, 1) = \begin{pmatrix} 2 - 1 & 0 \\ -1 & 1 - 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Since J is lower triangular, eigenvalues are the diagonal entries: $\lambda_1 = 1$, $\lambda_2 = -1$. Opposite signs $\implies$ **saddle point** (unstable). Species y alone at carrying capacity is unstable to invasion by species x.
Step 5: Classify $(2, 0)$.

$$J(2, 0) = \begin{pmatrix} 2 - 4 & -2 \\ 0 & 1 - 2 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}.$$

Upper triangular: $\lambda_1 = -2$, $\lambda_2 = -1$. Both negative $\implies$ **stable node (sink)**. Species x alone at carrying capacity is stable; species y cannot invade.

18.3 Trace-Determinant Classification

Just as for linear systems (section 12.6), the trace and determinant of the Jacobian at an equilibrium provide a quick classification without computing eigenvalues explicitly.

For $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ at an equilibrium:

$$\tau = \mathrm{tr}(J) = a + d, \qquad \Delta = \det(J) = ad - bc.$$

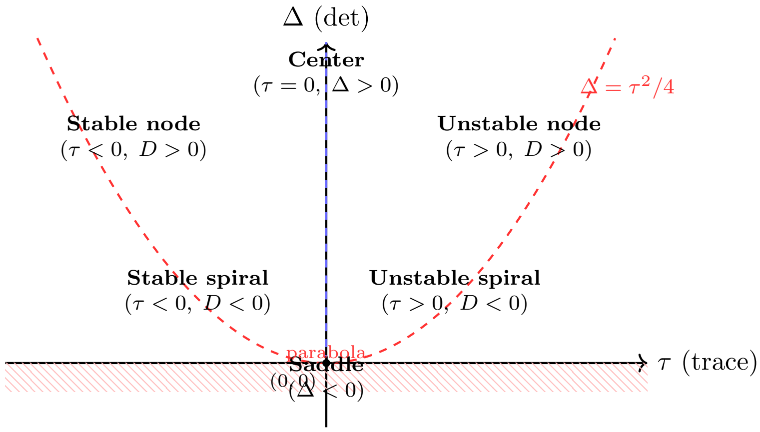
The eigenvalues satisfy $\lambda^2 - \tau\lambda + \Delta = 0$ with discriminant $D = \tau^2 - 4\Delta$.

Key Result

Trace-determinant classification for nonlinear equilibria. Evaluate $\tau = \operatorname{tr}(J)$ and $\Delta = \det(J)$ at the equilibrium point.

Region	**Conditions**	**Classification**
$\Delta < 0$	Below τ-axis	Saddle (unstable)
$\Delta > 0,\ \tau > 0,\ D > 0$	Right, above parabola	Unstable node
$\Delta > 0,\ \tau < 0,\ D > 0$	Left, above parabola	Stable node
$\Delta > 0,\ \tau > 0,\ D < 0$	Right, below parabola	Unstable spiral
$\Delta > 0,\ \tau < 0,\ D < 0$	Left, below parabola	Stable spiral
$\Delta > 0,\ \tau = 0$	On positive Δ-axis	Center (inconclusive)

The trace-determinant plane.



The parabola $\Delta = \tau^2/4$ separates real from complex eigenvalues. The Δ-axis ($\tau = 0$) separates stable from unstable systems. The τ-axis ($\Delta = 0$) separates saddles from nodes/spirals/centers.

Worked Example

Classify the equilibria of the competing species model from the previous section using the trace-determinant plane.

Recall the Jacobian:

$$J(x,y) = \begin{pmatrix} 2 - 2x - y & -x \\ -y & 1 - 2y - x \end{pmatrix}.$$

At $(0,0)$:

$$\tau = 2 + 1 = 3 > 0, \qquad \Delta = (2)(1) - (0) = 2 > 0.$$

Discriminant: $D = 9 - 8 = 1 > 0$. Region: $\tau > 0,\ \Delta > 0,\ D > 0 \implies$ **unstable node**. Consistent with $\lambda = 2, 1$.

At $(0,1)$:

$$\tau = 1 + (-1) = 0, \qquad \Delta = (1)(-1) - (0) = -1 < 0.$$

Region: $\Delta < 0 \implies$ **saddle point**. Consistent with $\lambda = \pm 1$.

At $(2,0)$:

$$\tau = (-2) + (-1) = -3 < 0, \qquad \Delta = (-2)(-1) - (0) = 2 > 0.$$

Discriminant: $D = 9 - 8 = 1 > 0$. Region: $\tau < 0,\ \Delta > 0,\ D > 0 \implies$ **stable node**. Consistent with $\lambda = -2, -1$.

18.4 Hartman–Grobman Theorem

The linearization tells us about the behavior of the nonlinear system in a *small neighborhood* of the equilibrium. The Hartman–Grobman theorem provides the rigorous justification for this connection.

Theorem 18.2 (Hartman–Grobman)**.** *Let (x^*, y^*) be an equilibrium point of the autonomous system $x' = f(x, y)$, $y' = g(x, y)$, where f and g are C^1 (continuously differentiable). Let J be the Jacobian matrix evaluated at (x^*, y^*).*

*If J has **no eigenvalues with zero real part** (i.e., the equilibrium is **hyperbolic**), then there exists a neighborhood of (x^*, y^*) in which the nonlinear system is **topologically conjugate** to its linearization $u' = au + bv$, $v' = cu + dv$.*

In particular, the local phase portrait of the nonlinear system near (x^*, y^*) has the same topological structure as that of the linearized system.

What topological conjugacy means. Two systems are topologically conjugate if there exists a continuous, invertible change of coordinates (a homeomorphism) that maps the trajectories of one system onto the trajectories of the other, preserving the direction of time. Practically, this means:

- A stable node of the linearization corresponds to a stable node of the nonlinear system.
- A saddle of the linearization corresponds to a saddle of the nonlinear system.
- The qualitative flow pattern (arrows, attraction, repulsion) is preserved.

The actual shapes of trajectories may differ — a nonlinear trajectory near a stable spiral may not be a perfect logarithmic spiral — but the essential features (spiraling inward, counterclockwise direction, etc.) are identical.

Limitations: non-hyperbolic equilibria. The theorem **does not apply** when the Jacobian has eigenvalues with zero real part:

- **Centers** ($\lambda = \pm i\beta$): The nonlinear system near a center can be a center, a stable spiral, or an unstable spiral. The linearization alone cannot distinguish between these cases. Higher-order terms in the Taylor expansion must be examined.
- **Non-hyperbolic points** with real zero eigenvalues ($\lambda = 0$): The dynamics can be extremely sensitive to nonlinear terms. Examples include semi-stable equilibria and bifurcation points.

Worked Example

Hartman–Grobman applies: stable spiral.
Consider $x' = -x + y + x(x^2 + y^2)$, $y' = -x - y + y(x^2 + y^2)$.
Equilibrium: $(0, 0)$ (the only one easily found).
Jacobian at $(0, 0)$*:*

$$J(0,0) = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \tau = -2,\ \Delta = 2,\ D = 4 - 8 = -4 < 0.$$

Eigenvalues: $\lambda = -1 \pm i$. Both have nonzero real part ($\Re(\lambda) = -1 \neq 0$).
Conclusion: The equilibrium is hyperbolic. By the Hartman–Grobman theorem, the nonlinear system near $(0, 0)$ is a **stable spiral**, topologically equivalent to its linearization.
(In fact, the nonlinear terms $x(x^2 + y^2)$ and $y(x^2 + y^2)$ cause outward spiraling for large radii, creating a stable limit cycle — but near the origin, the local portrait is exactly a stable spiral.)

Worked Example

Hartman–Grobman does not apply: center vs. spiral.
Consider the two systems:

$$\text{(A)} \quad x' = y, \quad y' = -x, \tag{123}$$
$$\text{(B)} \quad x' = y + x(x^2 + y^2), \quad y' = -x + y(x^2 + y^2). \tag{124}$$

Jacobian at $(0, 0)$ *for both systems:*

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda = \pm i.$$

Purely imaginary eigenvalues $\implies$ the equilibrium is **not hyperbolic**. Hartman–Grobman **does not apply**.
System (A): The linear system. Trajectories are circles $x^2 + y^2 = C$. This is a true **center**.
System (B): Convert to polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$):

$$r' = r^3, \quad \theta' = -1.$$

Since $r' = r^3 > 0$ for $r > 0$, the radius strictly increases. The origin is actually an **unstable spiral**, *not* a center, despite the linearization suggesting a center.
Lesson: When $\lambda = \pm i\beta$, the nonlinear terms determine the true behavior. Always use additional tools (Lyapunov functions, polar coordinates, first integrals) to resolve non-hyperbolic equilibria.

18.5 Limit Cycles

Definition 18.3 (Limit cycle)**.** A **limit cycle** is a **closed, isolated periodic orbit** of an autonomous system in the phase plane.

The key word is *isolated*: a limit cycle is a single closed trajectory that is not part of a continuous family of closed orbits. Nearby trajectories either spiral toward the limit cycle (stable limit cycle) or spiral away from it (unstable limit cycle).

This contrasts with the center in linear systems, where every trajectory near the equilibrium is a closed orbit forming a continuous family.

Theorem 18.4 (Poincaré–Bendixson)**.** *Let R be a closed, bounded (compact) region in the plane containing no equilibrium points, and let a trajectory enter R and remain there for all future time $t > 0$. Then the trajectory either:*

1. *approaches a periodic orbit (a closed trajectory) as $t \to \infty$, or*
2. *is itself a periodic orbit.*

Implications. The Poincaré–Bendixson theorem is a powerful tool for proving the existence of limit cycles in two-dimensional systems:

- If you can construct a trapping region R that contains no equilibria, any trajectory entering R must approach a periodic orbit.
- In the plane, the only possible long-term behaviors of bounded trajectories are: equilibrium points, periodic orbits (limit cycles), or trajectories that approach limit cycles.
- **No chaos in 2D autonomous systems:** Strange attractors and chaotic behavior require at least three dimensions.

Physical examples. Limit cycles model sustained oscillations in physical systems:

- **Electrical circuits:** The van der Pol oscillator models self-sustained oscillations in vacuum tube circuits.
- **Chemical reactions:** The Belousov–Zhabotinsky reaction exhibits periodic color changes due to a chemical limit cycle.
- **Biological rhythms:** Heartbeat, neural firing, and circadian rhythms can be modeled as limit cycles.
- **Mechanical oscillators:** A clock pendulum with a periodic driving force (the escapement mechanism) exhibits a limit cycle.

18.6 Lotka–Volterra Predator–Prey Model

The Lotka–Volterra model is the classic example of a nonlinear system with closed orbits, first introduced in the 1920s to model predator–prey population dynamics.

Biological motivation and model derivation. Consider two interacting populations:

- $x(t)$: number of **prey** (e.g., rabbits).
- $y(t)$: number of **predators** (e.g., foxes).

We make four biological assumptions:

1. In the absence of predators, prey grow exponentially at rate α: $x' = \alpha x$.
2. Predators eat prey at a rate proportional to encounters αxy. Combined with assumption 1: $x' = \alpha x - \beta xy$.
3. In the absence of prey, predators die off at rate γ: $y' = -\gamma y$.

4. Predators reproduce at a rate proportional to food consumption βxy. Combined with assumption 3: $y' = \delta xy - \gamma y$.

This yields the **Lotka–Volterra predator–prey equations**:

$$\boxed{\frac{dx}{dt} = \alpha x - \beta xy, \qquad \frac{dy}{dt} = \delta xy - \gamma y,} \tag{125}$$

where $\alpha, \beta, \gamma, \delta > 0$ are parameters.

Definition 18.5 (Lotka–Volterra model)**.** The Lotka–Volterra predator–prey system is the autonomous nonlinear system equation (125) with $\alpha, \beta, \gamma, \delta > 0$. It has two equilibrium points:

$$(0,0) \text{ (extinction)}, \qquad \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \text{ (coexistence)}.$$

Equilibrium analysis. Setting the right-hand sides to zero:

$$x(\alpha - \beta y) = 0, \tag{126}$$
$$y(\delta x - \gamma) = 0. \tag{127}$$

From equation (126), $x = 0$ or $y = \alpha/\beta$.

Case 1: $x = 0$. From equation (127), $y(-\gamma) = 0 \implies y = 0$. Equilibrium: $(0, 0)$.

Case 2: $y = \alpha/\beta$. From equation (127), $\delta x - \gamma = 0 \implies x = \gamma/\delta$. Equilibrium: $(\gamma/\delta, \alpha/\beta)$.

Jacobian and classification. The Jacobian of the system is

$$J(x,y) = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}.$$

At $(0,0)$**:**

$$J(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \lambda_1 = \alpha > 0,\ \lambda_2 = -\gamma < 0.$$

Opposite signs $\implies$ **saddle point** (unstable). Both populations at zero is unstable: a small number of prey grows, triggering predator response.

At $(\gamma/\delta, \alpha/\beta)$**:**

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\beta\gamma/\delta \\ \delta\alpha/\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{\beta\gamma}{\delta} \\ \dfrac{\delta\alpha}{\beta} & 0 \end{pmatrix}.$$

Trace: $\tau = 0$. Determinant: $\Delta = (0) - (-\beta\gamma/\delta)(\delta\alpha/\beta) = \alpha\gamma > 0$. Characteristic equation: $\lambda^2 + \alpha\gamma = 0$, so

$$\lambda = \pm i\sqrt{\alpha\gamma}.$$

Purely imaginary eigenvalues $\implies$ **center** by linearization. Hartman–Grobman does not apply. We must analyze the nonlinear system.

First integral (conservation law). The Lotka–Volterra system admits an exact first integral, proving that the orbits are indeed closed. Divide the two equations:

$$\frac{dy}{dx} = \frac{\delta xy - \gamma y}{\alpha x - \beta xy} = \frac{y(\delta x - \gamma)}{x(\alpha - \beta y)}.$$

Separate variables:

$$\frac{\alpha - \beta y}{y}\, dy = \frac{\delta x - \gamma}{x}\, dx.$$

Rewrite and integrate:

$$\left(\frac{\alpha}{y} - \beta\right) dy = \left(\delta - \frac{\gamma}{x}\right) dx.$$
$$\alpha \ln y - \beta y = \delta x - \gamma \ln x + C.$$

Rearranging gives the **first integral**:

$$V(x,y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y = \text{constant}. \tag{128}$$

Each value of C defines a closed orbit in the phase plane. The function $V(x,y)$ has a strict global minimum at the coexistence equilibrium $(\gamma/\delta, \alpha/\beta)$, and level curves $V(x,y) = C$ for $C > V_{\min}$ are closed curves surrounding this point.

Key Result

Lotka–Volterra closed orbits. The coexistence equilibrium $(\gamma/\delta, \alpha/\beta)$ is a true center for the nonlinear system. Every solution starting in the first quadrant $(x > 0, y > 0)$ traces a closed orbit around the coexistence point, determined by the initial conditions through the first integral equation (128).

Nullclines. The **nullclines** are curves in the phase plane where one of the derivatives vanishes:

- **x-nullcline** $(x' = 0)$: $x = 0$ or $y = \alpha/\beta$. On $x = 0$, the prey population is zero. On $y = \alpha/\beta$ (horizontal line), the prey population is momentarily stationary.
- **y-nullcline** $(y' = 0)$: $y = 0$ or $x = \gamma/\delta$. On $y = 0$, the predator population is zero. On $x = \gamma/\delta$ (vertical line), the predator population is momentarily stationary.

The two nontrivial nullclines intersect at the coexistence equilibrium $(\gamma/\delta, \alpha/\beta)$. The nullclines divide the first quadrant into four regions, each with a characteristic direction of motion:

- Region I $(x > \gamma/\delta, y < \alpha/\beta)$: $x' > 0$, $y' > 0 \implies$ motion up and right.
- Region II $(x < \gamma/\delta, y < \alpha/\beta)$: $x' > 0$, $y' < 0 \implies$ motion down and right.
- Region III $(x < \gamma/\delta, y > \alpha/\beta)$: $x' < 0$, $y' < 0 \implies$ motion down and left.
- Region IV $(x > \gamma/\delta, y > \alpha/\beta)$: $x' < 0$, $y' > 0 \implies$ motion up and left.

The resulting flow is counterclockwise around the equilibrium.

Worked Example

Full analysis of the Lotka–Volterra system with parameters $\alpha = 1.5$, $\beta = 1$, $\delta = 3$, $\gamma = 1$.
System:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1.5\,x - xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3\,xy - y.$$

Equilibria:

$$(0,0) \text{ and } \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \left(\frac{1}{3}, \frac{1.5}{1}\right) = \left(\frac{1}{3}, 1.5\right).$$

Jacobian:

$$J(x,y) = \begin{pmatrix} 1.5 - y & -x \\ 3y & 3x - 1 \end{pmatrix}.$$

At $(0,0)$**:**

$$J(0,0) = \begin{pmatrix} 1.5 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \lambda_1 = 1.5 > 0,\ \lambda_2 = -1 < 0.$$

Saddle point (unstable).
At $(1/3, 1.5)$**:**

$$J\left(\tfrac{1}{3}, 1.5\right) = \begin{pmatrix} 0 & -1/3 \\ 4.5 & 0 \end{pmatrix}.$$

$$\tau = 0, \quad \Delta = (0) - (-1/3)(4.5) = 1.5.$$

$$\lambda = \pm i\sqrt{1.5} \approx \pm 1.225\,i.$$

Purely imaginary eigenvalues. The coexistence point is a **center** (confirmed by the first integral, not just linearization).
Nullclines:

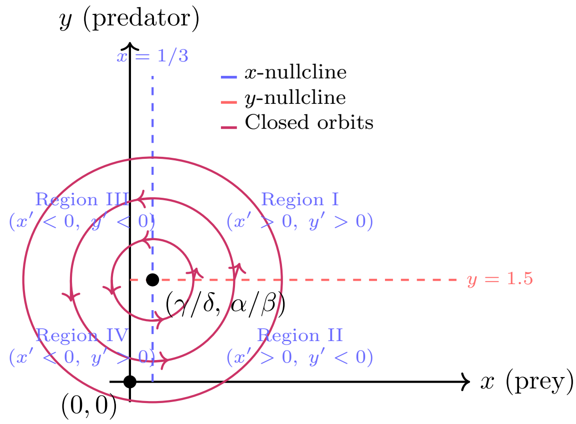
- x-nullcline: $x = 0$ or $y = 1.5$ (horizontal line).
- y-nullcline: $y = 0$ or $x = 1/3$ (vertical line).

First integral:

$$V(x,y) = 3x - \ln x + y - 1.5\,\ln y = \text{constant}.$$

Period of oscillation: Near the equilibrium, the period is approximately $T \approx 2\pi/\sqrt{\alpha\gamma} = 2\pi/\sqrt{1.5} \approx 5.13$ time units.

Phase portrait.



Interpretation of the oscillations. The closed orbits correspond to periodic predator–prey cycles:

1. Prey population grows (few predators) $\to$ predators have abundant food.
2. Predator population grows $\to$ predation pressure increases.
3. Prey population declines $\to$ predators face food shortage.
4. Predator population declines $\to$ prey can recover.
5. Cycle repeats.

The amplitude of the oscillation is determined by the initial conditions: the further the initial point from the equilibrium, the larger the orbit.

Limitations of the model. The Lotka–Volterra model has several simplifications:

- Unlimited prey growth in the absence of predators (no carrying capacity).
- Linear functional response (predation rate proportional to xy).
- No time delays in predator reproduction.
- Homogeneous mixing (well-mixed populations).

Realistic extensions include logistic prey growth, Holling-type functional responses, and more complex age-structured models. These modifications can create stable limit cycles (rather than neutral centers) and richer dynamics.

18.7 Summary

Nonlinear systems cannot be solved by superposition, but their qualitative behavior near equilibria is accessible through linearization. The Jacobian matrix, combined with the trace-determinant classification, provides a systematic framework for understanding local dynamics. The Hartman–Grobman theorem guarantees that this linear picture is topologically correct for hyperbolic equilibria. Global phenomena like limit cycles require additional tools: the Poincaré–Bendixson theorem, first integrals, and Lyapunov functions.

Hint

Problem-solving workflow for nonlinear autonomous systems.

1. Find all equilibrium points by solving $f(x, y) = 0$, $g(x, y) = 0$.
2. Compute the Jacobian matrix $J(x, y)$ and evaluate it at each equilibrium.
3. Classify each equilibrium using eigenvalues or the trace-determinant plane.
4. Apply the Hartman–Grobman theorem: if the equilibrium is hyperbolic, the local phase portrait matches the linear classification.
5. For non-hyperbolic equilibria ($\Re(\lambda) = 0$), use additional tools: first integrals, Lyapunov functions, or polar coordinates.

Table 25: Chapter 14 Summary: Nonlinear Systems

Concept	Key Formula/Method
Nonlinear autonomous system	$x' = f(x, y)$, $y' = g(x, y)$, no superposition
Equilibrium point	Solve $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$
Jacobian matrix	$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$
Linearized system	$\begin{pmatrix} u' \\ v' \end{pmatrix} = J(x^*, y^*) \begin{pmatrix} u \\ v \end{pmatrix}$, $u = x - x^*$, $v = y - y^*$
Trace-determinant	$\tau = \mathrm{tr}(J)$, $\Delta = \det(J)$, $D = \tau^2 - 4\Delta$
Hartman–Grobman theorem	Hyperbolic equilibria ($\Re(\lambda) \neq 0$): nonlinear $\cong$ linearization
Non-hyperbolic case	$\Re(\lambda) = 0$: linearization inconclusive; use first integrals, Lyapunov functions
Limit cycle	Closed, isolated periodic orbit
Poincaré–Bendixson theorem	Bounded trajectory in 2D with no equilibria $\implies$ periodic orbit
Lotka–Volterra model	$x' = \alpha x - \beta xy$, $y' = \delta xy - \gamma y$
LV equilibria	$(0, 0)$ saddle; $(\gamma/\delta, \alpha/\beta)$ center (closed orbits)
LV first integral	$V(x, y) = \delta x - \gamma \ln x + \beta y - \alpha \ln y = C$
LV nullclines	x-nullcline: $y = \alpha/\beta$; y-nullcline: $x = \gamma/\delta$

6. Draw nullclines to understand the flow direction in each region.
7. Look for limit cycles using the Poincaré–Bendixson theorem (requires constructing a trapping region).
8. Sketch the full phase portrait combining local and global information.

A Solution Method Summary

This appendix provides comprehensive quick-reference tables for every solution method covered in the handbook. Use these tables to identify the correct method for a given equation type and to recall key formulas.

A.1 First-Order ODE Methods

Table 26: First-Order Solution Methods (see section 6)

Method	Equation Form	Solution / Procedure	When to Use
Separable	$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)\,h(y)$	$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x + C$	RHS factors as $g(x) \cdot h(y)$
Linear	$y' + p(x)y = g(x)$	$y = \frac{\int \mu(x)g(x)\,\mathrm{d}x + C}{\mu(x)}$, $\mu = \exp\big(\int p(x)\,\mathrm{d}x\big)$	Can be written in standard linear form
Exact	$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	Find $\psi(x, y)$ such that $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$; solution is $\psi(x, y) = C$	Exactness test $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ passes
Bernoulli	$y' + p(x)y = g(x)\,y^n$	Substitute $v = y^{1-n}$; equation becomes linear in v	RHS is a power of y times a function of x
Homogeneous	$\frac{\mathrm{d}y}{\mathrm{d}x} = F\Big(\frac{y}{x}\Big)$	Substitute $y = vx$; equation becomes separable in v	RHS is a function of the ratio y/x

A.2 Second-Order Linear Homogeneous (Constant Coefficients)

Consider the equation $a\,y'' + b\,y' + c\,y = 0$ with $a, b, c \in \mathbb{R}$, $a \neq 0$.

Table 27: Characteristic Equation Cases (see section 8)

Discriminant	Roots of $ar^2 + br + c = 0$	General Solution	Phase / Behavior
$b^2 - 4ac > 0$ (distinct real)	$r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	Exponential growth/decay; overdamped
$b^2 - 4ac = 0$ (repeated)	$r = -\dfrac{b}{2a}$	$y = c_1 e^{rx} + c_2 x\, e^{rx}$	Critically damped
$b^2 - 4ac < 0$ (complex conjugate)	$r = \alpha \pm i\beta$, $\alpha = -\dfrac{b}{2a}$, $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$	$y = e^{\alpha x}\big[c_1 \cos(\beta x) + c_2 \sin(\beta x)\big]$	Damped oscillation; underdamped

Key Result

Superposition principle. For $L[y] = 0$ linear homogeneous, any linear combination of solutions is again a solution. A *fundamental set* $\{y_1, y_2\}$ has $W(y_1, y_2) \neq 0$, and the general solution is $y = c_1 y_1 + c_2 y_2$.

A.3 Second-Order Linear Nonhomogeneous

Consider $a\,y'' + b\,y' + c\,y = g(x)$. The general solution is $y = y_h + y_p$, where y_h is the homogeneous solution and y_p is a particular solution.

Table 28: Nonhomogeneous Solution Methods (see section 9)

Method	Formula / Procedure	When to Use
Undetermined Coefficients	See table 29 for guess table. Plug y_p into ODE, solve for unknown coefficients.	$g(x)$ is a polynomial, exponential, sine/cosine, or finite sums/products thereof
Variation of Parameters	$y_p(x) = -y_1(x)\int \frac{y_2(x)\,g(x)}{a\,W(x)}\,\mathrm{d}x + y_2(x)\int \frac{y_1(x)\,g(x)}{a\,W(x)}\,\mathrm{d}x$ where $W = y_1 y_2' - y_2 y_1'$.	Any $g(x)$ for which the integrals can be evaluated; requires the fundamental set $\{y_1, y_2\}$

Table 29: Guess Table for Undetermined Coefficients (see section 9)

Form of $g(x)$	Guess for $y_p(x)$
$P_n(x)$ (polynomial of degree n)	$A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$
$A\,e^{kx}$	$C\,e^{kx}$
$A \sin(\omega x)$ or $A \cos(\omega x)$	$C_1 \cos(\omega x) + C_2 \sin(\omega x)$
$A\,e^{kx} \sin(\omega x)$ or $A\,e^{kx} \cos(\omega x)$	$e^{kx}\big[C_1 \cos(\omega x) + C_2 \sin(\omega x)\big]$
Sums of the above	Sum of the corresponding guesses

Key Result

Modification rule. If any term of the guess y_p is already a solution of the homogeneous equation (i.e. duplicates a term in y_h), multiply the *entire* corresponding group by x. If x-multiplication still duplicates, multiply by x^2, and so on.

A.4 Laplace Transform Summary

Table 30: Common Laplace Transforms (see section 11)

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
$t^n \quad (n \in \mathbb{N})$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
$t\,\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$t\,\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
$u(t-a)$ (unit step)	$\frac{e^{-as}}{s}$
$\delta(t-a)$ (Dirac delta)	e^{-as}

Key Result

Solving IVPs with Laplace. (1) Take the Laplace transform of the ODE (using derivative formulas to incorporate initial conditions). (2) Solve the resulting algebraic equation for $Y(s)$. (3) Find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ via partial fractions and the transform table.

A.5 Systems of Linear ODEs

For $\mathbf{x}' = A\,\mathbf{x}$ with A a constant $n \times n$ matrix (see section 12):

Key Result

Matrix exponential. The solution to $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0$, where $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$. When A is diagonalizable, $e^{At} = P\,e^{\Lambda t}\,P^{-1}$ with Λ diagonal.

Table 31: Laplace Transform Properties (see section 11)

Property	**Formula**
Linearity	$\mathcal{L}\{a\,f(t) + b\,g(t)\} = a\,F(s) + b\,G(s)$
First Shift (exponential)	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Second Shift (time delay)	$\mathcal{L}\{u(t-a)\,f(t-a)\} = e^{-as}F(s)$
Derivatives	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ $\mathcal{L}\{f''(t)\} = s^2F(s) - s\,f(0) - f'(0)$ $\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Multiplication by t^n	$\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{\mathrm{d}^n}{\mathrm{d}s^n}F(s)$
Convolution	$\mathcal{L}\{f*g\} = F(s)\,G(s), \quad (f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,\mathrm{d}\tau$
Integral	$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \dfrac{F(s)}{s}$

Table 32: Eigenvalue Cases for 2×2 Systems (see section 12)

Eigenvalue Case	**Eigenvalues / Vectors**	**General Solution**
Real distinct	$\lambda_1 \neq \lambda_2 \in \mathbb{R}$; eigenvectors $\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{x}(t) = c_1e^{\lambda_1 t}\,\mathbf{v}_1 + c_2e^{\lambda_2 t}\,\mathbf{v}_2$
Complex conjugate	$\lambda = \alpha \pm i\beta$; eigenvector $\mathbf{a} \pm i\mathbf{b}$	$\mathbf{x}(t) = c_1\big[e^{\alpha t}\cos(\beta t)\,\mathbf{a} - e^{\alpha t}\sin(\beta t)\,\mathbf{b}\big] + c_2\big[e^{\alpha t}\sin(\beta t)\,\mathbf{a} + e^{\alpha t}\cos(\beta t)\,\mathbf{b}\big]$
Repeated (diagonalizable)	$\lambda_1 = \lambda_2 = \lambda$; two independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$	$\mathbf{x}(t) = c_1e^{\lambda t}\,\mathbf{v}_1 + c_2e^{\lambda t}\,\mathbf{v}_2$
Repeated (defective)	λ (single eigenvector $\mathbf{v}_1$; generalized eigenvector $\mathbf{v}_2$)	$\mathbf{x}(t) = c_1e^{\lambda t}\,\mathbf{v}_1 + c_2e^{\lambda t}\big(t\,\mathbf{v}_1 + \mathbf{v}_2\big)$

Table 33: Series Solution Methods (see section 13)

Method	**Equation Type**	**Solution Form**	**Key Details**
Power series	$p(x)$ and $q(x)$ analytic at ordinary point x_0 in $y'' + p(x)y' + q(x)y = 0$	$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$	Substitute into ODE, equate coefficients to find recurrence for a_n
Euler–Cauchy	$x^2y'' + \alpha xy' + \beta y = 0$, $x > 0$	Try $y = x^r$; indicial equation $r(r-1) + \alpha r + \beta = 0$	Distinct roots $r_1 \neq r_2$: $y = c_1x^{r_1} + c_2x^{r_2}$. Repeated: $y = x^r(c_1 + c_2\ln x)$
Frobenius	Regular singular point at x_0 in $y'' + p(x)y' + q(x)y = 0$	$y = x^r\sum_{n=0}^{\infty} a_nx^n, \quad a_0 \neq 0$	Indicial equation from lowest power. Cases depend on $r_1 - r_2$: non-integer, zero, or positive integer

Table 34: Linearized Classification via Jacobian (see section 18)

Type	Trace $T = \mathrm{tr}(J)$	Determinant $D = \det(J)$	Eigenvalue Pattern
Stable node	$T < 0$	$D > 0$, $T^2 - 4D > 0$	$\lambda_1, \lambda_2 < 0$, real
Unstable node	$T > 0$	$D > 0$, $T^2 - 4D > 0$	$\lambda_1, \lambda_2 > 0$, real
Saddle point	–	$D < 0$	$\lambda_1 < 0 < \lambda_2$, real
Stable spiral	$T < 0$	$D > 0$, $T^2 - 4D < 0$	$\alpha \pm i\beta$, $\alpha < 0$
Unstable spiral	$T > 0$	$D > 0$, $T^2 - 4D < 0$	$\alpha \pm i\beta$, $\alpha > 0$
Center	$T = 0$	$D > 0$	$\pm i\beta$ (pure imaginary)

A.6 Series Solutions

A.7 Nonlinear Systems

Key Result

Hartman–Grobman theorem. Near a hyperbolic equilibrium point ($\Re(\lambda_i) \neq 0$), the nonlinear system is topologically conjugate to its linearization. The phase portrait of the linearized system accurately describes the local behavior.

A.8 PDE Solution Methods

Table 35: PDE Methods and Solution Forms (see section 16, section 17)

Equation	Domain	Method	Solution Form
Heat equation $u_t = \alpha\, u_{xx}$	$0 < x < L$	Separation of variables + Fourier series	$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big) \exp\Big(-\alpha \frac{n^2\pi^2}{L^2} t\Big)$
Wave equation $u_{tt} = c^2\, u_{xx}$	$x \in \mathbb{R}$ (infinite)	d'Alembert's formula	$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$
Wave equation $u_{tt} = c^2\, u_{xx}$	$0 < x < L$ (finite, Dirichlet)	Separation + Fourier sine series	$u(x,t) = \sum_{n=1}^{\infty} \sin\Big(\frac{n\pi x}{L}\Big)\big[A_n \cos(\frac{n\pi ct}{L}) + B_n \sin(\frac{n\pi ct}{L})\big]$
Laplace $u_{xx} + u_{yy} = 0$	Rectangle $0 < x < a$, $0 < y < b$	Separation of variables	$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi x}{a}\Big) \sinh\Big(\frac{n\pi y}{a}\Big)$
Laplace $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$	Disk $r < a$	Polar separation	$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\big[A_n \cos(n\theta) + B_n \sin(n\theta)\big]$

Key Result

Separation of variables (general procedure). For a PDE $F(x,t,u,u_x,u_t,\dots) = 0$, assume $u(x,t) = X(x)T(t)$. Substitute and separate variables to obtain two ODEs, one for X and one for T, linked by a separation constant. Solve the spatial ODE as a Sturm–Liouville eigenvalue problem (section 15), then expand the initial/boundary data in the resulting eigenfunction basis.

Table 36: Damping Cases for Spring-Mass Systems (see section 10)

Case	**Condition**	**Solution** $y(t)$	**Behavior**
Undamped	$c = 0$	$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),\ \ \omega_0 = \sqrt{k/m}$	Persistent oscillation
Underdamped	$0 < c < 2\sqrt{km}$	$y(t) = e^{-ct/(2m)}\big[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\big],\ \ \omega_d = \sqrt{\omega_0^2 - (c/2m)^2}$	Decaying oscillation
Critically damped	$c = 2\sqrt{km}$	$y(t) = C_1 e^{-\omega_0 t} + C_2 t\, e^{-\omega_0 t}$	Fastest decay without oscillation
Overdamped	$c > 2\sqrt{km}$	$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},\ \ r_{1,2} = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$	Slow non-oscillatory decay

Table 37: Mechanical–Electrical Analogy (see section 10)

Mechanical	**Electrical (charge)**	**Mechanical**	**Electrical (flux)**
Mass m	Inductance L	Mass m	Capacitance C
Damping c	Resistance R	Damping c	Conductance G
Spring constant k	Inverse capacitance $1/C$	Spring constant k	Inverse inductance $1/L$
Force $F(t)$	Voltage $E(t)$	Force $F(t)$	Current $I(t)$
Displacement $y(t)$	Charge $q(t)$	Velocity $v(t)$	Flux $\phi(t)$

A.9 Mechanical and Electrical Applications

A.10 Qualitative Analysis and Numerical Methods

Table 38: Equilibrium Classification for $y' = f(y)$ (see section 7)

Condition at y^* (where $f(y^*) = 0$)	**Stability**	**Phase Line Behavior**
$f'(y^*) < 0$	Asymptotically stable (sink)	Solutions near y^* converge to y^*
$f'(y^*) > 0$	Unstable (source)	Solutions near y^* diverge away from y^*
$f'(y^*) = 0$	Inconclusive (semi-stable or higher-order)	Requires higher-order analysis; may be a node or saddle

A.11 Fourier Series Summary

B Transform and Integral Tables

This appendix provides comprehensive reference tables for Laplace transforms (section 11), Fourier series formulas (section 14), and common integrals used throughout the handbook. All entries are presented without derivation; see the referenced chapters for proofs and worked examples.

B.1 Laplace Transform Table

The Laplace transform is defined in equation (48) as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t,$$

provided the integral converges. The following table lists the most frequently encountered transforms.

Table 39: Euler's Method for Numerical Approximation (see section 7)

Algorithm
Given $y' = f(t, y)$, $y(t_0) = y_0$, step size h:
$t_{n+1} = t_n + h$
$y_{n+1} = y_n + h\, f(t_n, y_n)$
Local truncation error: $\mathcal{O}(h^2)$
Global error: $\mathcal{O}(h)$

Table 40: Fourier Series Coefficients (see section 14)

Series	**Coefficients**
Full range $[-L, L]$: $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big[a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\Big]$	$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x$
	$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
	$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
Even f (cosine series):	$b_n = 0,\ \ a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
Odd f (sine series):	$a_n = 0,\ \ b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$

B.2 Laplace Transform Properties

The following algebraic and operational properties make the Laplace transform a powerful tool for solving linear differential equations.

The convolution integral $(f * g)(t)$ appearing in the table above is defined by

$$(f * g)(t) = \int_0^t f(t - \tau)\, g(\tau)\,\mathrm{d}\tau.$$

B.3 Fourier Series Formulas

Fourier series decompose a periodic function $f(x)$ of period $2L$ into sine and cosine harmonics, as developed in section 14. The following tables collect the most essential formulas.

B.4 Common Integral Table

The following integrals are used throughout the handbook, particularly in separation of variables, integrating factor methods, and Fourier coefficient calculations.

Key Result

Integration by parts reminder. For products of functions that appear in Fourier coefficient computations:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

This formula underlies the xe^{ax}, $x\sin(ax)$, and $x\cos(ax)$ entries in table 44 and is essential for evaluating Fourier coefficients of piecewise-linear functions in section 14.

C Integral Tables

This appendix provides commonly used integral formulas encountered throughout the handbook. See section 6 for separable equations, section 8 and section 9 for second-order ODE techniques, section 11 for Laplace-domain integration, and section 14 for trigonometric integrals in series expansions.

Table 41: Common Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$	**Conditions**
1	$\frac{1}{s}$	$s > 0$
t	$\frac{1}{s^2}$	$s > 0$
t^n	$\frac{n!}{s^{n+1}}$	$s > 0,\ n \in \mathbb{N}$
e^{at}	$\frac{1}{s-a}$	$s > a$
$t\,e^{at}$	$\frac{1}{(s-a)^2}$	$s > a$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$	$s > \lvert b \rvert$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$	$s > \lvert b \rvert$
$t\ \sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$	$s > 0$
$t\ \cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$
$u_c(t)$ (unit step)	$\frac{e^{-cs}}{s}$	$s > 0$
$\delta(t-a)$ (Dirac)	e^{-as}	$a \geq 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a,\ n \in \mathbb{N}$
$\sin^2(bt)$	$\frac{2b^2}{s(s^2+4b^2)}$	$s > 0$
$\cos^2(bt)$	$\frac{s^2+2b^2}{s(s^2+4b^2)}$	$s > 0$

Table 42: Laplace Transform Properties

Property	Formula
Linearity	$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
First shift	$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$
Second shift	$\mathcal{L}\{u_c(t)\,f(t-c)\} = e^{-cs}F(s)$
Derivative	$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
Second derivative	$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
nth derivative	$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Integral	$\mathcal{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \dfrac{F(s)}{s}$
Convolution	$\mathcal{L}\{(f * g)(t)\} = F(s)\,G(s)$
t-multiplication	$\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{\mathrm{d}^n F}{\mathrm{d}s^n}$
Frequency differentiation	$\mathcal{L}\{t\,f(t)\} = -\dfrac{\mathrm{d}F}{\mathrm{d}s}$
Final value theorem	$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$
Initial value theorem	$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

C.1 Basic Integrals

C.2 Integrals Involving Exponential and Trigonometric Functions

C.3 Integrals Involving Inverse Trigonometric Functions

C.4 Hyperbolic Integrals

C.5 Additional Common Integrals

C.6 Integration by Parts Formula

The method of integration by parts, frequently used in solving ODEs (section 6), is based on the product rule:

$$\int u\,dv = uv - \int v\,du. \tag{129}$$

Choose u and dv so that $\int v\,du$ is simpler than the original integral.

C.7 Reduction Formulas

Reduction formulas allow integrals of higher powers to be expressed in terms of lower powers:

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx, \tag{130}$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx, \tag{131}$$

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx. \tag{132}$$

These formulas are especially useful when solving higher-order ODEs (sections 8 and 9) and evaluating Fourier coefficients (section 14).

D Notation Glossary

This appendix provides a glossary of notation used throughout the handbook.

Table 43: Fourier Series on $[-L, L]$

Formula	**Expression**
Full series	$f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \Big[a_n \cos\Big(\frac{n\pi x}{L}\Big) + b_n \sin\Big(\frac{n\pi x}{L}\Big) \Big]$
a_0	$a_0 = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x)\,\mathrm{d}x$
a_n	$a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
b_n	$b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
Half-range cosine	$a_n = \dfrac{2}{L} \displaystyle\int_{0}^{L} f(x) \cos\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
Half-range sine	$b_n = \dfrac{2}{L} \displaystyle\int_{0}^{L} f(x) \sin\Big(\frac{n\pi x}{L}\Big)\,\mathrm{d}x$
Complex coefficients	$c_n = \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\, e^{-in\pi x/L}\,\mathrm{d}x$
Complex series	$f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L}$
Parseval's identity	$\dfrac{1}{L} \displaystyle\int_{-L}^{L} \lvert f(x) \rvert^2\,\mathrm{d}x = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \big(a_n^2 + b_n^2\big)$

Table 44: Common Indefinite Integrals

Integrand	Result
$\int x^n \, \mathrm{d}x$	$\frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
$\int \frac{1}{x} \, \mathrm{d}x$	$\ln \lvert x \rvert + C$
$\int e^{ax} \, \mathrm{d}x$	$\frac{1}{a} e^{ax} + C$
$\int \sin(ax) \, \mathrm{d}x$	$-\frac{1}{a} \cos(ax) + C$
$\int \cos(ax) \, \mathrm{d}x$	$\frac{1}{a} \sin(ax) + C$
$\int \sec^2(ax) \, \mathrm{d}x$	$\frac{1}{a} \tan(ax) + C$
$\int \csc^2(ax) \, \mathrm{d}x$	$-\frac{1}{a} \cot(ax) + C$
$\int \tan(ax) \, \mathrm{d}x$	$-\frac{1}{a} \ln \lvert \cos(ax) \rvert + C$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x$	$\arcsin\left(\frac{x}{a}\right) + C$
$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
$\int \frac{1}{x\sqrt{x^2 - a^2}} \, \mathrm{d}x$	$\frac{1}{a} \operatorname{arcsec}\left(\frac{\lvert x \rvert}{a}\right) + C$
$\int \sqrt{a^2 - x^2} \, \mathrm{d}x$	$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin\left(\frac{x}{a}\right) + C$
$\int e^{ax} \sin(bx) \, \mathrm{d}x$	$\frac{e^{ax}}{a^2 + b^2}\big(a \sin(bx) - b \cos(bx)\big) + C$
$\int e^{ax} \cos(bx) \, \mathrm{d}x$	$\frac{e^{ax}}{a^2 + b^2}\big(a \cos(bx) + b \sin(bx)\big) + C$
$\int x e^{ax} \, \mathrm{d}x$	$\frac{e^{ax}}{a^2}\big(ax - 1\big) + C$
$\int x \sin(ax) \, \mathrm{d}x$	$\frac{\sin(ax)}{a^2} - \frac{x \cos(ax)}{a} + C$
$\int x \cos(ax) \, \mathrm{d}x$	$\frac{\cos(ax)}{a^2} + \frac{x \sin(ax)}{a} + C$
$\int \ln(x) \, \mathrm{d}x$	$x \ln(x) - x + C$

Table 45: Basic Integral Formulas

Integrand	**Result**
$\int x^n \, dx$	$\frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \frac{1}{x} \, dx$	$\ln\lvert x\rvert + C$
$\int e^{ax} \, dx$	$\frac{e^{ax}}{a} + C$
$\int a^x \, dx$	$\frac{a^x}{\ln(a)} + C$
$\int \sin(ax) \, dx$	$-\frac{\cos(ax)}{a} + C$
$\int \cos(ax) \, dx$	$\frac{\sin(ax)}{a} + C$
$\int \tan(ax) \, dx$	$-\frac{\ln\lvert\cos(ax)\rvert}{a} + C = \frac{\ln\lvert\sec(ax)\rvert}{a} + C$
$\int \sec^2(ax) \, dx$	$\frac{\tan(ax)}{a} + C$
$\int \csc^2(ax) \, dx$	$-\frac{\cot(ax)}{a} + C$
$\int \sec(ax)\tan(ax) \, dx$	$\frac{\sec(ax)}{a} + C$
$\int \csc(ax)\cot(ax) \, dx$	$-\frac{\csc(ax)}{a} + C$

Table 46: Exponential–Trigonometric Integrals

Integrand	**Result**
$\int e^{ax}\sin(bx) \, dx$	$\frac{e^{ax}\big(a\sin(bx) - b\cos(bx)\big)}{a^2+b^2} + C$
$\int e^{ax}\cos(bx) \, dx$	$\frac{e^{ax}\big(a\cos(bx) + b\sin(bx)\big)}{a^2+b^2} + C$
$\int xe^{ax} \, dx$	$e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right) + C$
$\int x\sin(ax) \, dx$	$\frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a} + C$
$\int x\cos(ax) \, dx$	$\frac{\cos(ax)}{a^2} + \frac{x\sin(ax)}{a} + C$
$\int \ln(x) \, dx$	$x\ln(x) - x + C$

Table 47: Inverse Trigonometric Integrals

Integrand	**Result**
$\int \frac{1}{\sqrt{a^2-x^2}} \, dx$	$\arcsin\left(\frac{x}{a}\right) + C$
$\int \frac{1}{a^2+x^2} \, dx$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\int \frac{1}{x\sqrt{x^2-a^2}} \, dx$	$\frac{1}{a}\operatorname{arcsec}\left(\frac{\lvert x\rvert}{a}\right) + C$
$\int \sqrt{a^2-x^2} \, dx$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{x^2+a^2}} \, dx$	$\ln\left\lvert x + \sqrt{x^2+a^2}\,\right\rvert + C = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$

Table 48: Hyperbolic Integral Formulas

Integrand	**Result**
$\int \sinh(ax)\,dx$	$\dfrac{\cosh(ax)}{a} + C$
$\int \cosh(ax)\,dx$	$\dfrac{\sinh(ax)}{a} + C$
$\int \tanh(ax)\,dx$	$\dfrac{\ln\lvert\cosh(ax)\rvert}{a} + C$
$\int \operatorname{sech}^2(ax)\,dx$	$\dfrac{\tanh(ax)}{a} + C$
$\int \operatorname{csch}^2(ax)\,dx$	$-\dfrac{\coth(ax)}{a} + C$
$\int \operatorname{sech}(ax)\tanh(ax)\,dx$	$-\dfrac{\operatorname{sech}(ax)}{a} + C$

Table 49: Additional Common Integral Formulas

Integrand	**Result**
$\int \dfrac{1}{x^2 - a^2}\,dx$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert + C$
$\int \dfrac{1}{a^2 - x^2}\,dx$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert + C$
$\int \dfrac{x}{\sqrt{a^2 + x^2}}\,dx$	$\sqrt{a^2 + x^2} + C$
$\int \dfrac{dx}{x\sqrt{a^2 + x^2}}$	$-\dfrac{1}{a}\ln\left\lvert\dfrac{a + \sqrt{a^2 + x^2}}{\lvert x\rvert}\right\rvert + C$
$\int \sqrt{x^2 + a^2}\,dx$	$\dfrac{x}{2}\sqrt{x^2 + a^2} + \dfrac{a^2}{2}\ln\lvert x + \sqrt{x^2 + a^2}\,\rvert + C$
$\int \sqrt{x^2 - a^2}\,dx$	$\dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\ln\lvert x + \sqrt{x^2 - a^2}\,\rvert + C$

Nomenclature

$*$	Convolution operator
α, β, γ	General constants; damping ratio
δ	Dirac delta function
λ	Eigenvalue; parameter in characteristic equation
$\mathbf{A}$	Coefficient matrix in systems
$\mathbf{u}, \mathbf{v}$	Vectors in phase plane
$\mathbf{x}(t)$	State vector
$\mathcal{F}$	Fourier transform operator
$\mathcal{L}$	Laplace transform operator
$\mathcal{L}^{-1}$	Inverse Laplace transform
μ	Separation constant; parameter
ω	Angular frequency
ω_0	Natural angular frequency
ϕ	Eigenfunction; angle
θ	Angle; phase shift
A, B, C	General constants of integration
a, b, c	Coefficients in differential equations
$e^{\mathbf{A}t}$	Matrix exponential
$F(s)$	Laplace transform of $f(t)$
$f(t)$	Input/forcing function
I_n	Identity matrix of size n
n, m, k	Integer indices
t	Independent variable (time)
$u(t)$	Unit step (Heaviside) function
$u(t-a)$	Shifted unit step function
$W(y_1, y_2)$	Wronskian of y_1 and y_2
x	Independent variable (spatial)
y	Dependent variable; unknown function
$y(t)$	Dependent variable as a function of time t
$y(x)$	Dependent variable as a function of spatial variable x
y_h	Homogeneous solution
y_p	Particular solution