AP Physics C Handbook

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Part I

Mechanics

Chapter 1

Mechanics

1.1 Kinematics

Kinematics describes motion by first choosing a reference frame and then tracking how an object's position vector $\vec{r}(t)$ changes with time. In AP Physics C: Mechanics, the focus is on motion in inertial frames, with 1D and 2D models that connect physical situations to equations, graphs, and vector components.

This unit develops the core chain of ideas for motion: define position and displacement, differentiate locally to obtain $\vec{v}$ and $\vec{a}$, interpret motion graphs, use constant-acceleration relationships when applicable, and extend the same framework to relative motion and projectile motion with negligible air resistance.

1.1.1 Scalars, Vectors, and Components

This subsection introduces the language used throughout Unit 1 for quantities that have magnitude only and for quantities that also have direction.

Definition 1.1.1: Scalars, vectors, and component form

A **scalar** quantity is specified completely by a magnitude. Examples include mass, time, temperature, energy, and speed.
A **vector** quantity is specified by both a magnitude and a direction. Examples include displacement, velocity, acceleration, and force.
In a chosen Cartesian coordinate system, let x, y, and z denote coordinates along the x-, y-, and z-axes, and let $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ denote unit vectors along those axes. If $\vec{v}$ is a vector with scalar components v_x, v_y, and v_z, then its unit-vector decomposition is

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}.$$

In two dimensions,

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath}.$$

The numbers v_x, v_y, and v_z are **components** of $\vec{v}$; they are scalars and can be positive, negative, or zero. The magnitude of $\vec{v}$ is written $|\vec{v}|$.

Note:-

Speed is a scalar, but velocity is a vector. A component such as v_x is a scalar, not a vector by itself. Also, the magnitude $|\vec{v}|$ is not the same thing as a component: $|\vec{v}| \geqslant 0$, while a component can be negative if the vector points partly in a negative coordinate direction. Likewise, distance is a scalar, while displacement $\Delta\vec{r}$ is a vector. Later in this unit, $\Delta\vec{r}$, $\vec{v}$, and $\vec{a}$ will all be handled with the same component ideas.

Proposition 1.1.1 Operational rules in components

Let

$$\vec{u} = u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{k} \qquad \text{and} \qquad \vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

be vectors written in the same Cartesian coordinate system.

① Vector addition and subtraction are done component-by-component:

$$\vec{u} + \vec{v} = (u_x + v_x)\hat{\imath} + (u_y + v_y)\hat{\jmath} + (u_z + v_z)\hat{k},$$

$$\vec{u} - \vec{v} = (u_x - v_x)\hat{\imath} + (u_y - v_y)\hat{\jmath} + (u_z - v_z)\hat{k}.$$

② In Cartesian coordinates, the magnitude of a vector comes from its components:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} \qquad \text{in 2D},$$

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \qquad \text{in 3D}.$$

③ In two dimensions, if θ is the direction angle of $\vec{v}$ measured from the positive x-axis, then

$$\tan\theta = \frac{v_y}{v_x},$$

provided $v_x \neq 0$. The signs of v_x and v_y determine the correct quadrant for θ. If $v_x = 0$, then the vector points straight along the positive or negative y-axis.

Question 1: Worked example

In an xy coordinate system, the positive x-axis points east and the positive y-axis points north. A student first moves with displacement

$$\Delta\vec{r}_1 = (3.0\hat{\imath} + 4.0\hat{\jmath})\,\text{m}$$

and then moves with displacement

$$\Delta\vec{r}_2 = (-1.0\hat{\imath} + 2.0\hat{\jmath})\,\text{m}.$$

Let $\Delta\vec{r}_{\text{tot}}$ denote the total displacement, and let d denote the total distance traveled.

(a) Identify whether $\Delta\vec{r}_{\text{tot}}$, the component $(\Delta\vec{r}_2)_x = -1.0\,\text{m}$, and d are scalars or vectors.

(b) Find $\Delta\vec{r}_{\text{tot}}$ in component form.

(c) Find $|\Delta\vec{r}_{\text{tot}}|$.

(d) Let θ be the direction of $\Delta\vec{r}_{\text{tot}}$ measured counterclockwise from the positive x-axis. Find θ.

Solution: For part (a), $\Delta\vec{r}_{\text{tot}}$ is a vector because it has both magnitude and direction. The quantity $(\Delta\vec{r}_2)_x = -1.0\,\text{m}$ is a scalar because it is one component of a vector. The quantity d is also a scalar because distance gives only a path length.

For part (b), add the two displacements by components:

$$\Delta\vec{r}_{\text{tot}} = \Delta\vec{r}_1 + \Delta\vec{r}_2.$$

Therefore,

$$\Delta\vec{r}_{\text{tot}} = (3.0 - 1.0)\hat{\imath} + (4.0 + 2.0)\hat{\jmath}.$$

So,

$$\Delta\vec{r}_{\text{tot}} = (2.0\hat{\imath} + 6.0\hat{\jmath})\,\text{m}.$$

For part (c), use the magnitude formula in two dimensions:

$$|\Delta\vec{r}_{\text{tot}}| = \sqrt{(2.0\,\text{m})^2 + (6.0\,\text{m})^2}.$$

Thus,

$$|\Delta\vec{r}_{\text{tot}}| = \sqrt{40}\,\text{m} = 6.32\,\text{m}.$$

For part (d), use the component ratio. Since both components of $\Delta\vec{r}_{\text{tot}}$ are positive, the vector lies in the first quadrant. Then

$$\tan\theta = \frac{6.0}{2.0} = 3.0.$$

So,

$$\theta = \tan^{-1}(3.0) = 71.6^\circ.$$

Therefore, the total displacement is

$$\Delta\vec{r}_{\text{tot}} = (2.0\hat{\imath} + 6.0\hat{\jmath})\,\text{m},$$

with magnitude 6.32 m and direction 71.6° counterclockwise from the positive x-axis.

As a useful comparison, the total distance traveled is

$$d = |\Delta\vec{r}_1| + |\Delta\vec{r}_2| = 5.0\,\text{m} + \sqrt{5}\,\text{m} = 7.24\,\text{m},$$

which is a scalar and is not equal to the magnitude of the total displacement.

1.1.2 Position, Displacement, Distance, and Reference Frames

Definition 1.1.2: Reference frame, position, displacement, and distance

A *reference frame* is a choice of origin O, coordinate axes x, y, and z, and a clock for measuring time t. In AP kinematics, we describe motion in an inertial reference frame.
The *position vector* of an object at time t is

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k},$$

where $x(t)$, $y(t)$, and $z(t)$ are the object's coordinates in the chosen frame.
If the object is at initial position $\vec{r}_i$ and later at final position $\vec{r}_f$, the *displacement* over that time interval is the vector

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i.$$

The *distance traveled*, denoted by d, is the total length of the path actually followed. Distance is a scalar, while displacement is a vector.

Note:-

The sign of a coordinate such as x depends on the chosen axis direction, and the value of a position such as $\vec{r}(t)$ depends on the chosen origin. Thus position is frame-dependent. Displacement $\Delta\vec{r}$ compares two positions in the same frame, so changing the origin alone changes $\vec{r}_i$ and $\vec{r}_f$ but not their difference. Distance d is not the same as $|\Delta\vec{r}|$ in general: d is the total path length, while $|\Delta\vec{r}|$ is the straight-line separation between the initial and final positions. If an object turns around or follows a curved path, then $d > |\Delta\vec{r}|$.

Proposition 1.1.2 Component formulas and the straight-line special case

If the initial position is

$$\vec{r}_i = x_i\,\hat{\imath} + y_i\,\hat{\jmath} + z_i\,\hat{k}$$

and the final position is

$$\vec{r}_f = x_f\,\hat{\imath} + y_f\,\hat{\jmath} + z_f\,\hat{k},$$

then the displacement is

$$\Delta\vec{r} = (x_f - x_i)\,\hat{\imath} + (y_f - y_i)\,\hat{\jmath} + (z_f - z_i)\,\hat{k}.$$

Its magnitude is

$$|\Delta\vec{r}| = \sqrt{(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2}.$$

If the object moves along a straight path without changing direction, then the distance traveled equals the magnitude of the displacement:

$$d = |\Delta\vec{r}|.$$

For any other path, the distance satisfies

$$d \geq |\Delta\vec{r}|.$$

Question 2: Worked example

In a laboratory reference frame, the origin is marked on the floor, the x-axis points east, and the y-axis points north. A robot starts at the position

$$\vec{r}_i = (2\,\text{m})\,\hat{\imath} + (1\,\text{m})\,\hat{\jmath}.$$

It then moves 5 m east, then 3 m south, and then 2 m west. Find the final position $\vec{r}_f$, the displacement $\Delta\vec{r}$, the magnitude $|\Delta\vec{r}|$, and the total distance traveled d.

Solution: The initial position is

$$\vec{r}_i = (2\,\text{m})\,\hat{\imath} + (1\,\text{m})\,\hat{\jmath}.$$

After the first motion, the robot moves 5 m east, so its position becomes

$$(7\,\text{m})\,\hat{\imath} + (1\,\text{m})\,\hat{\jmath}.$$

After the second motion, the robot moves 3 m south, so its position becomes

$$(7\,\text{m})\,\hat{\imath} + (-2\,\text{m})\,\hat{\jmath}.$$

After the third motion, the robot moves 2 m west, so the final position is

$$\vec{r}_f = (5\,\text{m})\,\hat{\imath} + (-2\,\text{m})\,\hat{\jmath}.$$

Therefore the displacement is

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \big[(5-2)\,\text{m}\big]\hat{\imath} + \big[(-2)-1\big]\text{m}\,\hat{\jmath}.$$

So

$$\Delta\vec{r} = (3\,\text{m})\,\hat{\imath} + (-3\,\text{m})\,\hat{\jmath}.$$

Its magnitude is

$$|\Delta\vec{r}| = \sqrt{(3\,\text{m})^2 + (-3\,\text{m})^2} = \sqrt{18}\,\text{m} = 3\sqrt{2}\,\text{m}.$$

The total distance traveled is the sum of the three path segments:

$$d = 5\,\text{m} + 3\,\text{m} + 2\,\text{m} = 10\,\text{m}.$$

Thus,

$$\vec{r}_f = (5\,\text{m})\,\hat{\imath} - (2\,\text{m})\,\hat{\jmath}, \qquad \Delta\vec{r} = (3\,\text{m})\,\hat{\imath} - (3\,\text{m})\,\hat{\jmath},$$

$$|\Delta\vec{r}| = 3\sqrt{2}\,\text{m}, \qquad d = 10\,\text{m}.$$

1.1.3 Velocity and Acceleration as Derivatives

This subsection describes motion locally: starting from the position vector $\vec{r}(t)$, velocity and acceleration are defined by derivatives at an instant. Later sections will reverse these local definitions with definite integrals to recover displacement and changes in velocity over a time interval.

Definition 1.1.3: Average and instantaneous velocity and acceleration

Let t denote time, let Δt denote a nonzero time interval, and let $\vec{r}(t)$ denote the position vector of a particle.
The *average velocity* from time t to time $t+\Delta t$ is

$$\vec{v}_{\text{avg}} = \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}.$$

If the limit exists, the *instantaneous velocity* at time t is the vector

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}.$$

Let $\vec{v}(t)$ denote the instantaneous velocity. Then the *average acceleration* from time t to time $t+\Delta t$ is

$$\vec{a}_{\text{avg}} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}.$$

If the limit exists, the *instantaneous acceleration* at time t is the vector

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}.$$

The *speed* at time t is the scalar magnitude $|\vec{v}(t)|$.

Theorem 1.1.1 Derivative relations in vector and component form

Let

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k},$$

where $x(t)$, $y(t)$, and $z(t)$ are coordinate functions.
Then the velocity vector is

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} + \frac{dz}{dt}\,\hat{k}.$$

If $v_x(t)$, $v_y(t)$, and $v_z(t)$ denote the components of $\vec{v}(t)$, then

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}.$$

The acceleration vector is

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{dv_x}{dt}\,\hat{\imath} + \frac{dv_y}{dt}\,\hat{\jmath} + \frac{dv_z}{dt}\,\hat{k}.$$

If $a_x(t)$, $a_y(t)$, and $a_z(t)$ denote the components of $\vec{a}(t)$, then

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}.$$

In two-dimensional motion, the same formulas hold with the z-terms omitted.

Why these formulas are true: By definition,

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}.$$

If

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k},$$

then differentiating component-by-component gives

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} + \frac{dz}{dt}\,\hat{k}.$$

Likewise,

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt}.$$

Differentiating the velocity components gives

$$a_x = \frac{dv_x}{dt}, \qquad a_y = \frac{dv_y}{dt}, \qquad a_z = \frac{dv_z}{dt},$$

and substituting $v_x = dx/dt$, $v_y = dy/dt$, and $v_z = dz/dt$ gives

$$a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}, \qquad a_z = \frac{d^2z}{dt^2}.$$

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Corollary 1.1.1 One-dimensional motion and a speed caution

If motion is confined to the x-axis, so that

$$\vec{r}(t) = x(t)\,\hat{\imath},$$

then

$$\vec{v}(t) = v_x(t)\,\hat{\imath} \qquad \text{and} \qquad \vec{a}(t) = a_x(t)\,\hat{\imath},$$

with

$$v_x = \frac{dx}{dt}, \qquad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}.$$

However, in two or three dimensions, constant speed $|\vec{v}|$ does not by itself imply zero acceleration, because the direction of $\vec{v}$ can change even when its magnitude stays the same.

Question 3: Worked example

Let t denote time in seconds. A particle moves in the xy-plane with position vector

$$\vec{r}(t) = \left(t^2 - 4t\right)\hat{\imath} + \left(3t - t^2\right)\hat{\jmath}\ \text{m}.$$

Let $x(t) = t^2 - 4t$ and let $y(t) = 3t - t^2$, where $x(t)$ and $y(t)$ are measured in meters.
Find $\vec{v}(t)$ and $\vec{a}(t)$. Then find $\vec{v}(1.0\,\text{s})$, $\vec{a}(1.0\,\text{s})$, and the speed at $t = 1.0\,\text{s}$. Interpret the signs of the velocity components at $t = 1.0\,\text{s}$.

Solution: From

$$x(t) = t^2 - 4t \qquad \text{and} \qquad y(t) = 3t - t^2,$$

the component formulas for velocity give

$$v_x = \frac{dx}{dt} = 2t - 4 \qquad \text{and} \qquad v_y = \frac{dy}{dt} = 3 - 2t.$$

Therefore,

$$\vec{v}(t) = \left(2t - 4\right)\hat{\imath} + \left(3 - 2t\right)\hat{\jmath}\ \text{m/s}.$$

Differentiate again to find the acceleration components:

$$a_x = \frac{dv_x}{dt} = 2 \qquad \text{and} \qquad a_y = \frac{dv_y}{dt} = -2.$$

So,

$$\vec{a}(t) = 2\hat{\imath} - 2\hat{\jmath}\ \text{m/s}^2.$$

At $t = 1.0\,\text{s}$,

$$\vec{v}(1.0\,\text{s}) = \left(2(1.0) - 4\right)\hat{\imath} + \left(3 - 2(1.0)\right)\hat{\jmath} = -2\hat{\imath} + \hat{\jmath}\ \text{m/s},$$

and

$$\vec{a}(1.0\,\text{s}) = 2\hat{\imath} - 2\hat{\jmath}\ \text{m/s}^2.$$

The speed at $t = 1.0\,\text{s}$ is the magnitude of the velocity vector:

$$|\vec{v}(1.0\,\text{s})| = \sqrt{(-2)^2 + (1)^2}\ \text{m/s} = \sqrt{5}\ \text{m/s}.$$

Because $v_x(1.0\,\text{s}) = -2\,\text{m/s}$, the particle is moving in the negative x-direction at that instant. Because $v_y(1.0\,\text{s}) = 1\,\text{m/s}$, the particle is moving in the positive y-direction at that instant. So at $t = 1.0\,\text{s}$ the particle is moving left and upward, with speed $\sqrt{5}\,\text{m/s}$.

1.1.4 Motion Graphs, Slopes, and Signed Areas

This subsection connects the local language of derivatives to the global language of accumulation. In one-dimensional motion, or when motion is analyzed along the x-axis, AP problems often ask you to translate among a verbal description, a graph such as $x(t)$, $v_x(t)$, or $a_x(t)$, and equations relating slope and area.

Definition 1.1.4: Common motion graphs and the meaning of slope

Let t denote time. Let $x(t)$ denote the position coordinate along the x-axis, let $v_x(t)$ denote the x-component of velocity, and let $a_x(t)$ denote the x-component of acceleration.
An $x(t)$ graph shows position as a function of time. A $v_x(t)$ graph shows velocity as a function of time. An $a_x(t)$ graph shows acceleration as a function of time.
If t_1 and t_2 are two times with $t_2 > t_1$, then the *average slope* of a graph of a quantity $q(t)$ over the interval from t_1 to t_2 is

$$\frac{q(t_2) - q(t_1)}{t_2 - t_1}.$$

This is the slope of the secant line through the two points on the graph.
The *slope at a point* is the slope of the tangent line at that time. For motion graphs, the slope at a point gives an instantaneous rate of change, while the average slope over an interval gives an average rate of change over that interval.

Note:-

Do not confuse a graph's *value* with its *slope*. On an $x(t)$ graph, a point high above the axis means large position, not large velocity. On a $v_x(t)$ graph, a point at $v_x = 0$ means zero velocity at that instant, while a horizontal tangent means zero acceleration at that instant. Likewise, zero slope is not the same as zero value. Also, signed area under a $v_x(t)$ graph gives displacement, not total distance traveled. If velocity changes sign, distance in one dimension is found from $\int |v_x|\,dt$, so areas below the axis must be counted with positive magnitude when finding distance.

Proposition 1.1.3 Operational graph rules for one-dimensional motion

Let t_1 and t_2 be times with $t_2 > t_1$. Let $\Delta x = x(t_2) - x(t_1)$ and let $\Delta v_x = v_x(t_2) - v_x(t_1)$.
For position and velocity graphs,

$$\text{slope of } x(t) \text{ at time } t = v_x(t),$$

so the average slope of the position graph over $[t_1, t_2]$ is the average velocity:

$$\frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{t_2 - t_1}.$$

For velocity and acceleration graphs,

$$\text{slope of } v_x(t) \text{ at time } t = a_x(t),$$

so the average slope of the velocity graph over $[t_1, t_2]$ is the average acceleration:

$$\frac{v_x(t_2) - v_x(t_1)}{t_2 - t_1} = \frac{\Delta v_x}{t_2 - t_1}.$$

The signed area under the velocity graph from t_1 to t_2 gives displacement:

$$\Delta x = \int_{t_1}^{t_2} v_x(t)\,dt.$$

Area above the time axis contributes positively, and area below the time axis contributes negatively. The signed area under the acceleration graph from t_1 to t_2 gives change in velocity:

$$\Delta v_x = \int_{t_1}^{t_2} a_x(t)\,dt.$$

In one dimension, the total distance traveled from t_1 to t_2 is

$$\text{distance} = \int_{t_1}^{t_2} |v_x(t)|\,dt,$$

which equals the total unsigned area between the $v_x(t)$ graph and the time axis.

Question 4: Worked example

A particle moves along the x-axis. Its velocity graph $v_x(t)$ is described as follows.
From $t = 0$ to $t = 2.0\,\text{s}$, the velocity increases linearly from 0 to $4.0\,\text{m/s}$. From $t = 2.0\,\text{s}$ to $t = 5.0\,\text{s}$, the velocity is constant at $4.0\,\text{m/s}$. From $t = 5.0\,\text{s}$ to $t = 7.0\,\text{s}$, the velocity decreases linearly from $4.0\,\text{m/s}$ to $-2.0\,\text{m/s}$.
Find the acceleration on each time interval, the displacement from $t = 0$ to $t = 7.0\,\text{s}$, the total distance traveled from $t = 0$ to $t = 7.0\,\text{s}$, and the average velocity over the full $7.0\,\text{s}$ interval. State when the particle moves in the negative x-direction.

Solution: Let a_x denote the slope of the velocity graph.

From $t = 0$ to $t = 2.0\,\text{s}$,

$$a_x = \frac{4.0 - 0}{2.0 - 0} = 2.0\,\text{m/s}^2.$$

From $t = 2.0\,\text{s}$ to $t = 5.0\,\text{s}$, the graph is horizontal, so

$$a_x = 0.$$

From $t = 5.0\,\text{s}$ to $t = 7.0\,\text{s}$,

$$a_x = \frac{-2.0 - 4.0}{7.0 - 5.0} = -3.0\,\text{m/s}^2.$$

Now find displacement from the signed area under the $v_x(t)$ graph.

From $t = 0$ to $t = 2.0\,\text{s}$, the area is a triangle with base $2.0\,\text{s}$ and height $4.0\,\text{m/s}$:

$$\Delta x_1 = \frac{1}{2}(2.0)(4.0) = 4.0\,\text{m}.$$

From $t = 2.0\,\text{s}$ to $t = 5.0\,\text{s}$, the area is a rectangle:

$$\Delta x_2 = (3.0)(4.0) = 12.0\,\text{m}.$$

From $t = 5.0\,\text{s}$ to $t = 7.0\,\text{s}$, the signed area is a trapezoid:

$$\Delta x_3 = \frac{4.0 + (-2.0)}{2}(2.0) = 2.0\,\text{m}.$$

Therefore the total displacement is

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 = 4.0 + 12.0 + 2.0 = 18.0\,\text{m}.$$

For total distance, any part of the velocity graph below the axis must be counted positively. The velocity becomes zero during the last interval, so first find that time. Starting from $v_x = 4.0\,\mathrm{m/s}$ at $t = 5.0\,\mathrm{s}$ with slope $-3.0\,\mathrm{m/s^2}$,

$$0 = 4.0 + (-3.0)(t - 5.0).$$

So,

$$t - 5.0 = \frac{4.0}{3.0}, \qquad t = \frac{19}{3}\,\mathrm{s}.$$

From $t = 5.0\,\mathrm{s}$ to $t = 19/3\,\mathrm{s}$, the graph is above the axis, giving a triangle of area

$$A_+ = \frac{1}{2}\left(\frac{4}{3}\right)(4.0) = \frac{8}{3}\,\mathrm{m}.$$

From $t = 19/3\,\mathrm{s}$ to $t = 7.0\,\mathrm{s}$, the graph is below the axis, giving a triangle with signed area $-\frac{2}{3}\,\mathrm{m}$ and magnitude

$$A_- = \frac{1}{2}\left(\frac{2}{3}\right)(2.0) = \frac{2}{3}\,\mathrm{m}.$$

Thus the total distance traveled is

$$d = 4.0 + 12.0 + \frac{8}{3} + \frac{2}{3} = 16.0 + \frac{10}{3} = \frac{58}{3}\,\mathrm{m} \approx 19.3\,\mathrm{m}.$$

The average velocity over the full interval is displacement divided by the total elapsed time of 7.0 s:

$$v_{x,\mathrm{avg}} = \frac{18.0\,\mathrm{m}}{7.0\,\mathrm{s}} \approx 2.57\,\mathrm{m/s}.$$

The particle moves in the negative x-direction when $v_x < 0$, which occurs after the graph crosses the axis. Therefore it moves in the negative x-direction for

$$\frac{19}{3}\,\mathrm{s} < t \le 7.0\,\mathrm{s}.$$

So the interval accelerations are $2.0\,\mathrm{m/s^2}$, 0, and $-3.0\,\mathrm{m/s^2}$; the displacement is 18.0 m; the total distance is 58/3 m; and the average velocity is about 2.57 m/s.

1.1.5 Constant-Acceleration Motion and Free Fall

This subsection treats motion over a time interval during which an acceleration component is constant. The standard kinematic formulas are not separate facts; they are the integrated consequences of the local derivative relation between acceleration and velocity.

Definition 1.1.5: Constant acceleration and the free-fall approximation

Let t denote time measured from a chosen initial instant $t = 0$. Let $x(t)$ and $y(t)$ denote position components, let $v_x(t) = dx/dt$ and $v_y(t) = dy/dt$ denote velocity components, and let $a_x(t) = dv_x/dt$ and $a_y(t) = dv_y/dt$ denote acceleration components.
Motion has *constant acceleration in the x-direction* if there is a constant scalar a_x such that $a_x(t) = a_x$ throughout the time interval of interest. Likewise, motion has *constant acceleration in the y-direction* if there is a constant scalar a_y such that $a_y(t) = a_y$ throughout the interval.
Near Earth's surface, and neglecting air resistance, *free fall* is motion with constant vertical acceleration of magnitude $g \approx 9.8\,\mathrm{m/s^2}$. If the positive y-axis is chosen upward, then $a_y = -g$. If the positive y-axis is chosen downward, then $a_y = +g$.

Theorem 1.1.2 Integrated kinematics in component form

Let t denote elapsed time from the initial instant $t = 0$. Let $x_0 = x(0)$ and $v_{x0} = v_x(0)$, and let $x = x(t)$ denote the later x-coordinate at time t. If a_x is constant, then

$$v_x = v_{x0} + a_x t,$$

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2,$$

and

$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0).$$

Likewise, let $y_0 = y(0)$ and $v_{y0} = v_y(0)$, and let $y = y(t)$ denote the later y-coordinate at time t. If a_y is constant, then

$$v_y = v_{y0} + a_y t,$$

$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2,$$

and

$$v_y^2 = v_{y0}^2 + 2a_y(y - y_0).$$

If both a_x and a_y are constant, then each component evolves independently according to these formulas.

Derivation from a constant acceleration component: Assume first that $a_x(t) = a_x$ is constant. By the local definition of acceleration,

$$\frac{dv_x}{dt} = a_x.$$

Integrate from the initial time 0 to a later time t:

$$\int_0^t \frac{dv_x}{dt'}\, dt' = \int_0^t a_x\, dt'.$$

The left side is $v_x - v_{x0}$, so

$$v_x - v_{x0} = a_x t,$$

which gives

$$v_x = v_{x0} + a_x t.$$

Now use $dx/dt = v_x$ and substitute the result just found:

$$\frac{dx}{dt} = v_{x0} + a_x t.$$

Integrating from 0 to t gives

$$\int_0^t \frac{dx}{dt'}\, dt' = \int_0^t \left(v_{x0} + a_x t'\right) dt'.$$

The left side is $x - x_0$, so

$$x - x_0 = v_{x0}t + \tfrac{1}{2}a_x t^2,$$

which gives

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2.$$

To eliminate time, apply the chain rule:

$$a_x = \frac{dv_x}{dt} = \frac{dv_x}{dx}\frac{dx}{dt} = v_x \frac{dv_x}{dx}.$$

Integrate from x_0 to x and from v_{x0} to v_x:

$$\int_{v_{x0}}^{v_x} v\, dv = \int_{x_0}^{x} a_x\, dx'.$$

This gives

$$\tfrac{1}{2}\left(v_x^2 - v_{x0}^2\right) = a_x(x - x_0),$$

so

$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0).$$

The y-component formulas follow in exactly the same way after replacing x by y, v_x by v_y, and a_x by a_y. 😊

Corollary 1.1.2 Vertical free-fall formulas and a sign caution

Choose the positive y-axis upward. Let $y_0 = y(0)$, let $v_{y0} = v_y(0)$, and let $y = y(t)$ be the height at time t. For free fall near Earth with negligible air resistance, $a_y = -g$, where $g > 0$. Therefore,

$$v_y = v_{y0} - gt,$$

$$y = y_0 + v_{y0}t - \tfrac{1}{2}gt^2,$$

and

$$v_y^2 = v_{y0}^2 - 2g(y - y_0).$$

Negative acceleration does not automatically mean an object is slowing down. An object slows down only when its velocity and acceleration point in opposite directions. Thus, in free fall with $a_y = -g$, an object moving upward has decreasing speed, but an object moving downward has increasing speed.

Question 5: Worked example

Choose the positive y-axis upward, and let y denote height above the ground. A ball is thrown straight upward from a balcony. Let the initial time be $t = 0$, let the initial height be $y_0 = 24.0\,\mathrm{m}$, let the initial vertical velocity be $v_{y0} = +12.0\,\mathrm{m/s}$, and let the constant vertical acceleration be $a_y = -9.8\,\mathrm{m/s^2}$. Neglect air resistance.
Find the time when the ball hits the ground and the vertical velocity just before impact.

Solution: The ball hits the ground when its height is $y = 0$. Using

$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2,$$

we substitute the stated values:

$$0 = 24.0\,\mathrm{m} + (12.0\,\mathrm{m/s})t + \tfrac{1}{2}(-9.8\,\mathrm{m/s^2})t^2.$$

So,

$$0 = 24.0 + 12.0t - 4.9t^2.$$

Rewriting into standard quadratic form gives

$$4.9t^2 - 12.0t - 24.0 = 0.$$

Using the quadratic formula,

$$t = \frac{12.0 \pm \sqrt{(-12.0)^2 - 4(4.9)(-24.0)}}{2(4.9)} = \frac{12.0 \pm \sqrt{614.4}}{9.8}.$$

This gives two mathematical roots,

$$t \approx -1.30\,\mathrm{s} \qquad \text{or} \qquad t \approx 3.75\,\mathrm{s}.$$

The negative time does not fit the physical situation after the throw, so the ball hits the ground at

$$t \approx 3.75\,\mathrm{s}.$$

Now find the vertical velocity at that time from

$$v_y = v_{y0} + a_y t.$$

Substituting the known values and the physical root gives

$$v_y = (12.0\,\mathrm{m/s}) + (-9.8\,\mathrm{m/s^2})(3.75\,\mathrm{s}) \approx -24.8\,\mathrm{m/s}.$$

Therefore, just before impact, the ball's vertical velocity is

$$v_y \approx -24.8\,\mathrm{m/s},$$

which means the ball is moving downward with speed 24.8 m/s. The sign is negative because the positive axis was chosen upward. During the descent, both v_y and a_y are negative, so the ball speeds up even though the acceleration is negative.

1.1.6 Relative Motion and Projectile Motion

This subsection combines two central AP kinematics ideas in an inertial frame: relative velocity between different observers, and two-dimensional projectile motion with negligible air resistance. In both settings, vectors are handled component-by-component, and careful notation keeps track of what is being measured.

Definition 1.1.6: Relative velocity and the projectile-motion setup

Let A and B denote moving objects, and let E denote an inertial reference frame such as Earth. If $\vec{v}_{A/E}$ denotes the velocity of object A measured in frame E, then the *relative velocity of A with respect to B* is the velocity vector of A as measured in the frame moving with B, written $\vec{v}_{A/B}$.

For projectile motion, choose Cartesian axes before writing equations. Let t denote time after launch, let $x(t)$ denote the horizontal coordinate, and let $y(t)$ denote the vertical coordinate measured upward. Let $x_0 = x(0)$ and $y_0 = y(0)$ denote the initial coordinates, let $v_x(t)$ and $v_y(t)$ denote the velocity components, let $\vec{v}(t) = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath}$ denote the velocity vector, and let $v_{0x} = v_x(0)$ and $v_{0y} = v_y(0)$ denote the initial velocity components. In the AP model, air resistance is neglected and the only acceleration is gravity, so the acceleration vector is

$$\vec{a} = -g\hat{\jmath},$$

where g denotes the positive magnitude of the gravitational acceleration. Thus the component equations are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g.$$

Theorem 1.1.3 Relative-velocity addition and projectile component formulas

Let A, B, and E denote objects or frames in classical mechanics. Then the relative-velocity addition law is

$$\vec{v}_{A/E} = \vec{v}_{A/B} + \vec{v}_{B/E}.$$

Equivalently,

$$\vec{v}_{A/B} = \vec{v}_{A/E} - \vec{v}_{B/E}.$$

These equations are vector equations, so they may be applied component-by-component in any chosen axes. For projectile motion with the setup in the definition,

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g.$$

Integrating once gives the velocity components

$$v_x(t) = v_{0x}, \qquad v_y(t) = v_{0y} - gt.$$

Integrating again gives the position components

$$x(t) = x_0 + v_{0x}t, \qquad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2.$$

If v_0 denotes the initial speed and θ denotes the launch angle measured above the positive x-axis, then

$$v_{0x} = v_0 \cos\theta, \qquad v_{0y} = v_0 \sin\theta.$$

The horizontal and vertical motions are independent in the sense that each component has its own equation, but they are linked by the same time variable t.

Example 1.1.1 (Illustrative relative-motion example)

Let A denote a student walking on a moving walkway, let B denote the walkway, and let E denote the ground frame. Suppose

$$\vec{v}_{A/B} = 3.0\hat{\jmath}\,\text{m/s} \qquad \text{and} \qquad \vec{v}_{B/E} = 4.0\hat{\imath}\,\text{m/s}.$$

Then

$$\vec{v}_{A/E} = \vec{v}_{A/B} + \vec{v}_{B/E} = 4.0\hat{\imath} + 3.0\hat{\jmath}\,\text{m/s}.$$

So the student moves relative to the ground with speed

$$|\vec{v}_{A/E}| = \sqrt{4.0^2 + 3.0^2}\,\text{m/s} = 5.0\,\text{m/s}.$$

Note:-

In $\vec{v}_{A/B}$, the first label tells *whose* velocity is being described, and the second label tells *which frame* measures it. Reversing the labels changes the meaning. In projectile motion, choose axes first so that the signs of v_{0x}, v_{0y}, and $-g$ are clear. The horizontal and vertical motions must be evaluated at the same time t; they are not separate motions with separate clocks. Common mistakes include adding speeds instead of velocity vectors in relative-motion problems, forgetting that v_x stays constant only when air resistance is neglected, and setting $v_y = 0$ for the entire flight instead of only at the top of the path.

Question 6: Worked example

A ball is launched from the top of a platform. Let t denote time in seconds after launch. Choose the x-axis horizontal and the y-axis vertical upward. Let $x(t)$ and $y(t)$ denote the coordinates of the ball in meters. At $t = 0$, let $x_0 = 0$, let $y_0 = 30.0\,\text{m}$, let $v_{0x} = 12.0\,\text{m/s}$, and let $v_{0y} = 5.0\,\text{m/s}$. Let $g = 10.0\,\text{m/s}^2$. Find the time when the ball hits the ground, where the ground is given by $y = 0$. Then find the horizontal distance traveled and the velocity vector just before impact.

Solution: From the projectile component formulas,

$$x(t) = x_0 + v_{0x}t = 12.0t,$$

and

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 = 30.0 + 5.0t - 5.0t^2.$$

The ball hits the ground when $y = 0$, so solve

$$30.0 + 5.0t - 5.0t^2 = 0.$$

Divide by 5.0:

$$6 + t - t^2 = 0.$$

Rearrange:

$$t^2 - t - 6 = 0.$$

Factor:

$$(t - 3)(t + 2) = 0.$$

Thus the two algebraic solutions are $t = 3.0\,\text{s}$ and $t = -2.0\,\text{s}$. The negative time is not physically relevant after launch, so the impact time is

$$t = 3.0\,\text{s}.$$

The horizontal distance traveled is the x-coordinate at this time:

$$x(3.0) = 12.0(3.0)\,\text{m} = 36.0\,\text{m}.$$

So the ball lands 36.0 m horizontally from the launch point.

Next find the velocity components. The horizontal component is constant:

$$v_x(t) = v_{0x} = 12.0\,\text{m/s}.$$

The vertical component is

$$v_y(t) = v_{0y} - gt = 5.0 - 10.0t.$$

At impact,

$$v_y(3.0) = 5.0 - 10.0(3.0)\,\text{m/s} = -25.0\,\text{m/s}.$$

Therefore the velocity vector just before impact is

$$\vec{v}(3.0\,\text{s}) = 12.0\hat{\imath} - 25.0\hat{\jmath}\,\text{m/s}.$$

Its negative y-component shows that the ball is moving downward at impact. If the impact speed is also desired, then

$$|\vec{v}(3.0\,\text{s})| = \sqrt{12.0^2 + (-25.0)^2}\,\text{m/s} = \sqrt{769}\,\text{m/s} \approx 27.7\,\text{m/s}.$$

Thus the ball hits the ground after 3.0 s, lands 36.0 m from the launch point, and has impact velocity

$$\vec{v} = 12.0\hat{\imath} - 25.0\hat{\jmath}\,\text{m/s}.$$

1.2 Force and Translational Dynamics

Dynamics explains changes in motion by relating the net external force on a chosen system to its acceleration through Newton's laws. In AP Physics C: Mechanics, this unit stays within inertial frames and emphasizes careful system definition, free-body reasoning, and consistent vector notation for quantities such as $\vec{F}$, $\vec{a}$, $\vec{g}$, $\vec{T}$, and $\vec{N}$.

The flow of the unit begins with Newton's laws and center-of-mass ideas, then develops common force models such as gravitation, normal force, tension, friction, and springs. It closes with velocity-dependent forces and radial dynamics, including circular motion and circular orbits, with idealized strings, pulleys, and springs used throughout where appropriate.

1.2.1 System Choice, Free-Body Diagrams, and Newton's Laws

This subsection gives the standard AP mechanics workflow: choose a system, identify the interactions on that system, draw a free-body diagram, choose axes, and then apply Newton's laws in an inertial frame.

Definition 1.2.1: System choice, interactions, free-body diagrams, and net external force

Let S denote a chosen **system**. In this subsection, S will usually be a single body treated as a particle or rigid object.
An **interaction** is a physical influence between the system S and something in the environment, such as gravity from Earth or contact with a surface, rope, or another object. Each interaction can exert a force on S.
A **free-body diagram** for S is a diagram that shows only the forces acting *on* S. If a force is exerted by the environment on S, that force belongs on the free-body diagram. Forces exerted by S on other objects do not belong on the free-body diagram of S.
Let $\vec{F}_{\text{ext}}$ denote the net external force on S. If the external forces acting on S are $\vec{F}_1$, $\vec{F}_2$, …, and $\vec{F}_n$, then

$$\vec{F}_{\text{ext}} = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n.$$

This is a vector sum. After axes are chosen, one may resolve a force into components for calculation, but those components are not additional physical forces to be added to the free-body diagram.

Theorem 1.2.1 Newton's laws in AP-usable form

Work in an inertial reference frame. Let m denote the mass of the chosen body, let $\vec{v}$ denote its velocity, let $\vec{a}$ denote its acceleration, and let $\vec{F}_{\text{ext}}$ denote the vector sum of all external forces on that body.

① **Newton I.** If $\vec{F}_{\text{ext}} = \vec{0}$, then $\vec{a} = \vec{0}$. Thus the body is either at rest or moves with constant velocity.

② **Newton II.** For the chosen body,

$$\vec{F}_{\text{ext}} = m\vec{a}.$$

This is the main working law for AP mechanics.

③ **Newton III.** If body A exerts a force $\vec{F}_{A\to B}$ on body B, then body B exerts a force $\vec{F}_{B\to A}$ on body A such that

$$\vec{F}_{B\to A} = -\vec{F}_{A\to B}.$$

These two forces act on different bodies, so they do not cancel on a single free-body diagram.

Why the vector law becomes component equations: Choose Cartesian axes with unit vectors $\hat{\imath}$ and $\hat{\jmath}$. Let the acceleration be

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath},$$

where a_x and a_y are scalar components. Let the net external force be

$$\vec{F}_{\text{ext}} = \left(\sum F_x\right)\hat{\imath} + \left(\sum F_y\right)\hat{\jmath},$$

where $\sum F_x$ and $\sum F_y$ are the scalar sums of force components along the chosen axes.

Substitute these into Newton II:

$$\left(\sum F_x\right)\hat{\imath} + \left(\sum F_y\right)\hat{\jmath} = ma_x\hat{\imath} + ma_y\hat{\jmath}.$$

Because the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are independent, the corresponding scalar components must match. Therefore,

$$\sum F_x = ma_x, \qquad \sum F_y = ma_y.$$

In practice, one first draws only the actual forces on the free-body diagram, then chooses axes, and only then resolves forces into components if that makes the equations simpler. ☺

Corollary 1.2.1 Zero net force and equilibrium

If $\vec{F}_{\text{ext}} = \vec{0}$, then Newton II gives $\vec{a} = \vec{0}$. This does *not* mean the velocity must be zero. A body can have zero net force while moving with a nonzero constant velocity. A special case is equilibrium: if a body is initially at rest and $\vec{F}_{\text{ext}} = \vec{0}$, then it remains at rest.

Question 7: Worked example

A block of mass $m = 5.0\,\text{kg}$ rests on a frictionless incline that makes an angle $\theta = 30^\circ$ with the horizontal. Let $g = 9.8\,\text{m/s}^2$ denote the magnitude of the gravitational field near Earth. Choose the system to be the block.

Find the acceleration of the block and the magnitude of the normal force exerted by the incline on the block.

Solution: The system is the block. The interactions on the block are the gravitational interaction with Earth and the contact interaction with the incline. Therefore the free-body diagram contains only two forces: the weight $\vec{W}$ exerted by Earth on the block and the normal force $\vec{N}$ exerted by the incline on the block. Let N denote the magnitude of $\vec{N}$.

Choose axes so that the x-axis is parallel to the incline and positive down the incline, and the y-axis is perpendicular to the incline and positive away from the surface. Let a_x and a_y denote the acceleration components in these directions.

The block stays in contact with the plane, so there is no acceleration perpendicular to the surface. Thus

$$a_y = 0.$$

Resolve the weight into components relative to the chosen axes. The component of $\vec{W}$ parallel to the incline has magnitude

$$W_x = mg\sin\theta,$$

and the component of $\vec{W}$ perpendicular to the incline has magnitude

$$W_y = mg\cos\theta.$$

These are components of the single force $\vec{W}$; they are not extra forces on the free-body diagram.

Now apply Newton II by components.

Along the incline,

$$\sum F_x = ma_x.$$

The only force component along the incline is $mg\sin\theta$ in the positive x-direction, so

$$mg\sin\theta = ma_x.$$

Cancel m:

$$a_x = g\sin\theta.$$

Substitute the stated values:

$$a_x = (9.8\,\mathrm{m/s^2})\sin 30^\circ = (9.8\,\mathrm{m/s^2})(0.50) = 4.9\,\mathrm{m/s^2}.$$

So the block accelerates at

$$4.9\,\mathrm{m/s^2}$$

down the incline.

Perpendicular to the incline,

$$\sum F_y = ma_y.$$

The positive y-direction is away from the surface, so the normal force is positive and the perpendicular component of the weight is negative. Since $a_y = 0$,

$$N - mg\cos\theta = 0.$$

Therefore,

$$N = mg\cos\theta.$$

Substitute the stated values:

$$N = (5.0\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\cos 30^\circ.$$

Using $\cos 30^\circ \approx 0.866$ gives

$$N \approx (49.0\,\mathrm{N})(0.866) = 42.4\,\mathrm{N}.$$

Therefore, the block's acceleration is

$$4.9\,\mathrm{m/s^2}$$

down the incline, and the magnitude of the normal force is

$$42.4\,\mathrm{N}.$$

1.2.2 Center of Mass and Translational Motion of Systems

This subsection extends Newton's second law from a single particle to a system of particles. The key idea is that the overall translational motion of the system is described by its center of mass, even when the particles exert complicated internal forces on one another.

Definition 1.2.2: Center of mass for a system

Consider N particles labeled by an index $i = 1, 2, \ldots, N$. Let m_i denote the mass of particle i, let $\vec{r}_i$ denote the position vector of particle i, and let

$$M = \sum_{i=1}^{N} m_i$$

denote the total mass of the system. The *center of mass* is the vector

$$\vec{r}_{\text{cm}} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i.$$

Its velocity and acceleration are

$$\vec{v}_{\text{cm}} = \frac{d\vec{r}_{\text{cm}}}{dt}, \qquad \vec{a}_{\text{cm}} = \frac{d\vec{v}_{\text{cm}}}{dt}.$$

For a continuous mass distribution, let dm denote a differential mass element located at position vector $\vec{r}$. Then the continuous analog is

$$\vec{r}_{\text{cm}} = \frac{1}{M} \int \vec{r}\, dm.$$

Theorem 1.2.2 Newton's second law for a system

For the system above, let $\vec{F}_{\text{ext},i}$ denote the net external force on particle i. Let

$$\sum \vec{F}_{\text{ext}} = \sum_{i=1}^{N} \vec{F}_{\text{ext},i}$$

denote the net external force on the whole system. Then

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}.$$

Therefore the center of mass moves as if all the system mass M were concentrated at the center of mass and acted on by the net external force.

Note:-

Internal forces can change the separations, shape, or rotation of the parts of a system, but they do not change the motion of the center of mass. When the internal forces are added over the whole system, they cancel in equal-and-opposite pairs. As a result, only external forces determine $\vec{a}_{\text{cm}}$. In particular, if $\sum \vec{F}_{\text{ext}} = \vec{0}$, then $\vec{a}_{\text{cm}} = \vec{0}$, so the center of mass moves with constant velocity even if an explosion or collision makes the individual parts fly apart.

Derivation by summing Newton's second law over the particles: For each particle i, let $\vec{a}_i$ denote its acceleration, and let $\vec{F}_{\text{int},i}$ denote the net internal force on that particle from the other particles in the system. Newton's second law for particle i gives

$$m_i \vec{a}_i = \vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i}.$$

Now sum over all particles:

$$\sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{\text{ext},i} + \sum_{i=1}^{N} \vec{F}_{\text{int},i}.$$

By Newton's third law, the internal forces cancel in pairs, so

$$\sum_{i=1}^{N} \vec{F}_{\text{int},i} = \vec{0}.$$

Thus,

$$\sum_{i=1}^{N} m_i \vec{a}_i = \sum \vec{F}_{\text{ext}}.$$

From the definition

$$\vec{r}_{\text{cm}} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i,$$

differentiate twice with respect to time:

$$\vec{a}_{\text{cm}} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{a}_i.$$

Multiplying by M gives

$$M\vec{a}_{\text{cm}} = \sum_{i=1}^{N} m_i \vec{a}_i.$$

Substitute this into the previous result to obtain

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}.$$

😄

Question 8: Worked example

Two carts move on a horizontal frictionless track. Cart 1 has mass $m_1 = 2.0\,\text{kg}$ and cart 2 has mass $m_2 = 3.0\,\text{kg}$. At the instant of interest, cart 1 is at position coordinate $x_1 = 0.40\,\text{m}$ and cart 2 is at position coordinate $x_2 = 1.60\,\text{m}$. The carts interact with each other through a light compressed spring between them, so the spring forces are internal to the two-cart system. A student pulls on cart 1 so that the net external force on the two-cart system is a constant horizontal force of magnitude $10.0\,\text{N}$ to the right. At that instant the center of mass is moving to the right with speed $v_{\text{cm},0} = 1.2\,\text{m/s}$.
Find the x-coordinate of the center of mass, the acceleration of the center of mass, and the speed of the center of mass $t = 2.0\,\text{s}$ later.

Solution: For one-dimensional motion along the x-axis, let x_{cm} denote the x-coordinate of $\vec{r}_{\text{cm}}$. Also let

$$M = m_1 + m_2 = 2.0\,\text{kg} + 3.0\,\text{kg} = 5.0\,\text{kg}.$$

Then the center-of-mass coordinate is

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{M},$$

Substitute the given values:

$$x_{\text{cm}} = \frac{(2.0\,\text{kg})(0.40\,\text{m}) + (3.0\,\text{kg})(1.60\,\text{m})}{5.0\,\text{kg}}.$$

Therefore,

$$x_{\text{cm}} = \frac{0.80 + 4.80}{5.0}\,\text{m} = 1.12\,\text{m}.$$

Now use Newton's second law for the system:

$$\sum \vec{F}_{\text{ext}} = M\vec{a}_{\text{cm}}.$$

The net external force has magnitude $10.0\,\text{N}$ to the right, so the center-of-mass acceleration has magnitude

$$a_{\text{cm}} = \frac{10.0\,\text{N}}{5.0\,\text{kg}} = 2.0\,\text{m/s}^2$$

to the right. Equivalently, $\vec{a}_{\text{cm}}$ points in the positive x-direction with magnitude $2.0\,\text{m/s}^2$.

Because the external force is constant, the center of mass has constant acceleration during the 2.0 s interval. Let v_{cm} denote the center-of-mass speed at the later time. Then

$$v_{\text{cm}} = v_{\text{cm},0} + a_{\text{cm}} t.$$

Substituting gives

$$v_{\text{cm}} = 1.2\,\text{m/s} + (2.0\,\text{m/s}^2)(2.0\,\text{s}) = 5.2\,\text{m/s}.$$

So the center of mass is at

$$x_{\text{cm}} = 1.12\,\text{m},$$

its acceleration has magnitude

$$a_{\text{cm}} = 2.0\,\text{m/s}^2,$$

with $\vec{a}_{\text{cm}}$ pointing in the positive x-direction, and its speed 2.0 s later is

$$v_{\text{cm}} = 5.2\,\text{m/s}.$$

Even if the spring pushes the carts apart strongly, that spring force is internal to the chosen system. It can change the individual motions of the carts, but it does not change these center-of-mass results.

1.2.3 Gravitation and the Inverse-Square Field

This subsection treats gravity from the local field viewpoint. A source mass creates a gravitational field $\vec{g}(\vec{r})$, and a test mass placed at position $\vec{r}$ experiences a gravitational force determined by that field.

Definition 1.2.3: Gravitational field and the source–test-mass viewpoint

Let $\vec{r}$ denote the position vector of a field point, and let a test mass m be placed at that point. If $\vec{F}_g(\vec{r})$ denotes the gravitational force on the test mass due to some source mass distribution, then the *gravitational field* at $\vec{r}$ is defined by

$$\vec{g}(\vec{r}) = \frac{\vec{F}_g(\vec{r})}{m}.$$

Thus the field is the gravitational force per unit mass. Equivalently, any test mass m placed at that point satisfies

$$\vec{F}_g = m\vec{g}.$$

The SI units of $\vec{g}$ are N/kg, which are equivalent to m/s^2. The test mass is assumed small enough that it does not significantly change the source field.

Theorem 1.2.3 Newton's law of gravitation and the inverse-square field

Let G denote the universal gravitational constant. Let a point mass or a spherically symmetric source have total mass M and center at the origin. Let $\vec{r}$ denote the position vector of a field point, let $r = |\vec{r}|$ denote its distance from the center, and let $\hat{r} = \vec{r}/r$ denote the outward radial unit vector. If the source is spherically symmetric, assume the field point lies outside the source. If a test mass m is placed at that point, then the gravitational force on the test mass is

$$\vec{F}_g(\vec{r}) = -\frac{GMm}{r^2}\hat{r}.$$

Therefore the gravitational field produced by the source is

$$\vec{g}(\vec{r}) = -\frac{GM}{r^2}\hat{r}.$$

Its magnitude is

$$g(r) = \frac{GM}{r^2}.$$

The negative sign shows that the field points toward the source, so gravity is attractive.

Connecting the force law to the field law: Let M denote the source mass, let m denote the test mass, and let r denote the separation between their centers. Newton's law of gravitation states that the magnitude of the gravitational force is

$$F_g = \frac{GMm}{r^2}.$$

Because gravity is attractive, the force on the test mass points toward the source. Since $\hat{r}$ points radially outward, the force direction is $-\hat{r}$. Therefore,

$$\vec{F}_g(\vec{r}) = -\frac{GMm}{r^2}\hat{r}.$$

Now apply the definition of gravitational field:

$$\vec{g}(\vec{r}) = \frac{\vec{F}_g(\vec{r})}{m} = -\frac{GM}{r^2}\hat{r}.$$

Thus the field depends on the source mass M and the location $\vec{r}$, but not on the test mass m. 😊

Corollary 1.2.2 Near-Earth constant-g approximation

Let Earth have mass M_E and radius R_E. Let a point above Earth's surface have altitude h, and let $r = R_E + h$ denote its distance from Earth's center. Then the gravitational field is

$$\vec{g}(\vec{r}) = -\frac{GM_E}{(R_E + h)^2}\hat{r}.$$

If $h \ll R_E$, then $R_E + h \approx R_E$, so the field magnitude is approximately constant:

$$g(r) \approx \frac{GM_E}{R_E^2} = g.$$

Over a small region near Earth's surface, the radial direction changes very little. If the positive y-axis is chosen upward, then locally

$$\vec{g} \approx -g\hat{\jmath},$$

which is the constant-acceleration free-fall model used near Earth's surface.

Question 9: Worked example

Let Earth have mass $M_E = 5.97\times10^{24}$ kg, let Earth have radius $R_E = 6.37\times10^6$ m, and let the gravitational constant be $G = 6.67\times10^{-11}\,\mathrm{N\,m^2/kg^2}$. A spacecraft has mass $m = 500$ kg and is at altitude $h = 4.00\times10^5$ m above Earth's surface. Let $r = R_E + h$ denote the spacecraft's distance from Earth's center, let $\hat{r}$ denote the outward radial unit vector, and let $g_0 = 9.80\,\mathrm{m/s^2}$ denote the near-surface gravitational field magnitude. Find the gravitational field magnitude $g(r)$ at the spacecraft, the gravitational force $\vec{F}_g$ on the spacecraft, and the percent by which the near-surface value g_0 overestimates the true field magnitude at that altitude.

Solution: First compute the distance from Earth's center:

$$r = R_E + h = 6.37 \times 10^6\,\mathrm{m} + 4.00 \times 10^5\,\mathrm{m} = 6.77 \times 10^6\,\mathrm{m}.$$

The inverse-square field magnitude is

$$g(r) = \frac{GM_E}{r^2}.$$

Substitute the given values:

$$g(r) = \frac{\left(6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}\right)\left(5.97 \times 10^{24}\,\mathrm{kg}\right)}{(6.77 \times 10^6\,\mathrm{m})^2}.$$

This gives

$$g(r) \approx 8.69\,\mathrm{m/s^2}.$$

So the gravitational field at the spacecraft has magnitude $8.69\,\mathrm{m/s^2}$ and points toward Earth's center.

Now use $\vec{F}_g = m\vec{g}$. Since $\vec{g} = -(8.69\,\mathrm{m/s^2})\hat{r}$ at that location,

$$\vec{F}_g = m\vec{g} = (500\,\mathrm{kg})\left[-(8.69\,\mathrm{m/s^2})\hat{r}\right].$$

Therefore,

$$\vec{F}_g \approx -4.34 \times 10^3 \hat{r}\,\mathrm{N}.$$

Its magnitude is

$$F_g \approx 4.34 \times 10^3\,\mathrm{N},$$

and the negative sign means the force points toward Earth's center.

Finally, compare this with the near-surface value $g_0 = 9.80\,\mathrm{m/s^2}$. The amount of overestimate is

$$g_0 - g(r) = 9.80 - 8.69 = 1.11\,\mathrm{m/s^2}.$$

Thus the percent overestimate is

$$\frac{g_0 - g(r)}{g(r)} \times 100\% = \frac{1.11}{8.69} \times 100\% \approx 12.8\%.$$

So at an altitude of $4.00 \times 10^5\,\mathrm{m}$, the true gravitational field magnitude is about $8.69\,\mathrm{m/s^2}$, the spacecraft's gravitational force is about $4.34 \times 10^3\,\mathrm{N}$ toward Earth, and the near-surface constant-g model overestimates the field by about 12.8%.

1.2.4 Normal Force, Tension, and Constrained Motion

This subsection treats normal force and tension as contact forces set by constraints. In AP mechanics, a surface constrains motion perpendicular to itself, and an ideal string constrains connected objects to have linked motions.

Definition 1.2.4: Normal force, tension, and mechanical constraints

Let a body be in contact with a surface whose outward unit normal is $\hat{n}$. The *normal force* on the body is the contact force exerted by the surface perpendicular to the surface, so

$$\vec{N} = N\hat{n},$$

where $N \geq 0$ is its magnitude.

Let a body be attached to a taut string, rope, or cable whose direction at the body is given by the unit vector $\hat{t}$. The *tension force* exerted by that connector acts along the connector, so

$$\vec{T} = T\hat{t},$$

where $T \geq 0$ is its magnitude. In the ideal AP model, the string is massless and inextensible, and any pulley is massless and frictionless.

A *constraint* is a geometric restriction on motion. Contact with a rigid surface constrains motion perpendicular to the surface, and an inextensible string constrains connected bodies to move so that their accelerations along the string are related. Therefore normal force and tension are found from the constraint together with Newton's second law, not chosen independently.

Note:-

The symbols N and T do not automatically mean mg. The equality $N = mg$ holds only in special cases such as a body on a horizontal surface with no vertical acceleration and no other vertical forces. Likewise, $T = mg$ holds only for special cases such as a hanging mass in equilibrium or moving with constant velocity. On an incline, N is often less than mg. In an accelerating elevator, N can be greater than or less than mg. In a connected multi-body system, T is usually set by the common acceleration of the system, so it is generally not equal to the weight of either mass.

Example 1.2.1 (Short example: normal force in an accelerating elevator)

Choose the positive y-axis upward. A student of mass $m = 60.0\,\text{kg}$ stands on a scale in an elevator that accelerates upward with magnitude $a = 2.0\,\text{m/s}^2$. Let N denote the magnitude of the scale's normal force on the student, and let $g = 9.8\,\text{m/s}^2$.

Newton's second law in the vertical direction gives

$$N - mg = ma.$$

Therefore,

$$N = m(g + a) = (60.0\,\text{kg})(11.8\,\text{m/s}^2) = 708\,\text{N}.$$

So here $N > mg$. The normal force is determined by the acceleration constraint, not by a rule that it must equal the weight.

Proposition 1.2.1 Operational rules for frictionless contact and ideal strings

Let $\vec{a}$ denote the acceleration of a body in contact with a frictionless surface, let $\hat{n}$ denote a unit vector perpendicular to the surface, and let $a_\perp = \vec{a} \cdot \hat{n}$ denote the acceleration component perpendicular to the surface. Then Newton's second law in the perpendicular direction is

$$\sum F_\perp = ma_\perp.$$

If the contact surface is a fixed plane and the body remains in contact without leaving that plane, then $a_\perp = 0$, so the perpendicular force equation determines N. For a plane at angle θ to the horizontal, if the only force with a perpendicular component besides $\vec{N}$ is the weight, then

$$N = mg\cos\theta.$$

But if another force has a perpendicular component, it must also be included, so N is not automatically $mg\cos\theta$.

Let two or more bodies be connected by one ideal string over massless, frictionless pulleys. Then the tension magnitude is the same everywhere in that string:

$$T_1 = T_2 = \cdots = T.$$

If a_1 and a_2 denote acceleration components of two connected bodies measured along their allowed directions of motion, then the inextensible-string constraint gives equal magnitudes:

$$|a_1| = |a_2|.$$

With a consistent choice of positive directions, this often becomes a signed relation such as

$$a_1 = a_2 \qquad \text{or} \qquad a_1 = -a_2.$$

Then write one Newton's second law equation for each body and solve those equations together with the constraint relation.

Question 10: Worked example

A block of mass $m_1 = 3.0\,\text{kg}$ rests on a frictionless incline that makes an angle $\theta = 30^\circ$ with the horizontal. The block is connected by a light inextensible string that passes over a massless frictionless pulley to a hanging block of mass $m_2 = 2.0\,\text{kg}$. Let $g = 9.8\,\text{m/s}^2$.

Find the magnitude and direction of the acceleration of the system, the tension T in the string, and the magnitude of the normal force N exerted on m_1 by the incline.

Solution: Choose the system to be both blocks, but write Newton's second-law equations separately for each block.

For block m_1, choose the positive x-axis up the incline and the positive y-axis perpendicular to the incline, away from the surface. Let a denote the common acceleration magnitude. If m_2 moves downward, then m_1 moves up the incline with the same acceleration magnitude because the string is ideal and inextensible.

For block m_2, choose the positive direction downward so that both bodies have acceleration component $+a$ along their chosen directions.

Now identify the forces.

On m_1, the forces are the weight $\vec{W}_1$, the normal force $\vec{N}$, and the tension $\vec{T}$. Resolve the weight into components relative to the incline:

$$W_{1,\parallel} = m_1 g \sin\theta, \qquad W_{1,\perp} = m_1 g \cos\theta.$$

On m_2, the forces are its weight $\vec{W}_2$ downward and the tension $\vec{T}$ upward.

First check the likely direction of motion. The hanging weight has magnitude

$$m_2 g = (2.0\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) = 19.6\,\mathrm{N},$$

while the component of m_1's weight down the incline has magnitude

$$m_1 g \sin\theta = (3.0\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\sin 30^\circ = 14.7\,\mathrm{N}.$$

Since $19.6\,\mathrm{N} > 14.7\,\mathrm{N}$, the system accelerates with m_2 downward and m_1 up the incline, consistent with the chosen positive directions.

Apply Newton's second law to m_1 along the incline:

$$\sum F_\parallel = m_1 a.$$

The positive direction is up the incline, so

$$T - m_1 g \sin\theta = m_1 a.$$

Apply Newton's second law to m_1 perpendicular to the incline:

$$\sum F_\perp = m_1 a_\perp.$$

Because the block stays in contact with the incline, $a_\perp = 0$. Therefore,

$$N - m_1 g \cos\theta = 0,$$

so

$$N = m_1 g \cos\theta.$$

Apply Newton's second law to m_2 in the downward positive direction:

$$\sum F = m_2 a.$$

Thus,

$$m_2 g - T = m_2 a.$$

Now solve the two equations containing a and T:

$$T - m_1 g \sin\theta = m_1 a,$$

$$m_2 g - T = m_2 a.$$

Add them to eliminate T:

$$m_2 g - m_1 g \sin\theta = (m_1 + m_2) a.$$

Hence

$$a = \frac{m_2 g - m_1 g \sin\theta}{m_1 + m_2}.$$

Substitute the values:

$$a = \frac{19.6 - 14.7}{3.0 + 2.0}\,\mathrm{m/s^2} = \frac{4.9}{5.0}\,\mathrm{m/s^2} = 0.98\,\mathrm{m/s^2}.$$

Now find the tension from either block's equation. Using block m_2,

$$m_2 g - T = m_2 a,$$

so

$$T = m_2 g - m_2 a.$$

Substitute:

$$T = (2.0\,\mathrm{kg})(9.8\,\mathrm{m/s^2}) - (2.0\,\mathrm{kg})(0.98\,\mathrm{m/s^2}) = 19.6\,\mathrm{N} - 1.96\,\mathrm{N} = 17.64\,\mathrm{N}.$$

Thus

$$T \approx 17.6\,\mathrm{N}.$$

Finally, compute the normal force:

$$N = m_1 g \cos\theta = (3.0\,\mathrm{kg})(9.8\,\mathrm{m/s^2})\cos 30^\circ.$$

Using $\cos 30^\circ \approx 0.866$ gives

$$N \approx (29.4\,\mathrm{N})(0.866) = 25.5\,\mathrm{N}.$$

Therefore the system accelerates with magnitude

$$0.98\,\mathrm{m/s^2},$$

with m_2 moving downward and m_1 moving up the incline, the string tension is

$$T \approx 17.6\,\mathrm{N},$$

and the normal force on the incline block is

$$N \approx 25.5\,\mathrm{N}.$$

1.2.5 Static and Kinetic Friction

This subsection gives the AP dry-friction model for a body in contact with a surface. Friction is a contact force parallel to the surface, tied to the tendency for slipping or to actual slipping, while the normal force is perpendicular to the surface.

Definition 1.2.5: Static friction, kinetic friction, and coefficients of friction

Let a body be in contact with a surface. Let $\vec{N}$ denote the normal force exerted by the surface on the body, let $N = |\vec{N}|$ denote its magnitude, and let $\vec{f}$ denote the friction force exerted by the surface on the body.

The friction force acts parallel to the contact surface and opposes the relative motion or the tendency of relative motion between the surfaces.

If the surfaces are not slipping relative to each other, the friction is called *static friction*. Let $\vec{f}_s$ denote the static-friction force and let $f_s = |\vec{f}_s|$ denote its magnitude. Then static friction can adjust in magnitude up to a maximum value:

$$f_s \le \mu_s N,$$

where μ_s is the coefficient of static friction.

If the surfaces are sliding relative to each other, the friction is called *kinetic friction*. Let $\vec{f}_k$ denote the kinetic-friction force and let $f_k = |\vec{f}_k|$ denote its magnitude. In the idealized AP dry-friction model,

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction.

The coefficients μ_s and μ_k are dimensionless constants for the pair of surfaces in the chosen model.

Note:-

Static friction is *not* always equal to $\mu_s N$. The quantity $\mu_s N$ is the *maximum possible* static-friction magnitude. The actual static-friction magnitude is whatever value is required by Newton's second law to prevent slipping, as long as that required value does not exceed $\mu_s N$. Only at the threshold of slipping does $f_s = \mu_s N$.

Example 1.2.2 (Illustrative example)

Let a block rest on a horizontal table. Let $\vec{P} = (3.0\,\mathrm{N})\hat{\imath}$ denote a horizontal applied force on the block, and let the maximum possible static-friction magnitude be $\mu_s N = 5.0\,\mathrm{N}$.
Because the required friction to prevent motion is only 3.0 N, the block remains at rest and the actual static-friction force is

$$\vec{f}_s = -(3.0\,\mathrm{N})\hat{\imath},$$

not $-(5.0\,\mathrm{N})\hat{\imath}$. If the applied-force magnitude were increased beyond 5.0 N, static friction could no longer hold the block at rest and slipping would begin.

Proposition 1.2.2 Operational laws and direction rules

Let $\vec{N}$ denote the normal force on a body, let $N = |\vec{N}|$, and let a tangent axis be chosen along the contact surface.

① If there is no slipping at the contact, then the friction is static. Its magnitude must satisfy

$$f_s \le \mu_s N.$$

Its direction is opposite the direction the body would move *relative to the surface* if friction were absent.

② If the body slides relative to the surface, then the friction is kinetic. Its magnitude is

$$f_k = \mu_k N,$$

and its direction is opposite the relative velocity of the sliding surfaces.

③ Friction and the normal force are different parts of the same contact interaction: $\vec{N}$ is perpendicular to the surface, while $\vec{f}$ is parallel to the surface.

④ On an incline that makes an angle θ with the horizontal, if the only forces perpendicular to the surface are the normal force and the perpendicular component of the weight, then

$$N = mg\cos\theta.$$

Thus the friction magnitudes are often written as

$$f_{s,\max} = \mu_s mg\cos\theta, \qquad f_k = \mu_k mg\cos\theta.$$

Question 11: Worked example

A block of mass $m = 6.0\,\mathrm{kg}$ is released from rest on a rough incline that makes an angle $\theta = 35^\circ$ with the horizontal. Let $g = 9.8\,\mathrm{m/s^2}$ denote the magnitude of the gravitational field. Let the coefficient of static friction be $\mu_s = 0.40$ and the coefficient of kinetic friction be $\mu_k = 0.30$.
Determine whether the block remains at rest or starts to slide. If it slides, find the magnitude and direction of its acceleration.

Solution: Choose axes so that the x-axis is parallel to the incline and positive down the incline, and the y-axis is perpendicular to the incline and positive away from the surface. Let a_x and a_y denote the acceleration components in these directions.

The forces on the block are the weight $\vec{W}$, the normal force $\vec{N}$ exerted by the incline, and a friction force

$\vec{f}$ exerted by the incline.

Because the block remains on the surface, there is no acceleration perpendicular to the incline, so

$$a_y = 0.$$

Resolve the weight into components relative to the chosen axes. The component parallel to the incline has magnitude

$$W_x = mg\sin\theta,$$

and the component perpendicular to the incline has magnitude

$$W_y = mg\cos\theta.$$

Apply Newton's second law perpendicular to the incline:

$$\sum F_y = ma_y.$$

Since the positive y-direction is away from the surface,

$$N - mg\cos\theta = 0.$$

Therefore,

$$N = mg\cos\theta.$$

Substitute the given values:

$$N = (6.0\,\text{kg})(9.8\,\text{m/s}^2)\cos 35^\circ \approx 48.2\,\text{N}.$$

Now check whether static friction can prevent motion. The component of the weight down the incline is

$$mg\sin\theta = (6.0\,\text{kg})(9.8\,\text{m/s}^2)\sin 35^\circ \approx 33.7\,\text{N}.$$

The maximum possible static-friction magnitude is

$$f_{s,\max} = \mu_s N = (0.40)(48.2\,\text{N}) \approx 19.3\,\text{N}.$$

To keep the block at rest, static friction would need to balance the 33.7 N downslope component of the weight by acting upslope with magnitude 33.7 N. But

$$33.7\,\text{N} > 19.3\,\text{N}.$$

So static friction is not large enough to hold the block at rest. The block starts to slide down the incline.

Once the block is sliding, the friction is kinetic and points up the incline. Its magnitude is

$$f_k = \mu_k N = (0.30)(48.2\,\text{N}) \approx 14.5\,\text{N}.$$

Now apply Newton's second law parallel to the incline:

$$\sum F_x = ma_x.$$

Taking down the incline as positive, the component of the weight is positive and the kinetic friction is negative, so

$$mg\sin\theta - f_k = ma_x.$$

Substitute the values:

$$33.7\,\text{N} - 14.5\,\text{N} = (6.0\,\text{kg})a_x.$$

Thus,

$$a_x = \frac{19.2\,\text{N}}{6.0\,\text{kg}} \approx 3.2\,\text{m/s}^2.$$

Therefore, the block does not remain at rest. It slides down the incline with acceleration

$$3.2\,\text{m/s}^2$$

down the incline.

1.2.6 Hooke's Law and Spring Models

This subsection uses the local displacement-from-equilibrium viewpoint for spring forces. That viewpoint makes the restoring nature of the force and the modeling of combined springs especially clear.

Definition 1.2.6: Spring constant and displacement from equilibrium

Let an ideal spring act along a line with positive direction given by the unit vector $\hat{u}$. Let x denote the signed displacement of the attached object from the spring's equilibrium position, measured along that line, so $x > 0$ means displacement in the $+\hat{u}$ direction and $x < 0$ means displacement in the opposite direction.
Let $k > 0$ denote the *spring constant*. It measures the stiffness of the spring: a larger k means a larger restoring force for the same displacement. The SI units of k are N/m.

Theorem 1.2.4 Hooke's law and equivalent spring constants

Let x denote the signed displacement from equilibrium for an ideal spring with spring constant k, measured along the spring's line of action with unit vector $\hat{u}$. Then the spring force on the attached object is

$$\vec{F}_s = -kx\hat{u}.$$

In one-dimensional scalar form,

$$F_s = -kx.$$

The negative sign means the spring force opposes the displacement from equilibrium.
If several ideal springs are modeled by one equivalent spring with constant k_{eq}, then:

① **Parallel:** if all springs undergo the same displacement x, then

$$k_{\text{eq}} = k_1 + k_2 + \cdots + k_n.$$

② **Series:** if all springs carry the same force magnitude, then

$$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}.$$

Why the sign and combination rules are correct: If $x > 0$, the object is displaced in the $+\hat{u}$ direction, so the restoring spring force must point in the $-\hat{u}$ direction. If $x < 0$, the restoring force must point in the $+\hat{u}$ direction. Both cases are captured by the single vector formula

$$\vec{F}_s = -kx\hat{u}.$$

For springs in parallel, each spring has the same displacement x, so the forces add:

$$\vec{F}_{\text{tot}} = -(k_1 + k_2 + \cdots + k_n)x\hat{u}.$$

Therefore $k_{\text{eq}} = k_1 + k_2 + \cdots + k_n$.

For springs in series, let F denote the common force magnitude through each spring, and let x_i denote the magnitude of the displacement of spring i. Since $F = k_i x_i$, we have

$$x_i = \frac{F}{k_i}.$$

The total displacement magnitude is then

$$x = x_1 + x_2 + \cdots + x_n = F\left(\frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}\right).$$

If the combination is replaced by one equivalent spring, then $F = k_{\text{eq}}x$, so

$$\frac{1}{k_{\text{eq}}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}.$$

😄

Corollary 1.2.3 Vertical equilibrium shift and the local spring equation

Let a mass m hang from a vertical spring of spring constant k. Choose downward as positive. Let y denote the downward displacement from the spring's unstretched length, let y_{eq} denote the equilibrium value of y, and let

$$x = y - y_{\text{eq}}$$

denote the displacement from equilibrium.
At static equilibrium,

$$mg - ky_{\text{eq}} = 0,$$

so

$$y_{\text{eq}} = \frac{mg}{k}.$$

Then the net force on the mass is

$$F_{\text{net}} = mg - ky = mg - k(y_{\text{eq}} + x) = -kx.$$

Thus, when motion is measured from equilibrium, gravity has already been accounted for, and the net force again has Hooke form.

Question 12: Worked example

A block of mass $m = 0.50\,\text{kg}$ hangs at rest from a vertical spring with spring constant $k = 200\,\text{N/m}$. Choose downward as positive. Let y denote the block's downward displacement from the spring's unstretched length, let y_{eq} denote the equilibrium displacement, and let $x = y - y_{\text{eq}}$ denote the displacement from equilibrium.

(a) Find y_{eq}.

(b) The block is pulled downward $0.030\,\text{m}$ from equilibrium and released from rest. Find the spring force and the net force at the instant of release.

(c) Find the acceleration at the instant of release.

Solution: At equilibrium, the acceleration is zero, so the net force is zero:

$$mg - ky_{\text{eq}} = 0.$$

Therefore,

$$y_{\text{eq}} = \frac{mg}{k} = \frac{(0.50\,\text{kg})(9.8\,\text{m/s}^2)}{200\,\text{N/m}} = 0.0245\,\text{m}.$$

So the equilibrium position is

$$2.45 \times 10^{-2}\,\text{m}$$

below the unstretched length.

At the instant of release, the block is displaced

$$x = +0.030\,\text{m}$$

from equilibrium, because downward is positive. The actual spring displacement from the unstretched length is

$$y = y_{\text{eq}} + x = 0.0245\,\text{m} + 0.030\,\text{m} = 0.0545\,\text{m}.$$

The spring force along the chosen vertical axis is

$$F_s = -ky = -(200\,\text{N/m})(0.0545\,\text{m}) = -10.9\,\text{N}.$$

The negative sign means the spring force is upward. Its magnitude is therefore

$$10.9\,\text{N}.$$

The weight is

$$mg = (0.50\,\text{kg})(9.8\,\text{m/s}^2) = 4.90\,\text{N}$$

in the positive direction. Hence the net force is

$$F_{\text{net}} = mg - ky = 4.90\,\text{N} - 10.9\,\text{N} = -6.0\,\text{N}.$$

Equivalently, using the local displacement-from-equilibrium form,

$$F_{\text{net}} = -kx = -(200\,\text{N/m})(0.030\,\text{m}) = -6.0\,\text{N},$$

which agrees.

Now apply Newton's second law:

$$F_{\text{net}} = ma.$$

Thus,

$$a = \frac{F_{\text{net}}}{m} = \frac{-6.0\,\text{N}}{0.50\,\text{kg}} = -12\,\text{m/s}^2.$$

The negative sign means the acceleration is upward, so the block's acceleration at release has magnitude

$$12\,\text{m/s}^2.$$

Therefore,

$$y_{\text{eq}} = 0.0245\,\text{m},$$

the spring force at release is

$$10.9\,\text{N}$$

upward, the net force is

$$6.0\,\text{N}$$

upward, and the acceleration is

$$12\,\text{m/s}^2$$

upward.

1.2.7 Drag Forces and Terminal Velocity

This subsection uses the AP linear-drag model, in which the resistive force depends on the object's instantaneous velocity. The local-first viewpoint is Newton's second law with a velocity-dependent force.

Definition 1.2.7: Linear drag and terminal velocity

Let an object move through a fluid with velocity $\vec{v}$ relative to the fluid, and let $b > 0$ denote the linear-drag coefficient. In the linear-drag model, the drag force exerted by the fluid on the object is

$$\vec{F}_D = -b\vec{v}.$$

The negative sign means that the drag force always points opposite the velocity.
If an object falls vertically through the fluid and eventually moves with constant velocity, that steady velocity is called the *terminal velocity*. At terminal velocity, the net force is zero, so the acceleration is zero.

Theorem 1.2.5 Vertical-fall ODE and terminal-speed result

Choose the vertical axis positive downward. Let $v(t)$ denote the downward velocity of an object of mass m at time t, let g denote the gravitational field strength, and let $b > 0$ denote the linear-drag coefficient. Then the vertical equation of motion is

$$m\frac{dv}{dt} = mg - bv.$$

The terminal speed v_T is the steady-state value obtained by setting the net force equal to zero:

$$mg - bv_T = 0,$$

so

$$v_T = \frac{mg}{b}.$$

If the initial velocity is $v(0) = v_0$, then

$$v(t) = v_T + (v_0 - v_T)e^{-bt/m}.$$

In particular, if the object is released from rest, then

$$v(t) = v_T\left(1 - e^{-bt/m}\right).$$

Short derivation of the velocity function: Start with Newton's second law for vertical fall in the downward-positive direction:

$$m\frac{dv}{dt} = mg - bv.$$

Define the terminal speed by

$$v_T = \frac{mg}{b}.$$

Then the differential equation becomes

$$\frac{dv}{dt} = \frac{b}{m}(v_T - v).$$

Separate variables:

$$\frac{dv}{v_T - v} = \frac{b}{m}\,dt.$$

Integrating gives

$$-\ln|v_T - v| = \frac{b}{m}t + C.$$

Therefore

$$v_T - v = Ce^{-bt/m}$$

for some constant C, so

$$v = v_T - Ce^{-bt/m}.$$

Now use the initial condition $v(0) = v_0$:

$$v_0 = v_T - C.$$

Thus $C = v_T - v_0$, and

$$v(t) = v_T - (v_T - v_0)e^{-bt/m} = v_T + (v_0 - v_T)e^{-bt/m}.$$

If $v_0 = 0$, this reduces to

$$v(t) = v_T\left(1 - e^{-bt/m}\right).$$

As $t \to \infty$, the exponential term approaches 0, so $v(t) \to v_T$. 😊

Example 1.2.3 (Illustrative example)

Choose downward as positive. A ball of mass $m = 0.20\,\text{kg}$ falls through air with linear drag coefficient $b = 0.50\,\text{N}\cdot\text{s/m}$. Find the terminal speed and the acceleration when the downward speed is $v = 2.0\,\text{m/s}$.
The terminal speed is

$$v_T = \frac{mg}{b} = \frac{(0.20\,\text{kg})(9.8\,\text{m/s}^2)}{0.50\,\text{N}\cdot\text{s/m}} = 3.92\,\text{m/s}.$$

When $v = 2.0\,\mathrm{m/s}$, Newton's second law gives

$$m\frac{dv}{dt} = mg - bv,$$

so the acceleration is

$$a = g - \frac{b}{m}v = 9.8 - \frac{0.50}{0.20}(2.0) = 4.8\,\mathrm{m/s^2}.$$

Since this value is positive in the downward-positive coordinate system, the ball is still accelerating downward.

Question 13: Worked example

A small package of mass $m = 0.40\,\mathrm{kg}$ is dropped from rest and falls vertically through air. Choose downward as positive. Let $v(t)$ denote the package's downward velocity at time t, let $g = 9.8\,\mathrm{m/s^2}$, and let the drag force be modeled by

$$\vec{F}_D = -b\vec{v}$$

with $b = 0.80\,\mathrm{N \cdot s/m}$.

(a) Write the differential equation for $v(t)$.

(b) Find the terminal speed.

(c) Find an explicit formula for $v(t)$.

(d) Find the package's velocity and acceleration at $t = 1.0\,\mathrm{s}$.

Solution: The forces on the package are its weight $\vec{W}$ downward and the drag force $\vec{F}_D$ upward because the package is moving downward. Since downward is chosen as positive, the scalar force equation is

$$mg - bv = m\frac{dv}{dt}.$$

For part (a), the differential equation is therefore

$$m\frac{dv}{dt} = mg - bv.$$

Substitute $m = 0.40\,\mathrm{kg}$ and $b = 0.80\,\mathrm{N \cdot s/m}$:

$$(0.40)\frac{dv}{dt} = (0.40)(9.8) - 0.80v.$$

So an equivalent form is

$$\frac{dv}{dt} = 9.8 - 2.0v.$$

For part (b), terminal speed occurs when the acceleration is zero, so

$$\frac{dv}{dt} = 0.$$

Then

$$mg - bv_T = 0,$$

which gives

$$v_T = \frac{mg}{b} = \frac{(0.40)(9.8)}{0.80} = 4.9\,\mathrm{m/s}.$$

For part (c), because the package is dropped from rest, the initial condition is

$$v(0) = 0.$$

Using the linear-drag result for release from rest,

$$v(t) = v_T \left(1 - e^{-bt/m}\right).$$

Substitute the values:

$$v(t) = 4.9 \left(1 - e^{-(0.80/0.40)t}\right).$$

Therefore,

$$v(t) = 4.9 \left(1 - e^{-2.0t}\right) \text{ m/s}.$$

For part (d), evaluate this at $t = 1.0\,\text{s}$:

$$v(1.0) = 4.9 \left(1 - e^{-2.0}\right) \text{ m/s}.$$

Since

$$e^{-2.0} \approx 0.135,$$

we get

$$v(1.0) \approx 4.9(0.865) = 4.24\,\text{m/s}.$$

So after 1.0 s the package is moving downward at about

$$4.24\,\text{m/s}.$$

Now find the acceleration. From the differential equation,

$$a = \frac{dv}{dt} = 9.8 - 2.0v.$$

At $t = 1.0\,\text{s}$,

$$a = 9.8 - 2.0(4.24) = 1.32\,\text{m/s}^2.$$

This is positive in the downward-positive coordinate system, so the acceleration is still downward.

Therefore the differential equation is

$$\frac{dv}{dt} = 9.8 - 2.0v,$$

the terminal speed is

$$4.9\,\text{m/s},$$

the velocity function is

$$v(t) = 4.9 \left(1 - e^{-2.0t}\right) \text{ m/s},$$

and at $t = 1.0\,\text{s}$ the package has velocity

$$4.24\,\text{m/s}$$

downward and acceleration

$$1.32\,\text{m/s}^2$$

downward.

1.2.8 Circular Motion and Orbital Dynamics

This subsection resolves motion on a circle into radial and tangential parts. In AP mechanics, the key idea is that even when the speed stays constant, the velocity direction changes, so there is still an inward acceleration. In a circular orbit, gravity supplies that inward acceleration.

Definition 1.2.8: Radial and tangential directions for circular motion

Let a particle move on a circle of radius r centered at a fixed point. Let $\vec{r}$ denote the particle's position vector from the center, let $\hat{r} = \vec{r}/r$ denote the outward radial unit vector, let $\hat{t}$ denote the unit vector tangent to the path in the direction of motion, let v denote the speed, and let $\vec{a}$ denote the acceleration. The acceleration can be decomposed into radial and tangential parts:

$$\vec{a} = a_r\hat{r} + a_t\hat{t}.$$

The tangential component a_t changes the speed, while the radial component changes the direction of the velocity. For circular motion, the radial acceleration points toward the center, so its direction is $-\hat{r}$.

Theorem 1.2.6 Centripetal acceleration and circular-orbit speed

Let a particle of mass m move on a circle of radius r with speed v. Let $\hat{r}$ denote the outward radial unit vector and let $\hat{t}$ denote the tangential unit vector in the direction of motion. Then the acceleration is

$$\vec{a} = -\frac{v^2}{r}\hat{r} + \frac{dv}{dt}\hat{t}.$$

In particular, for uniform circular motion, $dv/dt = 0$, so

$$\vec{a} = -\frac{v^2}{r}\hat{r}, \qquad a_r = \frac{v^2}{r} \text{ inward.}$$

Now let the same particle be in a circular orbit around a spherically symmetric body of mass M, with orbital radius r. If gravity is the only significant radial force, then

$$\frac{GMm}{r^2} = \frac{mv^2}{r},$$

so the orbital speed is

$$v = \sqrt{\frac{GM}{r}}.$$

Short derivation from radial force balance: Let F_r denote the net inward radial force on the particle. For circular motion of radius r and speed v, the required inward acceleration has magnitude v^2/r, so Newton's second law in the radial direction gives

$$F_r = m\frac{v^2}{r}.$$

Because inward is the $-\hat{r}$ direction,

$$\vec{a}_r = -\frac{v^2}{r}\hat{r}.$$

If the speed changes, the tangential component is

$$\vec{a}_t = \frac{dv}{dt}\hat{t}.$$

Therefore,

$$\vec{a} = \vec{a}_r + \vec{a}_t = -\frac{v^2}{r}\hat{r} + \frac{dv}{dt}\hat{t}.$$

For a circular orbit, gravity provides the entire inward force, so

$$F_r = \frac{GMm}{r^2} = m\frac{v^2}{r}.$$

Canceling m and solving for v gives

$$v = \sqrt{\frac{GM}{r}}.$$

😆

Corollary 1.2.4 Circular-orbit period

Let T denote the period of a circular orbit of radius r around a spherically symmetric body of mass M. Since one orbit has circumference $2\pi r$,

$$v = \frac{2\pi r}{T}.$$

Combine this with

$$v = \sqrt{\frac{GM}{r}}$$

to obtain

$$T = 2\pi\sqrt{\frac{r^3}{GM}}.$$

Near Earth's surface, if $r \approx R_E$ and $g = GM/R_E^2$, this becomes

$$T \approx 2\pi\sqrt{\frac{R_E}{g}}.$$

Question 14: Worked example

A satellite moves in a circular orbit at altitude $h = 4.00 \times 10^5$ m above Earth's surface. Let Earth have mass $M_E = 5.97 \times 10^{24}$ kg, radius $R_E = 6.37 \times 10^6$ m, and let the gravitational constant be $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2}$.
Find the satellite's orbital radius r, orbital speed v, orbital period T, and centripetal acceleration magnitude a_r.

Solution: First compute the orbital radius from Earth's center:

$$r = R_E + h = 6.37 \times 10^6\,\mathrm{m} + 4.00 \times 10^5\,\mathrm{m} = 6.77 \times 10^6\,\mathrm{m}.$$

For a circular orbit, gravity supplies the centripetal force, so

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}.$$

Cancel m and solve for v:

$$v = \sqrt{\frac{GM_E}{r}}.$$

Substitute the given values:

$$v = \sqrt{\frac{(6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2})\,(5.97 \times 10^{24}\,\mathrm{kg})}{6.77 \times 10^6\,\mathrm{m}}}.$$

This gives

$$v \approx 7.67 \times 10^3\,\mathrm{m/s}.$$

Now find the period from

$$T = \frac{2\pi r}{v}.$$

Thus,

$$T = \frac{2\pi\left(6.77 \times 10^6\,\mathrm{m}\right)}{7.67 \times 10^3\,\mathrm{m/s}} \approx 5.55 \times 10^3\,\mathrm{s}.$$

In minutes,

$$T \approx \frac{5.55 \times 10^3\,\mathrm{s}}{60} \approx 92.4\,\mathrm{min}.$$

Finally, the centripetal acceleration magnitude is

$$a_r = \frac{v^2}{r}.$$

Using the speed just found,

$$a_r = \frac{\left(7.67 \times 10^3\,\mathrm{m/s}\right)^2}{6.77 \times 10^6\,\mathrm{m}} \approx 8.69\,\mathrm{m/s^2}.$$

Therefore the satellite's orbital radius is

$$r = 6.77 \times 10^6\,\mathrm{m},$$

its orbital speed is

$$v \approx 7.67 \times 10^3\,\mathrm{m/s},$$

its orbital period is

$$T \approx 5.55 \times 10^3\,\mathrm{s} \approx 92.4\,\mathrm{min},$$

and its centripetal acceleration magnitude is

$$a_r \approx 8.69\,\mathrm{m/s^2}.$$

1.3 Work, Energy, and Power

In this unit, we develop the energy viewpoint for mechanics. We begin with work as the accumulated effect of a force acting through a displacement, using the line-integral form $W = \int \vec{F} \cdot d\vec{r}$ as the calculus backbone and then connecting net work to changes in kinetic energy K.

We then shift to conservative forces and potential energy U, which lets us track mechanical energy with $E_{\mathrm{mech}} = K + U$. With that accounting framework in place, we finish by defining power P as the rate at which energy is transferred or transformed. The emphasis throughout stays on AP-level mechanical energy, including one-dimensional potential-energy graphs and clear bookkeeping of energy changes.

1.3.1 Work as a Line Integral

This subsection develops work from the local relation between force and an infinitesimal displacement. In AP mechanics, this gives a clean calculus-based way to handle variable forces and to track when a force helps, opposes, or does no work on the motion.

Definition 1.3.1: Infinitesimal work and total work along a path

Let a particle move along a path C. Let $d\vec{r}$ denote an infinitesimal displacement vector of the particle, let $ds = |d\vec{r}|$ denote the corresponding infinitesimal path length, and let $\vec{F}$ denote the force acting on the particle at that location.
The *infinitesimal work* done by the force during that displacement is

$$dW = \vec{F} \cdot d\vec{r}.$$

If $\hat{t}$ denotes the unit tangent to the path and $F_{\|} = \vec{F} \cdot \hat{t}$ denotes the component of the force parallel to the motion, then

$$dW = F_{\|}\,ds.$$

Therefore the *total work* done by the force as the particle moves along the path C is

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C F_{\|}\,ds.$$

Theorem 1.3.1 Line-integral form of work and the constant-force special case

Let a particle move from an initial point to a final point along a path C under a force $\vec{F}$. Then the work done by that force is

$$W = \int_C \vec{F} \cdot d\vec{r}.$$

Only the component of $\vec{F}$ parallel to the instantaneous displacement contributes, so equivalently

$$W = \int_C F_{\|}\,ds.$$

If the force is constant in magnitude and direction, then

$$W = \vec{F} \cdot \Delta\vec{r},$$

where $\Delta\vec{r}$ is the total displacement from the initial point to the final point. In particular, if the motion is along a straight line parallel to the constant force, then

$$W = F\,\Delta s.$$

Note:-

Work is positive when the force has a component in the same direction as the displacement, negative when that component is opposite the displacement, and zero when the force is perpendicular to the displacement. For a general force, the value of $W = \int_C \vec{F}\cdot d\vec{r}$ can depend on the path C, not just on the endpoints. In AP problems, this often appears when the force changes with position or when different paths make the parallel component $F_{\parallel}$ different. Typical zero-work cases include a normal force on motion along a surface or a centripetal force in uniform circular motion, because those forces are perpendicular to the instantaneous displacement.

Why the line integral gives total work: Break the path into many small displacement vectors $\Delta\vec{r}_1, \Delta\vec{r}_2, \ldots, \Delta\vec{r}_n$. Over each small segment, the work is approximately

$$\Delta W_k \approx \vec{F}_k \cdot \Delta\vec{r}_k.$$

Summing over the path gives

$$W \approx \sum_{k=1}^{n} \vec{F}_k \cdot \Delta\vec{r}_k.$$

In the limit as the segments become infinitesimal, this Riemann sum becomes

$$W = \int_C \vec{F}\cdot d\vec{r}.$$

If $\vec{F}$ is constant, then it can be taken outside the integral:

$$W = \vec{F}\cdot\int_C d\vec{r} = \vec{F}\cdot\Delta\vec{r}.$$

😄

Question 15: Worked example

A cart moves along a straight horizontal track from $x_i = 0$ to $x_f = 5.0\,\text{m}$. Let x denote the cart's position coordinate, and let the applied force on the cart be

$$\vec{F}(x) = (6.0 - 2.0x)\hat{\imath}\,\text{N}.$$

Find the work done by this force on the cart over the interval from $x = 0$ to $x = 5.0\,\text{m}$. State where the force does negative work.

Solution: Because the motion is along the x-axis, the displacement element is

$$d\vec{r} = dx\,\hat{\imath}.$$

Therefore,

$$dW = \vec{F}\cdot d\vec{r} = [(6.0 - 2.0x)\hat{\imath}]\cdot(dx\,\hat{\imath}) = (6.0 - 2.0x)\,dx.$$

So the total work is

$$W = \int_0^{5.0} (6.0 - 2.0x)\,dx.$$

Evaluate the integral:

$$W = \left[6.0x - x^2\right]_0^{5.0}.$$

Substitute the limits:

$$W = (6.0)(5.0) - (5.0)^2 - \left[(6.0)(0) - 0^2\right] = 30.0 - 25.0 = 5.0\,\text{J}.$$

Thus the force does

$$5.0\,\mathrm{J}$$

of net work on the cart.

To identify where the force does negative work, find where the force component along the motion becomes negative:

$$6.0 - 2.0x < 0.$$

This occurs when

$$x > 3.0\,\mathrm{m}.$$

So from $x = 3.0\,\mathrm{m}$ to $x = 5.0\,\mathrm{m}$, the force points opposite the displacement and does negative work. From $x = 0$ to $x = 3.0\,\mathrm{m}$, it does positive work.

1.3.2 Kinetic Energy and the Work-Energy Theorem

This subsection introduces kinetic energy as the energy of motion and the work-energy theorem as the main AP bridge from force and displacement to speed without solving for time explicitly.

Definition 1.3.2: Kinetic energy and net work

Let m denote the mass of a particle or body, let $\vec{v}$ denote its velocity, and let $v = |\vec{v}|$ denote its speed. The *kinetic energy* of the body is

$$K = \tfrac{1}{2}mv^2.$$

If forces $\vec{F}_1, \vec{F}_2, \ldots,$ and $\vec{F}_n$ act on the body while it undergoes an infinitesimal displacement $d\vec{r}$, then the differential work done by force i is

$$dW_i = \vec{F}_i \cdot d\vec{r}.$$

Let the net force be

$$\vec{F}_{\mathrm{net}} = \sum_{i=1}^{n} \vec{F}_i.$$

Then the *net work* is the sum of the works done by all forces:

$$dW_{\mathrm{net}} = \sum_{i=1}^{n} dW_i = \vec{F}_{\mathrm{net}} \cdot d\vec{r}.$$

Over a finite motion,

$$W_{\mathrm{net}} = \int \vec{F}_{\mathrm{net}} \cdot d\vec{r} = \sum_{i=1}^{n} W_i.$$

The SI unit of both work and kinetic energy is the joule, where $1\,\mathrm{J} = 1\,\mathrm{N\,m}$.

Theorem 1.3.2 Work-energy theorem

Let m denote the mass of a body, let $\vec{v}$ denote its velocity, let $v = |\vec{v}|$ denote its speed, and let $d\vec{r}$ denote an infinitesimal displacement of the body. Then

$$dK = \vec{F}_{\mathrm{net}} \cdot d\vec{r}.$$

Integrating from an initial state to a final state gives the work-energy theorem:

$$W_{\mathrm{net}} = \Delta K = K_f - K_i = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2.$$

Thus the net work done on a body equals the change in its kinetic energy.

Note:-

The work-energy theorem is often more efficient than combining Newton's second law with kinematics when the problem asks for a speed after a known displacement or after a known amount of work. It avoids solving for time and often avoids solving for acceleration explicitly. In AP mechanics, *net work* means the algebraic sum of the work done by all forces on the chosen system. A force parallel to the displacement does positive work, a force opposite the displacement does negative work, and a force perpendicular to the displacement does zero work.

Short derivation from Newton II: . Let m denote the constant mass of the body, let $\vec{v}$ denote its velocity, and let $d\vec{r}$ denote its infinitesimal displacement. Start with Newton's second law,

$$\vec{F}_{\text{net}} = m\frac{d\vec{v}}{dt}.$$

Since

$$d\vec{r} = \vec{v}\,dt,$$

dot both sides of Newton's second law with $d\vec{r}$:

$$\vec{F}_{\text{net}} \cdot d\vec{r} = m\frac{d\vec{v}}{dt} \cdot (\vec{v}\,dt) = m\vec{v} \cdot d\vec{v}.$$

Now use

$$v^2 = \vec{v} \cdot \vec{v},$$

so

$$d(v^2) = 2\vec{v} \cdot d\vec{v}.$$

Therefore,

$$\vec{F}_{\text{net}} \cdot d\vec{r} = m\vec{v} \cdot d\vec{v} = d\left(\tfrac{1}{2}mv^2\right) = dK.$$

Integrating gives

$$W_{\text{net}} = \int \vec{F}_{\text{net}} \cdot d\vec{r} = \int dK = K_f - K_i = \Delta K.$$

☺

Question 16: Worked example

Choose the positive x-axis to the right. A crate of mass $m = 4.0\,\text{kg}$ moves to the right on a horizontal floor with initial speed $v_i = 3.0\,\text{m/s}$. A constant applied force of magnitude $F_A = 20.0\,\text{N}$ acts to the right while the crate moves a horizontal distance $\Delta x = 6.0\,\text{m}$. Kinetic friction of magnitude $f_k = 4.0\,\text{N}$ acts to the left. The normal force and the weight act vertically.
Find the crate's final speed v_f.

Solution: Use the work-energy theorem:

$$W_{\text{net}} = \Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2.$$

Compute the work done by each force.
The applied force is parallel to the displacement, so its work is positive:

$$W_A = F_A\Delta x = (20.0\,\text{N})(6.0\,\text{m}) = 120\,\text{J}.$$

The friction force is opposite the displacement, so its work is negative:

$$W_f = -f_k\Delta x = -(4.0\,\text{N})(6.0\,\text{m}) = -24\,\text{J}.$$

The normal force and the weight are perpendicular to the horizontal displacement, so each does zero work:

$$W_N = 0, \qquad W_g = 0.$$

Therefore the net work is

$$W_{\text{net}} = W_A + W_f + W_N + W_g = 120\,\text{J} - 24\,\text{J} = 96\,\text{J}.$$

Now find the initial kinetic energy:

$$K_i = \tfrac{1}{2}mv_i^2 = \tfrac{1}{2}(4.0\,\text{kg})(3.0\,\text{m/s})^2 = 18\,\text{J}.$$

So the final kinetic energy is

$$K_f = K_i + W_{\text{net}} = 18\,\text{J} + 96\,\text{J} = 114\,\text{J}.$$

Use $K_f = \tfrac{1}{2}mv_f^2$:

$$\tfrac{1}{2}(4.0\,\text{kg})v_f^2 = 114\,\text{J}.$$

Thus

$$2.0\,v_f^2 = 114, \qquad v_f^2 = 57, \qquad v_f = \sqrt{57}\,\text{m/s} \approx 7.5\,\text{m/s}.$$

Therefore the crate's final speed is

$$v_f \approx 7.5\,\text{m/s}.$$

This method is shorter than solving for the acceleration and then using a kinematics equation, because the theorem connects net work directly to the change in speed.

1.3.3 Conservative Forces and Potential Energy

This subsection introduces conservative forces through path-independent work and uses that idea to define potential energy differences for an interacting system.

Definition 1.3.3: Conservative force and potential energy difference

Let $\vec{F}_c$ denote a force associated with some interaction, let i and f denote initial and final positions, and let C denote a path from i to f. The force $\vec{F}_c$ is called *conservative* if the work

$$W_c(i \to f) = \int_C \vec{F}_c \cdot d\vec{r}$$

depends only on the endpoints i and f, not on the path C.
For a conservative force, the corresponding *potential energy difference* of the interacting system is defined by

$$\Delta U = U_f - U_i = -\int_i^f \vec{F}_c \cdot d\vec{r}.$$

Thus the work done by the conservative force is

$$W_c = -\Delta U.$$

Theorem 1.3.3 Equivalent conservative-force relations

Let $\vec{F}_c$ denote a conservative force and let $d\vec{r}$ denote an infinitesimal displacement. Then the following relations hold:

$$\oint \vec{F}_c \cdot d\vec{r} = 0,$$

so the work done by $\vec{F}_c$ around any closed loop is zero.
Equivalently, for any two paths C_1 and C_2 connecting the same endpoints,

$$\int_{C_1} \vec{F}_c \cdot d\vec{r} = \int_{C_2} \vec{F}_c \cdot d\vec{r}.$$

The local potential-energy relation is

$$dU = -\vec{F}_c \cdot d\vec{r}.$$

For one-dimensional motion along the x-axis,

$$F_x = -\frac{dU}{dx}.$$

More generally, one may write lightly

$$\vec{F}_c = -\nabla U.$$

Note:-

Potential energy is a property of a *system*, not of a single isolated object. For example, gravitational potential energy belongs to the Earth-object system, and spring potential energy belongs to the block-spring system. A conservative force can transfer energy between kinetic and potential forms without making the potential difference depend on the path. By contrast, nonconservative forces such as kinetic friction and air resistance have path-dependent work, so a single-valued potential-energy function for that interaction is not defined in this AP sense.

Why zero closed-loop work gives a well-defined ΔU: Assume that for every closed path,

$$\oint \vec{F}_c \cdot d\vec{r} = 0.$$

Take two paths C_1 and C_2 from the same initial point i to the same final point f. Traverse C_1 from i to f and then traverse C_2 backward from f to i. This makes a closed loop, so

$$\int_{C_1} \vec{F}_c \cdot d\vec{r} + \int_{f \to i \text{ on } C_2} \vec{F}_c \cdot d\vec{r} = 0.$$

Reversing the limits changes the sign of the second integral, giving

$$\int_{C_1} \vec{F}_c \cdot d\vec{r} = \int_{C_2} \vec{F}_c \cdot d\vec{r}.$$

So the work depends only on the endpoints. Therefore the quantity

$$U_f - U_i = -\int_i^f \vec{F}_c \cdot d\vec{r}$$

is path independent and is a well-defined potential-energy difference. 😄

Question 17: Worked example

A block of mass $m = 0.50\,\text{kg}$ is attached to an ideal horizontal spring of spring constant $k = 200\,\text{N/m}$ on a frictionless track. Let x denote the block's displacement from the spring's equilibrium position, with positive x to the right. Initially the block is held at rest at $x_i = +0.15\,\text{m}$ and then released. Find:

1. the change in spring potential energy ΔU_s as the block moves to $x_f = 0$,
2. the work done by the spring during that motion, and
3. the block's speed v_f when it passes through equilibrium.

Solution: For an ideal spring, the spring force is

$$\vec{F}_s = -kx\,\hat{\imath}.$$

Since the motion is one-dimensional, the potential-energy relation gives

$$F_x = -\frac{dU_s}{dx}.$$

So

$$-kx = -\frac{dU_s}{dx} \qquad \Rightarrow \qquad \frac{dU_s}{dx} = kx.$$

Integrate with respect to x:

$$U_s(x) = \int kx\,dx = \tfrac{1}{2}kx^2 + C.$$

Choose the usual reference $U_s = 0$ at $x = 0$, so $C = 0$ and

$$U_s(x) = \tfrac{1}{2}kx^2.$$

At the initial position,

$$U_{s,i} = \tfrac{1}{2}(200)(0.15)^2 = 100(0.0225) = 2.25\,\text{J}.$$

At the final position $x_f = 0$,

$$U_{s,f} = \tfrac{1}{2}(200)(0)^2 = 0.$$

Therefore the change in spring potential energy is

$$\Delta U_s = U_{s,f} - U_{s,i} = 0 - 2.25\,\text{J} = -2.25\,\text{J}.$$

Because the spring force is conservative,

$$W_s = -\Delta U_s = +2.25\,\text{J}.$$

So the spring does positive work on the block as the spring relaxes toward equilibrium.

The track is frictionless, so the spring is the only force doing work on the block in the horizontal direction. Thus

$$W_{\text{net}} = \Delta K.$$

Since the block starts from rest,

$$K_i = 0, \qquad K_f = W_{\text{net}} = 2.25\,\text{J}.$$

Then

$$\tfrac{1}{2}mv_f^2 = 2.25.$$

Substitute $m = 0.50\,\text{kg}$:

$$\tfrac{1}{2}(0.50)v_f^2 = 2.25 \qquad \Rightarrow \qquad 0.25v_f^2 = 2.25 \qquad \Rightarrow \qquad v_f^2 = 9.0.$$

Hence

$$v_f = 3.0\,\text{m/s}.$$

So the results are

$$\Delta U_s = -2.25\,\text{J}, \qquad W_s = +2.25\,\text{J}, \qquad v_f = 3.0\,\text{m/s}.$$

1.3.4 Mechanical Energy Conservation

This subsection packages work and potential energy into an energy-accounting method. In AP mechanics, the key step is to choose a system first, then decide which interactions are represented by potential energy and which must be tracked as nonconservative work.

Definition 1.3.4: Mechanical energy and nonconservative work

Let a chosen system move between an initial state and a final state. Let K denote the total kinetic energy of the system, and let U denote the total potential energy associated with all conservative interactions included in the system, such as gravitational and spring interactions. The *mechanical energy* of the system is

$$E_{\text{mech}} = K + U.$$

Let W_{nc} denote the total work done on the system by forces or processes that are not represented by a potential-energy function in the chosen model. With this sign convention,

$$W_{\text{nc}} > 0 \text{ increases } E_{\text{mech}}, \qquad W_{\text{nc}} < 0 \text{ decreases } E_{\text{mech}}.$$

Typical examples include kinetic friction, air drag, and an external applied force not absorbed into U.

Theorem 1.3.4 Mechanical energy equation

Let K_i and U_i denote the initial kinetic and potential energies of a chosen system, and let K_f and U_f denote the corresponding final quantities. If W_{nc} is the total nonconservative work done on the system, then

$$\Delta E_{\text{mech}} = \Delta(K+U) = W_{\text{nc}}.$$

Equivalently,

$$K_i + U_i + W_{\text{nc}} = K_f + U_f.$$

If the motion is governed only by conservative forces already accounted for in U, then $W_{\text{nc}} = 0$ and mechanical energy is conserved:

$$\Delta(K+U) = 0, \qquad K_i + U_i = K_f + U_f.$$

Note:-

Choose the system before writing any energy equation. If the system is *object + Earth*, then gravitational potential energy belongs in U and gravity should not also be counted as separate work. If the system is *object + spring*, then spring potential energy belongs in U. If the system is *object + Earth + spring*, then both U_g and U_s belong in U. Mechanical energy is conserved only when no nonconservative work changes $K+U$ for that chosen system. When friction, drag, or an external agent transfers energy into or out of the system, use

$$\Delta(K+U) = W_{\text{nc}}$$

instead of setting $\Delta(K+U)$ equal to zero. Total energy is still conserved overall; it is specifically *mechanical* energy that may change.

Derivation from the work-energy theorem: Let W_{net} denote the net work done on the chosen system. By the work-energy theorem,

$$\Delta K = W_{\text{net}}.$$

Split the net work into conservative and nonconservative parts:

$$W_{\text{net}} = W_c + W_{\text{nc}}.$$

For the conservative forces represented by the potential energy U,

$$W_c = -\Delta U.$$

Therefore,

$$\Delta K = -\Delta U + W_{\text{nc}}.$$

Rearranging gives

$$\Delta K + \Delta U = W_{\text{nc}},$$

so

$$\Delta(K+U) = W_{\text{nc}}.$$

If $W_{\text{nc}} = 0$, then

$$\Delta(K+U) = 0,$$

which is the conservation of mechanical energy. ☺

Question 18: Worked example

A block of mass $m = 2.0\,\text{kg}$ is released from rest at a height $h = 1.20\,\text{m}$ above the bottom of a ramp. The ramp is frictionless. After reaching the bottom, the block crosses a rough horizontal surface of length $d = 0.80\,\text{m}$ with coefficient of kinetic friction $\mu_k = 0.25$. The block then compresses a horizontal spring of spring constant $k = 400\,\text{N/m}$ on a frictionless section of track and momentarily comes to rest at maximum compression. Let x denote the maximum spring compression.
Find x.

Solution: Choose the system to be *block + Earth + spring.* Then gravity and the spring are accounted for through potential energy, and the only nonconservative work is the work done by kinetic friction on the rough horizontal section.

Take the initial state to be the release point and the final state to be the instant of maximum compression. Let the gravitational potential energy be zero at the bottom of the ramp, and let the spring potential energy be zero when the spring is uncompressed.

Use the mechanical energy equation

$$K_i + U_i + W_{\text{nc}} = K_f + U_f.$$

At the initial state, the block is released from rest, so

$$K_i = 0.$$

Its gravitational potential energy is

$$U_i = mgh = (2.0\,\text{kg})(9.8\,\text{m/s}^2)(1.20\,\text{m}) = 23.52\,\text{J}.$$

At the final state, the block momentarily stops at maximum compression, so

$$K_f = 0.$$

Its gravitational potential energy is zero because it is at the bottom level, and its spring potential energy is

$$U_f = \tfrac{1}{2}kx^2.$$

Now compute the nonconservative work. On the rough horizontal section, the kinetic friction force has magnitude

$$f_k = \mu_k mg = (0.25)(2.0\,\text{kg})(9.8\,\text{m/s}^2) = 4.9\,\text{N}.$$

Because friction opposes the motion over the distance $d = 0.80\,\text{m}$, the work done by friction is

$$W_{\text{nc}} = -f_k d = -(4.9\,\text{N})(0.80\,\text{m}) = -3.92\,\text{J}.$$

Substitute into the energy equation:

$$0 + 23.52\,\text{J} - 3.92\,\text{J} = 0 + \tfrac{1}{2}(400\,\text{N/m})x^2.$$

So

$$19.60\,\text{J} = 200x^2.$$

Hence

$$x^2 = 0.0980, \qquad x = \sqrt{0.0980}\,\text{m} \approx 0.313\,\text{m}.$$

Therefore the maximum spring compression is

$$x \approx 0.31\,\text{m}.$$

If the rough section were absent, then $W_{\text{nc}} = 0$ and mechanical energy would be conserved exactly. Here the negative friction work reduces the mechanical energy before the block reaches the spring.

1.3.5 Power and Instantaneous Power

This subsection introduces power as the rate at which work is done. In AP mechanics, the key idea is local: the instantaneous power delivered by a force depends on the component of that force along the motion, so the sign of the power tells whether the force is adding energy to the object or removing it.

Definition 1.3.5: Average and instantaneous power

Let ΔW denote the work done by a force during a time interval $\Delta t > 0$. The *average power* over that interval is

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t}.$$

The *instantaneous power* at time t is the limit of the average power over shorter and shorter time intervals:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}.$$

Power is a scalar quantity. Its SI unit is the watt:

$$1\,\text{W} = 1\,\text{J/s}.$$

If $P > 0$, the force is doing positive work and transferring energy to the object. If $P < 0$, the force is doing negative work and removing mechanical energy from the object.

Theorem 1.3.5 Mechanical power from force and velocity

Let $\vec{F}$ denote the force acting on a particle, let $\vec{v} = d\vec{r}/dt$ denote the particle's velocity, let $F = |\vec{F}|$ denote the magnitude of the force, let $v = |\vec{v}|$ denote the speed, let θ denote the angle between $\vec{F}$ and $\vec{v}$, and let $F_{\parallel}$ denote the component of the force parallel to the motion. Then the instantaneous power delivered by the force is

$$P = \vec{F} \cdot \vec{v} = Fv\cos\theta = F_{\parallel} v.$$

Therefore only the component of the force along the motion contributes to power. If $\theta < 90°$, then $P > 0$; if $\theta > 90°$, then $P < 0$; and if $\theta = 90°$, then $P = 0$. In one-dimensional motion along the x-axis,

$$P = F_x v_x.$$

Short derivation from $P = dW/dt$**:** Let $d\vec{r}$ denote the particle's infinitesimal displacement. From the local definition of work,

$$dW = \vec{F} \cdot d\vec{r}.$$

Divide by dt to obtain the instantaneous rate at which work is done:

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}.$$

Since

$$\frac{d\vec{r}}{dt} = \vec{v},$$

it follows that

$$P = \vec{F} \cdot \vec{v}.$$

If θ is the angle between $\vec{F}$ and $\vec{v}$, then the dot-product form gives

$$P = Fv\cos\theta = F_{\parallel} v.$$

😄

Corollary 1.3.1 Perpendicular forces do zero instantaneous power

If a force is perpendicular to the velocity at a given instant, then

$$\vec{F} \cdot \vec{v} = 0,$$

so

$$P = 0.$$

Thus a force can change the direction of motion without transferring energy through work at that instant. Common AP examples are a normal force on frictionless motion along a surface and the centripetal force in uniform circular motion.

Question 19: Worked example

A student pulls a sled across level snow with a rope of tension magnitude $T = 85\,\text{N}$ at an angle $\theta = 25°$ above the horizontal. The sled moves horizontally with constant speed $v = 1.8\,\text{m/s}$ for a time interval $\Delta t = 40\,\text{s}$. Let $\vec{T}$ denote the tension force and let $\vec{v}$ denote the sled's velocity.
(a) Find the instantaneous power delivered by the tension.
(b) Find the work done by the tension during the 40 s interval.
(c) Find the average power delivered by the tension over that interval.

Solution: Because the sled's velocity is horizontal, only the horizontal component of the tension contributes to the power and the work. The component of the tension along the motion is

$$T_{\|} = T\cos\theta = (85\,\text{N})\cos 25° \approx 77.0\,\text{N}.$$

For part (a), the instantaneous power is

$$P = \vec{T}\cdot\vec{v} = Tv\cos\theta.$$

Substitute the given values:

$$P = (85\,\text{N})(1.8\,\text{m/s})\cos 25° \approx 1.39\times 10^2\,\text{W}.$$

So the instantaneous power delivered by the tension is

$$P \approx 1.4\times 10^2\,\text{W}.$$

The power is positive because the tension has a component in the same direction as the motion.

For part (b), the sled's horizontal displacement during the interval is

$$\Delta x = v\Delta t = (1.8\,\text{m/s})(40\,\text{s}) = 72\,\text{m}.$$

The work done by the tension is

$$W = T\Delta x\cos\theta.$$

Therefore,

$$W = (85\,\text{N})(72\,\text{m})\cos 25° \approx 5.55\times 10^3\,\text{J}.$$

So the tension does

$$W \approx 5.6\times 10^3\,\text{J}$$

of positive work on the sled. The vertical component of the tension does no work because the displacement is horizontal.

For part (c), the average power is

$$P_{\text{avg}} = \frac{\Delta W}{\Delta t} = \frac{5.55\times 10^3\,\text{J}}{40\,\text{s}} \approx 1.39\times 10^2\,\text{W}.$$

Thus,

$$P_{\text{avg}} \approx 1.4\times 10^2\,\text{W}.$$

This matches the instantaneous power because the force magnitude, the angle, and the speed are all constant throughout the motion.

1.4 Linear Momentum and Collisions

This unit develops the momentum viewpoint for interactions between objects and systems. We begin by defining linear momentum $\vec{p}$ for a particle and total momentum $\vec{P}$ for a system, using consistent vector notation and emphasizing that momentum conservation is applied componentwise in both one and two dimensions.

We then connect external forces to momentum transfer through impulse $\vec{J}$, use isolated-system reasoning to establish conservation of momentum, and apply those ideas to collisions, recoil, and explosions. Throughout, the AP focus is on classifying collisions correctly, recognizing that kinetic energy is not always conserved, and relating system motion to the center-of-mass velocity $\vec{v}_{\mathrm{cm}}$ when helpful.

1.4.1 Linear Momentum

This subsection introduces linear momentum as the vector state variable for translational motion. In AP mechanics, momentum is the quantity that naturally leads into impulse and conservation ideas.

Definition 1.4.1: Particle and system momentum

Let m denote the mass of a particle, let $\vec{v}$ denote its velocity measured in a chosen inertial reference frame, and let $\vec{p}$ denote its linear momentum. The *linear momentum* of the particle is

$$\vec{p} = m\vec{v}.$$

Because m is a scalar and $\vec{v}$ is a vector, $\vec{p}$ is a vector in the same direction as $\vec{v}$. Its SI unit is

$$1\,\mathrm{kg}\cdot\mathrm{m/s}.$$

For a system of N particles labeled by $i = 1, 2, \ldots, N$, let m_i denote the mass of particle i, let $\vec{v}_i$ denote its velocity, and let $\vec{p}_i = m_i\vec{v}_i$ denote its momentum. The total linear momentum of the system is

$$\vec{p}_{\mathrm{sys}} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i\vec{v}_i.$$

Note:-

Momentum is not the same thing as speed. Speed is the scalar $v = |\vec{v}|$, while momentum is the vector $\vec{p} = m\vec{v}$. Two objects can have the same speed but different momenta if their masses are different or if they move in different directions. Momentum also depends on the chosen reference frame because velocity does: an object at rest in one frame has $\vec{p} = \vec{0}$ in that frame, but it can have nonzero momentum in another frame. In one-dimensional motion, a negative momentum component means motion in the negative coordinate direction; it does not mean negative mass or negative speed.

Proposition 1.4.1 Component and system relations

Let

$$\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$$

be the velocity of a particle of mass m, and let

$$\vec{p} = p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k}$$

be its momentum. For a system of particles, let

$$M = \sum_{i=1}^{N} m_i$$

denote the total mass, and let $\vec{v}_{\mathrm{cm}}$ denote the center-of-mass velocity.

① Momentum components are found component-by-component from the velocity:

$$p_x = mv_x, \qquad p_y = mv_y, \qquad p_z = mv_z.$$

In one-dimensional motion along the x-axis,

$$p_x = mv_x.$$

② For a single particle, the momentum magnitude is

$$p = |\vec{p}| = m|\vec{v}| = mv.$$

In two dimensions,

$$p = \sqrt{p_x^2 + p_y^2},$$

and in three dimensions,

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2}.$$

③ System momentum is the vector sum of the particle momenta:

$$\vec{p}_{\text{sys}} = \sum_{i=1}^{N} \vec{p}_i.$$

Therefore its components also add:

$$p_{\text{sys},x} = \sum_{i=1}^{N} p_{i,x}, \qquad p_{\text{sys},y} = \sum_{i=1}^{N} p_{i,y}, \qquad p_{\text{sys},z} = \sum_{i=1}^{N} p_{i,z}.$$

Equivalently,

$$\vec{p}_{\text{sys}} = M\vec{v}_{\text{cm}}.$$

Question 20: Worked example

In a laboratory frame, two pucks slide on nearly frictionless ice. Puck 1 has mass $m_1 = 2.0\,\text{kg}$ and velocity

$$\vec{v}_1 = (1.5\hat{\imath} + 0.50\hat{\jmath})\,\text{m/s}.$$

Puck 2 has mass $m_2 = 1.0\,\text{kg}$ and velocity

$$\vec{v}_2 = (-1.0\hat{\imath} + 4.0\hat{\jmath})\,\text{m/s}.$$

Let $\vec{p}_1$ and $\vec{p}_2$ denote the individual momenta, let $\vec{p}_{\text{sys}}$ denote the total momentum, let M denote the total mass, and let θ denote the direction of $\vec{p}_{\text{sys}}$ measured counterclockwise from the positive x-axis.
Find $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_{\text{sys}}$, the magnitude $|\vec{p}_{\text{sys}}|$, the direction θ, and the center-of-mass velocity $\vec{v}_{\text{cm}}$.

Solution: Use $\vec{p} = m\vec{v}$ for each puck.

For puck 1,

$$\vec{p}_1 = m_1\vec{v}_1 = (2.0\,\text{kg})(1.5\hat{\imath} + 0.50\hat{\jmath})\,\text{m/s}.$$

Therefore,

$$\vec{p}_1 = (3.0\hat{\imath} + 1.0\hat{\jmath})\,\text{kg}\cdot\text{m/s}.$$

For puck 2,

$$\vec{p}_2 = m_2\vec{v}_2 = (1.0\,\text{kg})(-1.0\hat{\imath} + 4.0\hat{\jmath})\,\text{m/s}.$$

So,

$$\vec{p}_2 = (-1.0\hat{\imath} + 4.0\hat{\jmath})\,\text{kg}\cdot\text{m/s}.$$

Now add the momenta component-by-component:

$$\vec{p}_{\text{sys}} = \vec{p}_1 + \vec{p}_2.$$

Hence,

$$\vec{p}_{\text{sys}} = (3.0 - 1.0)\hat{\imath} + (1.0 + 4.0)\hat{\jmath}.$$

Therefore,

$$\vec{p}_{\text{sys}} = (2.0\hat{\imath} + 5.0\hat{\jmath})\,\text{kg}\cdot\text{m/s}.$$

Its magnitude is

$$|\vec{p}_{\text{sys}}| = \sqrt{(2.0)^2 + (5.0)^2}\,\text{kg}\cdot\text{m/s} = \sqrt{29}\,\text{kg}\cdot\text{m/s}.$$

Numerically,

$$|\vec{p}_{\text{sys}}| \approx 5.39\,\text{kg}\cdot\text{m/s}.$$

To find the direction, use the component ratio. Since both components of $\vec{p}_{\text{sys}}$ are positive, the vector lies in the first quadrant. Thus,

$$\tan\theta = \frac{p_{\text{sys},y}}{p_{\text{sys},x}} = \frac{5.0}{2.0} = 2.5.$$

So,

$$\theta = \tan^{-1}(2.5) \approx 68.2^\circ.$$

Now find the center-of-mass velocity. The total mass is

$$M = m_1 + m_2 = 2.0\,\text{kg} + 1.0\,\text{kg} = 3.0\,\text{kg}.$$

Using

$$\vec{p}_{\text{sys}} = M\vec{v}_{\text{cm}},$$

we get

$$\vec{v}_{\text{cm}} = \frac{\vec{p}_{\text{sys}}}{M} = \frac{(2.0\hat{\imath} + 5.0\hat{\jmath})\,\text{kg}\cdot\text{m/s}}{3.0\,\text{kg}}.$$

Therefore,

$$\vec{v}_{\text{cm}} = \left(\frac{2.0}{3.0}\hat{\imath} + \frac{5.0}{3.0}\hat{\jmath}\right)\,\text{m/s}$$

or numerically,

$$\vec{v}_{\text{cm}} \approx (0.667\hat{\imath} + 1.67\hat{\jmath})\,\text{m/s}.$$

So the individual and system momenta are

$$\vec{p}_1 = (3.0\hat{\imath} + 1.0\hat{\jmath})\,\text{kg}\cdot\text{m/s}, \qquad \vec{p}_2 = (-1.0\hat{\imath} + 4.0\hat{\jmath})\,\text{kg}\cdot\text{m/s},$$

$$\vec{p}_{\text{sys}} = (2.0\hat{\imath} + 5.0\hat{\jmath})\,\text{kg}\cdot\text{m/s},$$

with magnitude

$$|\vec{p}_{\text{sys}}| \approx 5.39\,\text{kg}\cdot\text{m/s},$$

direction

$$\theta \approx 68.2^\circ,$$

and center-of-mass velocity

$$\vec{v}_{\text{cm}} \approx (0.667\hat{\imath} + 1.67\hat{\jmath})\,\text{m/s}.$$

This example shows why momentum must be handled as a vector: the total momentum is found by adding components, not by adding speeds.

1.4.2 Impulse and Momentum Transfer

This subsection connects the local momentum law $\vec{F}_{\text{net}} = d\vec{p}/dt$ to finite-time interactions such as hits, kicks, and collisions, where a large force acts for a short time and changes momentum by a measurable amount.

Definition 1.4.2: Momentum change and impulse

Let a body of constant mass m have velocity $\vec{v}$. Its linear momentum is

$$\vec{p} = m\vec{v}.$$

Let $\vec{p}_i$ and $\vec{p}_f$ denote the initial and final momenta over some time interval from t_i to t_f. The change in momentum is

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i.$$

Let $\vec{F}_{\text{net}}(t)$ denote the net external force on the body during that interval. The impulse delivered to the body is

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt.$$

If the net force is constant, then this reduces to

$$\vec{J} = \vec{F}_{\text{net}}\Delta t, \qquad \Delta t = t_f - t_i.$$

The SI unit of impulse is N s, which is equivalent to kg m/s.

Theorem 1.4.1 Impulse-momentum theorem

Let $\vec{p}(t)$ denote the momentum of a body and let $\vec{F}_{\text{net}}(t)$ denote the net external force on it. Over any time interval from t_i to t_f,

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt = \Delta\vec{p} = \vec{p}_f - \vec{p}_i.$$

Thus the net impulse on a body equals its change in momentum. In one dimension along the x-axis, let J_x, $F_{\text{net},x}$, and p_x denote the corresponding x-components. Then

$$J_x = \int_{t_i}^{t_f} F_{\text{net},x}\, dt = \Delta p_x.$$

Note:-

Impulse is a vector, so its direction is the direction of $\Delta\vec{p}$. A body can be moving in one direction while the impulse points in the opposite direction if the interaction slows or reverses the motion. Impulse depends on both force and time: a large force acting briefly can produce the same impulse as a smaller force acting longer. If an average net force $\vec{F}_{\text{avg}}$ over a time interval Δt is defined so that it has the same effect as the actual time-varying force, then

$$\vec{J} = \vec{F}_{\text{avg}}\Delta t.$$

On a force-versus-time graph, the signed area under the curve gives impulse. In component form, the signed area under an $F_x(t)$ graph gives $J_x = \Delta p_x$.

Short derivation from $d\vec{p}/dt$: Start with the local momentum law

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}.$$

Multiply by dt:

$$\vec{F}_{\text{net}}\, dt = d\vec{p}.$$

Now integrate from t_i to t_f:

$$\int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt = \int_{t_i}^{t_f} d\vec{p}.$$

The left side is the impulse $\vec{J}$, and the right side is the total change in momentum:

$$\vec{J} = \vec{p}_f - \vec{p}_i = \Delta\vec{p}.$$

This is the impulse-momentum theorem. 😄

Question 21: Worked example

Choose the positive x-axis to the right. A tennis ball of mass $m = 0.150\,\text{kg}$ moves horizontally with initial velocity

$$\vec{v}_i = -18.0\,\hat{\imath}\,\text{m/s}.$$

Let $F_x(t)$ denote the net horizontal force on the ball during contact. The force is directed to the right and has the following force-versus-time graph: it increases linearly from 0 at $t = 0$ to 900 N at $t = 4.0\,\text{ms}$, then decreases linearly back to 0 at $t = 8.0\,\text{ms}$.
Find (a) the impulse delivered to the ball, (b) the ball's final momentum, (c) the ball's final velocity, and (d) the average net force during contact.

Solution: Let $\Delta t = 8.0 \times 10^{-3}\,\text{s}$ denote the contact time. Because the force is entirely in the $+x$ direction, the impulse is the area under the triangular $F_x(t)$ graph in the positive direction:

$$\vec{J} = \left(\tfrac{1}{2}\right)(\Delta t)(900\,\text{N})\,\hat{\imath}.$$

Substitute $\Delta t = 8.0 \times 10^{-3}\,\text{s}$:

$$\vec{J} = \left(\tfrac{1}{2}\right)(8.0 \times 10^{-3}\,\text{s})(900\,\text{N})\,\hat{\imath} = 3.6\,\hat{\imath}\,\text{N s}.$$

Using $1\,\text{N s} = 1\,\text{kg m/s}$,

$$\vec{J} = 3.6\,\hat{\imath}\,\text{kg m/s}.$$

Now compute the initial momentum:

$$\vec{p}_i = m\vec{v}_i = (0.150\,\text{kg})(-18.0\,\hat{\imath}\,\text{m/s}) = -2.70\,\hat{\imath}\,\text{kg m/s}.$$

From the impulse-momentum theorem,

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i,$$

so

$$\vec{p}_f = \vec{p}_i + \vec{J}.$$

Substitute the values:

$$\vec{p}_f = (-2.70 + 3.60)\,\hat{\imath}\,\text{kg m/s} = 0.90\,\hat{\imath}\,\text{kg m/s}.$$

Then the final velocity is

$$\vec{v}_f = \frac{\vec{p}_f}{m} = \frac{0.90\,\hat{\imath}\,\text{kg m/s}}{0.150\,\text{kg}} = 6.0\,\hat{\imath}\,\text{m/s}.$$

The positive sign shows that the ball leaves moving to the right.

For the average net force, use

$$\vec{J} = \vec{F}_{\text{avg}}\Delta t.$$

Therefore,

$$\vec{F}_{\text{avg}} = \frac{\vec{J}}{\Delta t} = \frac{3.6\,\hat{\imath}\,\text{N s}}{8.0 \times 10^{-3}\,\text{s}} = 4.5 \times 10^2\,\hat{\imath}\,\text{N}.$$

So the results are

$$\vec{J} = 3.6\,\hat{\imath}\,\text{N s}, \qquad \vec{p}_f = 0.90\,\hat{\imath}\,\text{kg m/s},$$

$$\vec{v}_f = 6.0\,\hat{\imath}\,\text{m/s}, \qquad \vec{F}_{\text{avg}} = 4.5 \times 10^2\,\hat{\imath}\,\text{N}.$$

The graph interpretation is essential here: the triangular area under $F_x(t)$ gives the impulse directly, and that impulse determines the momentum transfer.

1.4.3 Conservation of Momentum for Systems

This subsection treats momentum conservation as a statement about a chosen system: internal forces can transfer momentum between parts of the system, but if the net external impulse is zero, the total momentum stays constant.

Definition 1.4.3: System momentum and momentum-isolated systems

Consider a system of N objects labeled by an index $i = 1, 2, \ldots, N$. Let m_i denote the mass of object i, let $\vec{v}_i$ denote its velocity, and let

$$\vec{p}_i = m_i \vec{v}_i$$

denote its momentum. The total momentum of the system is the vector sum

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i.$$

For an interval from time t_i to time t_f, let $\sum \vec{F}_{\text{ext}}$ denote the net external force on the whole system, and let

$$\vec{J}_{\text{ext}} = \int_{t_i}^{t_f} \sum \vec{F}_{\text{ext}} \, dt$$

denote the net external impulse on the system.

A system is called *closed* over that interval if the set of objects in the system does not change during the interaction. A closed system is *isolated for momentum* over that interval if the net external impulse is zero or negligible:

$$\vec{J}_{\text{ext}} = \vec{0}.$$

Theorem 1.4.2 Conservation of momentum for a system

For a closed system,

$$\Delta \vec{P} = \vec{P}_f - \vec{P}_i = \vec{J}_{\text{ext}}.$$

Therefore, if the net external impulse on the system is zero,

$$\vec{J}_{\text{ext}} = \vec{0},$$

then the total momentum is conserved:

$$\vec{P}_f = \vec{P}_i.$$

In component form, this means each Cartesian component is conserved separately, such as

$$P_{x,f} = P_{x,i} \qquad \text{and} \qquad P_{y,f} = P_{y,i}$$

when the external impulse is zero.

Note:-

The most important step is choosing the system boundary correctly. If two objects collide and both objects are included in the system, then the contact forces between them are internal forces and cancel in the total momentum balance. Forces from outside the system, such as a large friction force from the floor or a push from a person, are external and can change the system momentum. In two-dimensional problems, write momentum conservation separately in the x- and y-directions and solve the resulting scalar equations. Also, momentum conservation does *not* require kinetic energy to be conserved: in an inelastic collision, the total momentum can stay constant even though the kinetic energy changes.

Derivation from Newton's laws: For each object i, let $\vec{F}_{\text{ext},i}$ denote the net external force on that object, and let $\vec{F}_{\text{int},i}$ denote the net internal force on it from the other objects in the system. Then Newton's second law gives

$$\frac{d\vec{p}_i}{dt} = \vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i}.$$

Now sum over all objects:

$$\sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{N} \vec{F}_{\text{ext},i} + \sum_{i=1}^{N} \vec{F}_{\text{int},i}.$$

Since

$$\sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = \frac{d\vec{P}}{dt},$$

and the internal forces cancel in equal-and-opposite pairs by Newton's third law,

$$\sum_{i=1}^{N} \vec{F}_{\text{int},i} = \vec{0},$$

we obtain

$$\frac{d\vec{P}}{dt} = \sum \vec{F}_{\text{ext}}.$$

Integrate from t_i to t_f:

$$\vec{P}_f - \vec{P}_i = \int_{t_i}^{t_f} \sum \vec{F}_{\text{ext}}\, dt = \vec{J}_{\text{ext}}.$$

If $\vec{J}_{\text{ext}} = \vec{0}$, then $\vec{P}_f = \vec{P}_i$, so total momentum is conserved. 😄

Question 22: Worked example

On a frictionless horizontal air table, puck A has mass $m_A = 0.20\,\text{kg}$ and initial velocity

$$\vec{v}_{A,i} = (6.0\hat{\imath})\,\text{m/s}.$$

Puck B has mass $m_B = 0.30\,\text{kg}$ and initial velocity

$$\vec{v}_{B,i} = (4.0\hat{\jmath})\,\text{m/s}.$$

The pucks collide and stick together. Let

$$\vec{v}_f = v_{f,x}\hat{\imath} + v_{f,y}\hat{\jmath}$$

denote their common final velocity.
Find $\vec{v}_f$, its magnitude and direction, and determine whether kinetic energy is conserved.

Solution: Choose the system to be *puck A + puck B*. During the short collision, the table is frictionless, so the net external horizontal impulse on this two-puck system is zero. Therefore,

$$\vec{P}_i = \vec{P}_f.$$

Because this is a two-dimensional problem, conserve momentum separately in the x- and y-directions.
First write the initial momentum components.
For puck A,

$$\vec{p}_{A,i} = m_A\vec{v}_{A,i} = (0.20\,\text{kg})(6.0\hat{\imath}\,\text{m/s}) = (1.2\hat{\imath})\,\text{kg}\cdot\text{m/s}.$$

For puck B,

$$\vec{p}_{B,i} = m_B\vec{v}_{B,i} = (0.30\,\text{kg})(4.0\hat{\jmath}\,\text{m/s}) = (1.2\hat{\jmath})\,\text{kg}\cdot\text{m/s}.$$

So the total initial momentum is

$$\vec{P}_i = (1.2\hat{\imath} + 1.2\hat{\jmath})\,\text{kg}\cdot\text{m/s}.$$

After the collision, the pucks stick together, so the total mass is

$$M = m_A + m_B = 0.20\,\mathrm{kg} + 0.30\,\mathrm{kg} = 0.50\,\mathrm{kg}.$$

Their final momentum is

$$\vec{P}_f = M\vec{v}_f = (0.50\,\mathrm{kg})(v_{f,x}\hat{\imath} + v_{f,y}\hat{\jmath}).$$

Now conserve momentum by components.

In the x-direction,

$$P_{x,i} = P_{x,f}.$$

Thus,

$$1.2\,\mathrm{kg}\cdot\mathrm{m/s} = (0.50\,\mathrm{kg})v_{f,x},$$

so

$$v_{f,x} = 2.4\,\mathrm{m/s}.$$

In the y-direction,

$$P_{y,i} = P_{y,f}.$$

Thus,

$$1.2\,\mathrm{kg}\cdot\mathrm{m/s} = (0.50\,\mathrm{kg})v_{f,y},$$

so

$$v_{f,y} = 2.4\,\mathrm{m/s}.$$

Therefore, the common final velocity is

$$\vec{v}_f = (2.4\hat{\imath} + 2.4\hat{\jmath})\,\mathrm{m/s}.$$

Its magnitude is

$$|\vec{v}_f| = \sqrt{(2.4\,\mathrm{m/s})^2 + (2.4\,\mathrm{m/s})^2} = 3.39\,\mathrm{m/s}.$$

Let θ denote the direction of $\vec{v}_f$ measured counterclockwise from the positive x-axis. Then

$$\tan\theta = \frac{v_{f,y}}{v_{f,x}} = \frac{2.4}{2.4} = 1,$$

so

$$\theta = 45^\circ.$$

Now check the kinetic energy.

Initially,

$$K_i = \tfrac{1}{2}m_A|\vec{v}_{A,i}|^2 + \tfrac{1}{2}m_B|\vec{v}_{B,i}|^2 = \tfrac{1}{2}(0.20)(6.0)^2 + \tfrac{1}{2}(0.30)(4.0)^2 = 6.0\,\mathrm{J}.$$

Finally,

$$K_f = \tfrac{1}{2}M|\vec{v}_f|^2 = \tfrac{1}{2}(0.50)(3.39)^2 \approx 2.88\,\mathrm{J}.$$

Since

$$K_f \neq K_i,$$

kinetic energy is not conserved. That is expected because the pucks stick together, so the collision is inelastic.

Thus the final velocity is

$$\vec{v}_f = (2.4\hat{\imath} + 2.4\hat{\jmath})\,\mathrm{m/s},$$

with magnitude

$$|\vec{v}_f| = 3.39\,\mathrm{m/s},$$

directed

$$45^\circ$$

above the positive x-axis, while momentum is conserved but kinetic energy is not.

1.4.4 Elastic, Inelastic, and Perfectly Inelastic Collisions

This subsection classifies collisions by what happens to the system's kinetic energy. In AP mechanics, the first step is still to apply momentum conservation to an isolated system; only elastic collisions add a kinetic-energy conservation equation.

Definition 1.4.4: Elastic, inelastic, and perfectly inelastic collisions

Consider two objects of masses m_1 and m_2. Let $\vec{v}_{1,i}$ and $\vec{v}_{2,i}$ denote their velocities just before a collision, and let $\vec{v}_{1,f}$ and $\vec{v}_{2,f}$ denote their velocities just after it. Let the total kinetic energy before and after the collision be

$$K_i = \frac{1}{2}m_1|\vec{v}_{1,i}|^2 + \frac{1}{2}m_2|\vec{v}_{2,i}|^2$$

and

$$K_f = \frac{1}{2}m_1|\vec{v}_{1,f}|^2 + \frac{1}{2}m_2|\vec{v}_{2,f}|^2.$$

A collision is called *elastic* if the total kinetic energy is unchanged, so

$$K_f = K_i.$$

It is called *inelastic* if the total kinetic energy changes, so

$$K_f \neq K_i.$$

In the usual AP collision problems without stored energy being released during impact, an inelastic collision has $K_f < K_i$.
A collision is *perfectly inelastic* if the objects stick together after the collision, so they share one final velocity:

$$\vec{v}_{1,f} = \vec{v}_{2,f} = \vec{v}_f.$$

Every perfectly inelastic collision is inelastic.

Note:-

Do not decide whether momentum is conserved by asking whether the collision is elastic. Those are different ideas. Momentum conservation depends on the net external impulse on the chosen system. If the system is isolated during the collision, then total momentum is conserved for elastic, inelastic, and perfectly inelastic collisions alike. Kinetic energy supplies an *extra* condition only in the elastic case. In two-dimensional AP problems, conserve momentum separately in the x- and y-directions.

Example 1.4.1 (Illustrative example)

On a frictionless track, cart 1 has mass $m_1 = 0.40\,\text{kg}$ and initial velocity

$$\vec{v}_{1,i} = (3.0\hat{\imath})\,\text{m/s}.$$

Cart 2 has mass $m_2 = 0.20\,\text{kg}$ and initial velocity

$$\vec{v}_{2,i} = \vec{0}.$$

After the collision, the carts move together with common velocity

$$\vec{v}_f = (2.0\hat{\imath})\,\text{m/s}.$$

Because the carts move together after impact, the collision is perfectly inelastic. The initial kinetic energy is

$$K_i = \frac{1}{2}(0.40)(3.0)^2 = 1.8\,\text{J},$$

while the final kinetic energy is

$$K_f = \frac{1}{2}(0.60)(2.0)^2 = 1.2\,\text{J}.$$

Since $K_f < K_i$, the collision is inelastic, as expected. Momentum is still conserved because

$$\vec{P}_i = (0.40)(3.0\hat{\imath}) = (1.2\hat{\imath})\,\mathrm{kg\cdot m/s}$$

and

$$\vec{P}_f = (0.60)(2.0\hat{\imath}) = (1.2\hat{\imath})\,\mathrm{kg\cdot m/s}.$$

Proposition 1.4.2 Practical relations for isolated collisions

Consider two objects that form an isolated system during a short collision. Let

$$\vec{P}_i = m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i}$$

denote the total initial momentum and let

$$\vec{P}_f = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}$$

denote the total final momentum.

1. For any isolated collision,

$$\vec{P}_i = \vec{P}_f,$$

so

$$m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} = m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f}.$$

2. In two dimensions, let $P_{x,i}$ and $P_{y,i}$ denote the initial momentum components, and let $P_{x,f}$ and $P_{y,f}$ denote the final momentum components. Then write two separate component equations:

$$P_{x,i} = P_{x,f}, \qquad P_{y,i} = P_{y,f}.$$

3. If the collision is perfectly inelastic, then $\vec{v}_{1,f} = \vec{v}_{2,f} = \vec{v}_f$, so momentum conservation gives the shared final velocity directly:

$$\vec{v}_f = \frac{m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i}}{m_1 + m_2}.$$

4. If the collision is elastic, then in addition to momentum conservation,

$$\tfrac{1}{2}m_1|\vec{v}_{1,i}|^2 + \tfrac{1}{2}m_2|\vec{v}_{2,i}|^2 = \tfrac{1}{2}m_1|\vec{v}_{1,f}|^2 + \tfrac{1}{2}m_2|\vec{v}_{2,f}|^2.$$

For a one-dimensional elastic collision, this is equivalent to the relative-speed relation

$$v_{1,i} - v_{2,i} = -(v_{1,f} - v_{2,f}),$$

where each v is an x-component.

Question 23: Worked example

Choose the positive x-axis to the right. On a frictionless track, cart 1 has mass $m_1 = 1.0\,\mathrm{kg}$ and initial velocity

$$\vec{v}_{1,i} = (4.0\hat{\imath})\,\mathrm{m/s}.$$

Cart 2 has mass $m_2 = 3.0\,\mathrm{kg}$ and is initially at rest, so

$$\vec{v}_{2,i} = \vec{0}.$$

After the collision, the carts lock together and move with common final velocity $\vec{v}_f$.
Find $\vec{v}_f$, determine whether the collision is elastic, inelastic, or perfectly inelastic, and calculate the change in kinetic energy $\Delta K = K_f - K_i$.

Solution: Because the carts lock together, this is a perfectly inelastic collision by definition. During the short collision the track is frictionless, so the two-cart system is isolated horizontally. Therefore total momentum is conserved:

$$\vec{P}_i = \vec{P}_f.$$

The initial momentum is

$$\vec{P}_i = m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i}.$$

Substitute the given values:

$$\vec{P}_i = (1.0\,\text{kg})(4.0\hat{\imath}\,\text{m/s}) + (3.0\,\text{kg})(\vec{0}) = 4.0\hat{\imath}\,\text{kg}\cdot\text{m/s}.$$

After the collision the carts move together, so their total mass is

$$m_1 + m_2 = 4.0\,\text{kg}$$

and their final momentum is

$$\vec{P}_f = (m_1 + m_2)\vec{v}_f.$$

Thus,

$$4.0\hat{\imath}\,\text{kg}\cdot\text{m/s} = (4.0\,\text{kg})\vec{v}_f,$$

so

$$\vec{v}_f = (1.0\hat{\imath})\,\text{m/s}.$$

Now compare the kinetic energies.

Initially,

$$K_i = \tfrac{1}{2}m_1|\vec{v}_{1,i}|^2 + \tfrac{1}{2}m_2|\vec{v}_{2,i}|^2 = \tfrac{1}{2}(1.0)(4.0)^2 + \tfrac{1}{2}(3.0)(0)^2 = 8.0\,\text{J}.$$

Finally,

$$K_f = \tfrac{1}{2}(m_1 + m_2)|\vec{v}_f|^2 = \tfrac{1}{2}(4.0)(1.0)^2 = 2.0\,\text{J}.$$

Therefore,

$$\Delta K = K_f - K_i = 2.0\,\text{J} - 8.0\,\text{J} = -6.0\,\text{J}.$$

The negative sign means 6.0 J of kinetic energy was transformed into other forms of energy during the collision. Since the carts stick together and the kinetic energy decreases, the collision is inelastic, more specifically perfectly inelastic.

So the results are

$$\vec{v}_f = (1.0\hat{\imath})\,\text{m/s}, \qquad \Delta K = -6.0\,\text{J},$$

and the collision is perfectly inelastic rather than elastic.

1.4.5 Recoil, Explosions, and the Center-of-Mass Viewpoint

This subsection treats recoil and explosions as momentum-redistribution processes. In AP mechanics, internal energy may be released during the interaction, but if the net external impulse on the chosen system is zero, the total momentum and the center-of-mass motion do not change.

Definition 1.4.5: Recoil/explosion interactions and the center-of-mass viewpoint

Consider a closed system of particles labeled by $i = 1, 2, \ldots, N$. Let m_i denote the mass of particle i, let $\vec{v}_i$ denote its velocity, let

$$\vec{p}_i = m_i \vec{v}_i$$

denote its momentum, and let

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i$$

denote the total momentum. Let

$$M = \sum_{i=1}^{N} m_i$$

denote the total mass, and let $\vec{v}_{\text{cm}}$ denote the center-of-mass velocity. Then

$$\vec{P} = M \vec{v}_{\text{cm}}.$$

A *recoil* or *explosion* interaction is a short internal interaction in which parts of the system push apart or are driven apart by released internal energy. If the net external impulse on the system during the interaction is zero or negligible, then

$$\vec{P}_f = \vec{P}_i,$$

so equivalently

$$\vec{v}_{\text{cm},f} = \vec{v}_{\text{cm},i}.$$

If the system is initially at rest, then $\vec{P}_i = \vec{0}$ and $\vec{v}_{\text{cm}} = \vec{0}$ both before and after the interaction.

Note:-

In recoil and explosion problems, the internal forces between the parts of the system can be very large, but they occur in equal-and-opposite pairs and therefore only redistribute momentum within the system. They can change the individual velocities and can increase the total kinetic energy if internal energy is released, but they do not change the total momentum of an isolated system. From the center-of-mass viewpoint, the whole event is just internal rearrangement: the center of mass continues to remain at rest or to move with constant velocity if the external impulse is zero.

Example 1.4.2 (Illustrative example)

A student of mass $m_s = 60.0\,\text{kg}$ stands at rest on a frictionless skateboard and throws a backpack of mass $m_b = 5.0\,\text{kg}$ horizontally backward with velocity

$$\vec{v}_b = (-8.0\hat{\imath})\,\text{m/s}.$$

Let $\vec{v}_s = v_s \hat{\imath}$ denote the student's recoil velocity after the throw.
Because the student-backpack system starts at rest and the external horizontal impulse is negligible,

$$\vec{P}_f = \vec{0}.$$

Thus,

$$m_s \vec{v}_s + m_b \vec{v}_b = \vec{0}.$$

Substitute the values:

$$(60.0\,\text{kg}) v_s \hat{\imath} + (5.0\,\text{kg})(-8.0\hat{\imath}\,\text{m/s}) = \vec{0}.$$

So,

$$60.0 v_s - 40.0 = 0,$$

which gives

$$v_s = 0.667\,\text{m/s}.$$

Therefore,

$$\vec{v}_s = (0.667\hat{\imath})\,\text{m/s}.$$

The student recoils forward while the backpack moves backward, and the total momentum remains zero.

Proposition 1.4.3 Useful recoil and center-of-mass relations

Let a system of total mass M have velocity $\vec{V}_0$ just before a recoil or explosion event. Let the final pieces have masses $m_1, m_2, \ldots$ and velocities $\vec{v}_1, \vec{v}_2, \ldots$. If the net external impulse during the short interaction is zero or negligible, then:

1. Total momentum is conserved:

$$\sum \vec{p}_f = \sum \vec{p}_i, \qquad \sum m_k \vec{v}_k = M\vec{V}_0.$$

2. The center-of-mass velocity stays constant:

$$\vec{v}_{\text{cm}} = \frac{\vec{P}}{M} = \vec{V}_0.$$

   If the system starts from rest, then $\vec{v}_{\text{cm}} = \vec{0}$ and

$$\sum m_k \vec{v}_k = \vec{0}.$$

3. For a two-piece explosion from rest,

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{0},$$

   so the two final momenta are equal in magnitude and opposite in direction:

$$m_1\vec{v}_1 = -m_2\vec{v}_2.$$

4. In two dimensions, conserve momentum component-by-component:

$$\sum p_{x,f} = \sum p_{x,i}, \qquad \sum p_{y,f} = \sum p_{y,i}.$$

   This is usually the most direct way to find unknown fragment velocities.

Question 24: Worked example

A firework shell of total mass $M = 4.0\,\text{kg}$ is moving with velocity

$$\vec{V}_0 = (10.0\hat{\imath} + 6.0\hat{\jmath})\,\text{m/s}$$

just before it explodes into two fragments. Fragment 1 has mass $m_1 = 1.5\,\text{kg}$ and velocity

$$\vec{v}_1 = (18.0\hat{\imath} + 2.0\hat{\jmath})\,\text{m/s}$$

immediately after the explosion. Fragment 2 has mass $m_2 = 2.5\,\text{kg}$ and velocity

$$\vec{v}_2 = v_{2,x}\hat{\imath} + v_{2,y}\hat{\jmath}.$$

Assume the net external impulse during the explosion is negligible.
Find $\vec{v}_2$, find the center-of-mass velocity after the explosion, and determine whether the explosion could have increased the total kinetic energy.

Solution: Because the explosion is brief and the net external impulse is negligible, the shell-plus-fragments system conserves momentum:

$$\vec{P}_i = \vec{P}_f.$$

First compute the initial total momentum. Since the shell has total mass $M = 4.0\,\mathrm{kg}$ and velocity $\vec{V}_0$ just before the explosion,

$$\vec{P}_i = M\vec{V}_0.$$

Substitute the given values:

$$\vec{P}_i = (4.0\,\mathrm{kg})(10.0\hat{\imath} + 6.0\hat{\jmath})\,\mathrm{m/s}.$$

Therefore,

$$\vec{P}_i = (40.0\hat{\imath} + 24.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}.$$

Next compute the momentum of fragment 1:

$$\vec{p}_1 = m_1\vec{v}_1 = (1.5\,\mathrm{kg})(18.0\hat{\imath} + 2.0\hat{\jmath})\,\mathrm{m/s}.$$

So,

$$\vec{p}_1 = (27.0\hat{\imath} + 3.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}.$$

Let $\vec{p}_2 = m_2\vec{v}_2$ denote the momentum of fragment 2. Since

$$\vec{P}_f = \vec{p}_1 + \vec{p}_2,$$

momentum conservation gives

$$\vec{p}_2 = \vec{P}_i - \vec{p}_1.$$

Thus,

$$\vec{p}_2 = (40.0\hat{\imath} + 24.0\hat{\jmath}) - (27.0\hat{\imath} + 3.0\hat{\jmath}).$$

Therefore,

$$\vec{p}_2 = (13.0\hat{\imath} + 21.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}.$$

Now divide by $m_2 = 2.5\,\mathrm{kg}$ to find the velocity of fragment 2:

$$\vec{v}_2 = \frac{\vec{p}_2}{m_2} = \frac{(13.0\hat{\imath} + 21.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}}{2.5\,\mathrm{kg}}.$$

Hence,

$$\vec{v}_2 = (5.2\hat{\imath} + 8.4\hat{\jmath})\,\mathrm{m/s}.$$

Now find the center-of-mass velocity after the explosion. The total momentum after the explosion is still

$$\vec{P}_f = \vec{P}_i = (40.0\hat{\imath} + 24.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}.$$

Using

$$\vec{v}_{\mathrm{cm}} = \frac{\vec{P}}{M},$$

we get

$$\vec{v}_{\mathrm{cm},f} = \frac{\vec{P}_f}{M} = \frac{(40.0\hat{\imath} + 24.0\hat{\jmath})\,\mathrm{kg}\cdot\mathrm{m/s}}{4.0\,\mathrm{kg}}.$$

So,

$$\vec{v}_{\mathrm{cm},f} = (10.0\hat{\imath} + 6.0\hat{\jmath})\,\mathrm{m/s}.$$

This is exactly the same as the pre-explosion velocity $\vec{V}_0$, which is the center-of-mass viewpoint: the fragments fly apart, but the center of mass continues with the same velocity because the external impulse is negligible.

Finally, check whether the total kinetic energy could have increased.

Before the explosion,

$$K_i = \tfrac{1}{2}M|\vec{V}_0|^2 = \tfrac{1}{2}(4.0)\left[(10.0)^2 + (6.0)^2\right].$$

Thus,

$$K_i = 2.0(136) = 272\,\mathrm{J}.$$

After the explosion,

$$K_f = \frac{1}{2}m_1|\vec{v}_1|^2 + \frac{1}{2}m_2|\vec{v}_2|^2.$$

For fragment 1,

$$|\vec{v}_1|^2 = (18.0)^2 + (2.0)^2 = 328,$$

so

$$K_1 = \frac{1}{2}(1.5)(328) = 246\,\text{J}.$$

For fragment 2,

$$|\vec{v}_2|^2 = (5.2)^2 + (8.4)^2 = 27.04 + 70.56 = 97.60,$$

so

$$K_2 = \frac{1}{2}(2.5)(97.60) = 122\,\text{J}.$$

Therefore,

$$K_f = 246\,\text{J} + 122\,\text{J} = 368\,\text{J}.$$

Since

$$K_f > K_i,$$

the total kinetic energy increased by

$$\Delta K = 368\,\text{J} - 272\,\text{J} = 96\,\text{J}.$$

That is possible because the explosion released internal energy. Momentum stayed constant because the system was isolated during the brief explosion, but kinetic energy did not have to remain constant.

So the results are

$$\vec{v}_2 = (5.2\hat{\imath} + 8.4\hat{\jmath})\,\text{m/s},$$

and

$$\vec{v}_{\text{cm},f} = (10.0\hat{\imath} + 6.0\hat{\jmath})\,\text{m/s}.$$

The explosion changed the fragment velocities and increased the total kinetic energy, but it did not change the total momentum or the motion of the center of mass.

1.5 Torque and Rotational Dynamics

This unit extends translational mechanics to fixed-axis rotational motion. We begin with angular position θ, angular velocity ω, and angular acceleration α, then connect those quantities to the corresponding linear quantities for points on a rigid body.

We next develop the rotational analogs of force and mass through torque $\vec{\tau}$ and moment of inertia I, apply equilibrium ideas to situations with no angular acceleration, and conclude with the rotational dynamics relation $\sum \tau = I\alpha$ for planar rigid-body rotation in AP Physics scope.

1.5.1 Angular Position, Velocity, and Acceleration

This subsection describes rigid-body rotation about a fixed axis using signed angular variables. Once a positive sense of rotation is chosen, the formulas parallel one-dimensional kinematics.

Definition 1.5.1: Angular kinematic variables for fixed-axis rotation

Consider a rigid body rotating about a fixed axis. Choose a positive direction along the axis with unit vector $\hat{k}$ so that positive rotation is counterclockwise by the right-hand rule. Let $\theta(t)$ denote the signed angular position of the body, let $\omega(t)$ denote its angular velocity, and let $\alpha(t)$ denote its angular acceleration. Then

$$\omega = \frac{d\theta}{dt}, \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Over a time interval Δt, let θ_i and θ_f denote the initial and final angular positions, and let ω_i and ω_f denote the initial and final angular velocities. Then

$$\omega_{\text{avg}} = \frac{\Delta\theta}{\Delta t}, \qquad \alpha_{\text{avg}} = \frac{\Delta\omega}{\Delta t},$$

where $\Delta\theta = \theta_f - \theta_i$ and $\Delta\omega = \omega_f - \omega_i$. In vector form for fixed-axis rotation,

$$\vec{\omega} = \omega\hat{k}, \qquad \vec{\alpha} = \alpha\hat{k}.$$

A positive value means rotation or change in rotation in the chosen positive sense, and a negative value means the opposite sense.

Note:-

Angular position is measured in radians, with $2\pi\,\text{rad} = 1$ revolution. Because a radian is a ratio of lengths, it is technically dimensionless, but radians should still be written in angular answers to make the meaning clear. The angular variables θ, ω, and α describe the entire rigid body, whereas the linear quantities of an individual point on the body depend on its distance r from the axis. Also, θ is a signed coordinate, not a distance, so it can be negative and can exceed 2π after multiple turns.

Proposition 1.5.1 Core fixed-axis relations

Let t denote elapsed time from an initial instant $t = 0$. Let $\theta_0 = \theta(0)$ and $\omega_0 = \omega(0)$. Let $\theta = \theta(t)$ and $\omega = \omega(t)$ at a later time t. For a point on the rigid body at perpendicular distance r from the axis, let s denote its arc length from the chosen reference line, let v_t denote its tangential velocity component in the positive tangential direction, and let a_t denote its tangential acceleration component in that direction.

1. Angular and linear quantities are related by

$$s = r\theta, \qquad v_t = r\omega, \qquad a_t = r\alpha.$$

Thus all points on a rigid body have the same θ, ω, and α, but points farther from the axis have larger $|s|$, $|v_t|$, and $|a_t|$.

2. If α is constant over the interval, then the angular kinematic equations are

$$\omega = \omega_0 + \alpha t,$$
$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2,$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0).$$

3. For constant α, the average angular velocity over the interval is

$$\omega_{\text{avg}} = \frac{\omega_0 + \omega}{2},$$

so the angular displacement can also be written as

$$\Delta\theta = \omega_{\text{avg}} t = \frac{\omega_0 + \omega}{2}\, t.$$

These equations are the exact rotational analogs of the one-dimensional constant-acceleration formulas.

Question 25: Worked example

A bicycle wheel rotates about a fixed axle. Choose counterclockwise as positive. At the instant the brakes are applied, let $t = 0$, let the wheel's angular position be $\theta_0 = 0$, and let its angular velocity be $\omega_0 = +18.0\,\mathrm{rad/s}$. While braking, the wheel has constant angular acceleration $\alpha = -3.00\,\mathrm{rad/s^2}$ until it stops. The wheel radius is $r = 0.340\,\mathrm{m}$.
Find:

(a) the time required to stop,

(b) the angular displacement before stopping,

(c) the number of revolutions made while stopping, and

(d) the initial tangential speed of a point on the rim and the tangential acceleration of the rim during braking.

Solution: First use constant angular acceleration to find the stopping time. At the instant the wheel stops, $\omega = 0$. From

$$\omega = \omega_0 + \alpha t,$$

we have

$$0 = 18.0\,\mathrm{rad/s} + (-3.00\,\mathrm{rad/s^2})t.$$

So

$$t = \frac{18.0\,\mathrm{rad/s}}{3.00\,\mathrm{rad/s^2}} = 6.00\,\mathrm{s}.$$

Therefore the wheel stops after

$$t = 6.00\,\mathrm{s}.$$

Now find the angular displacement. Using

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2,$$

with $\theta_0 = 0$, $t = 6.00\,\mathrm{s}$, $\omega_0 = 18.0\,\mathrm{rad/s}$, and $\alpha = -3.00\,\mathrm{rad/s^2}$,

$$\theta = (0) + (18.0)(6.00) + \tfrac{1}{2}(-3.00)(6.00)^2.$$

Thus

$$\theta = 108 - 54 = 54.0\,\mathrm{rad}.$$

Since $\theta_0 = 0$, the angular displacement is

$$\Delta\theta = 54.0\,\mathrm{rad}.$$

Convert this to revolutions using $2\pi\,\mathrm{rad} = 1$ revolution:

$$N = \frac{\Delta\theta}{2\pi} = \frac{54.0}{2\pi} \approx 8.59.$$

So the wheel makes

$$N \approx 8.59 \text{ revolutions}$$

before stopping.

For the rim's initial tangential speed,

$$v_t = r\omega_0 = (0.340\,\mathrm{m})(18.0\,\mathrm{rad/s}) = 6.12\,\mathrm{m/s}.$$

For the tangential acceleration,

$$a_t = r\alpha = (0.340\,\mathrm{m})(-3.00\,\mathrm{rad/s^2}) = -1.02\,\mathrm{m/s^2}.$$

So the tangential acceleration is opposite the positive tangential direction, consistent with the wheel slowing down.

The results are

$$t = 6.00\,\mathrm{s}, \qquad \Delta\theta = 54.0\,\mathrm{rad}, \qquad N \approx 8.59,$$
$$v_{t,0} = 6.12\,\mathrm{m/s}, \qquad a_t = -1.02\,\mathrm{m/s^2}.$$

Because $\omega_0 > 0$ and $\alpha < 0$, the brake torque reduces the wheel's counterclockwise rotation rate until the angular velocity reaches zero.

1.5.2 Linear and Rotational Kinematic Correspondence

This subsection connects the angular variables of a rigid body in fixed-axis rotation to the linear motion of any specific point on that body. Once a positive sense of rotation is chosen, the tangential quantities behave like signed one-dimensional variables along the circular path, while the radial acceleration always points inward.

Definition 1.5.2: Arc length, tangential speed, and radial/tangential acceleration for a rotating point

Consider a rigid body rotating about a fixed axis with unit vector $\hat{k}$. Let P be a point on the body at constant perpendicular distance r from the axis, and let $\vec{r} = r\hat{r}$ be the position vector from the axis to P. Choose the positive tangential direction $\hat{t}$ to be the direction of positive rotation. Let $\theta(t)$ denote the signed angular position of the body, let $\omega = d\theta/dt$ denote its angular velocity, and let $\alpha = d\omega/dt$ denote its angular acceleration.

The signed arc-length coordinate of P along its circular path is

$$s = r\theta.$$

The tangential velocity component and tangential acceleration component of P are

$$v_t = \frac{ds}{dt}, \qquad a_t = \frac{dv_t}{dt}.$$

Let $a_r \geq 0$ denote the magnitude of the radial component of the acceleration. Then the acceleration vector of P can be written as

$$\vec{a} = -a_r\hat{r} + a_t\hat{t},$$

because the radial part points toward the axis.

Theorem 1.5.1 Linear/rotational correspondence for fixed-axis rotation

For the point P above at radius r,

$$s = r\theta, \qquad v_t = r\omega, \qquad a_t = r\alpha, \qquad a_r = \frac{v_t^2}{r} = r\omega^2.$$

Thus

$$\vec{v} = v_t\hat{t}, \qquad \vec{a} = -r\omega^2\hat{r} + r\alpha\hat{t}.$$

If $\vec{\omega} = \omega\hat{k}$, then the tangential-velocity relation may also be written as

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

The first three formulas are the direct linear analogs of angular position, angular velocity, and angular acceleration for a point on the rotating body, while a_r gives the inward acceleration required to keep the point on a circular path.

Short derivation from $s = r\theta$: Because the point stays a fixed distance r from the axis, its arc-length coordinate satisfies

$$s = r\theta.$$

Differentiate with respect to time:

$$v_t = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega.$$

Differentiate again:

$$a_t = \frac{dv_t}{dt} = r\frac{d\omega}{dt} = r\alpha.$$

For the radial part, the point is moving instantaneously on a circle of radius r with speed $|v_t|$, so the inward radial acceleration has magnitude

$$a_r = \frac{v_t^2}{r}.$$

Substitute $v_t = r\omega$:

$$a_r = \frac{(r\omega)^2}{r} = r\omega^2.$$

Therefore

$$s = r\theta, \qquad v_t = r\omega, \qquad a_t = r\alpha, \qquad a_r = r\omega^2.$$

😄

Corollary 1.5.1 Same ω, different linear motion at different radii

Let two points P_1 and P_2 lie on the same rigid body at radii r_1 and r_2 from the same fixed axis. At any instant they share the same angular quantities θ, ω, and α, but their linear quantities scale with radius:

$$\frac{s_2}{s_1} = \frac{r_2}{r_1}, \qquad \frac{v_{t,2}}{v_{t,1}} = \frac{r_2}{r_1}, \qquad \frac{a_{t,2}}{a_{t,1}} = \frac{r_2}{r_1}, \qquad \frac{a_{r,2}}{a_{r,1}} = \frac{r_2}{r_1}.$$

So points farther from the axis move faster, have larger tangential acceleration for the same α, and require larger inward acceleration for the same ω.

Question 26: Worked example

A rigid wheel rotates counterclockwise about a fixed axle. At the instant of interest, the wheel has angular position $\theta = 1.20\,\text{rad}$, angular velocity $\omega = 6.0\,\text{rad/s}$, and angular acceleration $\alpha = -2.0\,\text{rad/s}^2$. Point A is painted on the wheel at radius $r_A = 0.050\,\text{m}$, and point B is painted at radius $r_B = 0.150\,\text{m}$.
For each point, find:

(a) the signed arc-length coordinate s,

(b) the tangential velocity component v_t,

(c) the tangential acceleration component a_t, and

(d) the radial acceleration magnitude a_r.

Solution: Choose the positive tangential direction to be counterclockwise, since the wheel's rotation is positive in that sense. Use

$$s = r\theta, \qquad v_t = r\omega, \qquad a_t = r\alpha, \qquad a_r = r\omega^2.$$

For point A, $r_A = 0.050\,\text{m}$.
Its arc-length coordinate is

$$s_A = r_A\theta = (0.050\,\text{m})(1.20) = 0.060\,\text{m}.$$

Its tangential velocity component is

$$v_{t,A} = r_A\omega = (0.050\,\text{m})(6.0\,\text{rad/s}) = 0.30\,\text{m/s}.$$

Its tangential acceleration component is

$$a_{t,A} = r_A\alpha = (0.050\,\text{m})(-2.0\,\text{rad/s}^2) = -0.10\,\text{m/s}^2.$$

The negative sign means the tangential acceleration is opposite the positive tangential direction at that instant.
Its radial acceleration magnitude is

$$a_{r,A} = r_A\omega^2 = (0.050\,\text{m})(6.0\,\text{rad/s})^2 = (0.050)(36) = 1.8\,\text{m/s}^2.$$

This radial part points inward, toward the axle.
For point B, $r_B = 0.150\,\text{m}$.

Its arc-length coordinate is

$$s_B = r_B\theta = (0.150\,\mathrm{m})(1.20) = 0.180\,\mathrm{m}.$$

Its tangential velocity component is

$$v_{t,B} = r_B\omega = (0.150\,\mathrm{m})(6.0\,\mathrm{rad/s}) = 0.90\,\mathrm{m/s}.$$

Its tangential acceleration component is

$$a_{t,B} = r_B\alpha = (0.150\,\mathrm{m})(-2.0\,\mathrm{rad/s^2}) = -0.30\,\mathrm{m/s^2}.$$

Its radial acceleration magnitude is

$$a_{r,B} = r_B\omega^2 = (0.150\,\mathrm{m})(6.0\,\mathrm{rad/s})^2 = (0.150)(36) = 5.4\,\mathrm{m/s^2}.$$

Therefore,

$$s_A = 0.060\,\mathrm{m}, \qquad v_{t,A} = 0.30\,\mathrm{m/s}, \qquad a_{t,A} = -0.10\,\mathrm{m/s^2}, \qquad a_{r,A} = 1.8\,\mathrm{m/s^2},$$

$$s_B = 0.180\,\mathrm{m}, \qquad v_{t,B} = 0.90\,\mathrm{m/s}, \qquad a_{t,B} = -0.30\,\mathrm{m/s^2}, \qquad a_{r,B} = 5.4\,\mathrm{m/s^2}.$$

Point B, which is three times farther from the axis than point A, has three times the arc length, tangential speed, tangential-acceleration component, and radial-acceleration magnitude.

1.5.3 Torque and Lever Arm

This subsection introduces torque as the rotational effect of a force about a chosen point or axis. In AP mechanics, the key computational idea is that only the part of the force with a nonzero lever arm contributes to the turning effect.

Definition 1.5.3: Torque vector, lever arm, and fixed-axis sign convention

Let O denote the pivot or reference point. Let $\vec{r}$ denote the position vector from O to the point where a force $\vec{F}$ is applied. The *torque vector* of $\vec{F}$ about O is the vector quantity that measures the tendency of the force to cause rotation about O.

Let ϕ denote the angle between $\vec{r}$ and $\vec{F}$. The *lever arm* ℓ is the perpendicular distance from the pivot to the line of action of the force, so

$$\ell = r\sin\phi,$$

where $r = |\vec{r}|$.

For a fixed-axis problem, choose an axis with unit vector $\hat{k}$ and declare a positive sense of rotation by the right-hand rule. The corresponding signed scalar torque is

$$\tau = (\vec{r}\times\vec{F})\cdot\hat{k}.$$

If the force tends to rotate the object in the chosen positive sense, then $\tau > 0$; if it tends to rotate the object in the opposite sense, then $\tau < 0$.

Theorem 1.5.2 Torque formulas

Let $\vec{r}$ denote the position vector from the pivot to the point of application of a force $\vec{F}$. Let ϕ denote the angle between $\vec{r}$ and $\vec{F}$, let $r = |\vec{r}|$, let $F = |\vec{F}|$, and let ℓ denote the lever arm. Then the torque vector is

$$\vec{\tau} = \vec{r}\times\vec{F},$$

and its magnitude is

$$|\vec{\tau}| = rF\sin\phi = F\ell.$$

For rotation about a chosen fixed axis with unit vector $\hat{k}$,

$$\tau = (\vec{r}\times\vec{F})\cdot\hat{k},$$

so the scalar τ is positive or negative according to the declared sign convention.

Why $rF\sin\phi$ equals $F\ell$: From the cross-product magnitude formula,

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF\sin\phi.$$

But by geometry, the lever arm is the perpendicular distance from the pivot to the line of action of the force, so

$$\ell = r\sin\phi.$$

Substituting this into the previous expression gives

$$|\vec{\tau}| = rF\sin\phi = F\ell.$$

Thus torque can be found either from the perpendicular component of the force or from the full force multiplied by the lever arm. 😄

Corollary 1.5.2 If the line of action passes through the pivot, the torque is zero

If the line of action of $\vec{F}$ passes through the pivot, then the perpendicular distance from the pivot to that line is $\ell = 0$. Therefore

$$|\vec{\tau}| = F\ell = 0.$$

Equivalently, in this case $\phi = 0$ or $\phi = \pi$, so $rF\sin\phi = 0$. Thus a force directed exactly through the pivot can change the net force on an object without producing any torque about that pivot.

Question 27: Worked example

A door rotates about a vertical hinge through its left edge. View the door from above and choose counterclockwise rotation as positive. Let $r = 0.90\,\text{m}$ be the distance from the hinge to the point where the force is applied at the outer edge. A student pushes with force magnitude $F = 40\,\text{N}$. The force lies in the horizontal plane and makes an angle $\phi = 30°$ with the door, so it tends to rotate the door counterclockwise. Find:

(a) the lever arm ℓ,

(b) the signed torque τ about the hinge, and

(c) the magnitude of a force applied perpendicular to the door at the same point that would produce the same torque.

Solution: Because the position vector from the hinge to the point of application lies along the door, the given angle $\phi = 30°$ is the angle between $\vec{r}$ and $\vec{F}$.

For part (a), the lever arm is

$$\ell = r\sin\phi = (0.90\,\text{m})\sin 30°.$$

Since $\sin 30° = 0.50$,

$$\ell = (0.90)(0.50) = 0.45\,\text{m}.$$

So the lever arm is

$$\ell = 0.45\,\text{m}.$$

For part (b), the torque magnitude is

$$|\tau| = rF\sin\phi = F\ell.$$

Using either form,

$$|\tau| = (0.90\,\text{m})(40\,\text{N})\sin 30° = (40\,\text{N})(0.45\,\text{m}).$$

Thus

$$|\tau| = 18\,\text{N}\cdot\text{m}.$$

Because the force tends to rotate the door counterclockwise and counterclockwise was chosen as positive,

$$\tau = +18\,\mathrm{N\cdot m}.$$

For part (c), let $F_\perp$ denote the force magnitude that would act perpendicular to the door at the same point. A perpendicular force has lever arm equal to the full distance $r = 0.90\,\mathrm{m}$, so its torque magnitude is

$$|\tau| = rF_\perp.$$

Set this equal to the required $18\,\mathrm{N\cdot m}$:

$$18\,\mathrm{N\cdot m} = (0.90\,\mathrm{m})F_\perp.$$

Therefore,

$$F_\perp = \frac{18}{0.90} = 20\,\mathrm{N}.$$

The results are

$$\ell = 0.45\,\mathrm{m}, \qquad \tau = +18\,\mathrm{N\cdot m}, \qquad F_\perp = 20\,\mathrm{N}.$$

This example shows that the same torque can be produced either by a larger force with a shorter lever arm or by a smaller force applied more effectively.

1.5.4 Moment of Inertia and Mass Distribution

This subsection introduces the rotational analog of mass for fixed-axis motion. In AP mechanics, the key idea is that both the amount of mass and how far that mass lies from the chosen axis determine how strongly an object resists angular acceleration.

Definition 1.5.4: Moment of inertia for discrete and continuous mass distributions

Consider a rigid body about a chosen fixed axis. For a discrete collection of particles, let particle i have mass m_i, let $\vec{r}_i$ denote its position vector relative to a point on the axis, and let $r_{\perp,i}$ denote its perpendicular distance to the axis. The *moment of inertia* of the body about that axis is

$$I = \sum_i m_i r_{\perp,i}^2.$$

For a continuous mass distribution, let dm denote a small mass element and let $r_\perp$ denote that element's perpendicular distance to the same axis. Then

$$I = \int r_\perp^2\,dm.$$

For a single point mass,

$$I = mr_\perp^2.$$

The SI unit of moment of inertia is $\mathrm{kg\cdot m^2}$. Because the distance to the axis is squared, mass farther from the axis contributes much more strongly to I.

Theorem 1.5.3 Key fixed-axis relations and axis dependence

Consider a rigid body rotating about a fixed axis with unit vector $\hat{k}$. Let $\vec{\alpha} = \alpha\hat{k}$ denote its angular acceleration, let $\vec{\tau}_{\text{net}} = \tau_{\text{net}}\hat{k}$ denote the net external torque about that axis, and let I denote the moment of inertia about that same axis. Then

$$\tau_{\text{net}} = I\alpha, \qquad \vec{\tau}_{\text{net}} = I\vec{\alpha}.$$

Thus, for the same net torque, a larger moment of inertia gives a smaller angular acceleration.
Now let M denote the total mass of the body, let I_{cm} denote the moment of inertia about an axis through the center of mass, and let d denote the perpendicular distance from that axis to a second axis parallel to it. Then the parallel-axis theorem states

$$I = I_{\text{cm}} + Md^2.$$

Therefore moment of inertia depends on the chosen axis as well as on the mass distribution. Moving the axis farther from the center of mass increases I.

Example 1.5.1 (Illustrative example)

A light turntable carries two small clay balls, each of mass $m = 0.40\,\text{kg}$. In arrangement A, each ball is at distance $r_A = 0.10\,\text{m}$ from the axis. In arrangement B, each ball is at distance $r_B = 0.20\,\text{m}$ from the axis. For arrangement A,

$$I_A = 2mr_A^2 = 2(0.40)(0.10)^2 = 8.0 \times 10^{-3}\,\text{kg} \cdot \text{m}^2.$$

For arrangement B,

$$I_B = 2mr_B^2 = 2(0.40)(0.20)^2 = 3.2 \times 10^{-2}\,\text{kg} \cdot \text{m}^2.$$

So doubling the distance from the axis makes the moment of inertia four times as large, even though the total mass is unchanged.

Note:-

Equal mass does not guarantee equal rotational response. Two objects can have the same total mass but different moments of inertia if their mass is distributed differently or if the axis is changed. In fixed-axis rotation, the object with larger I has smaller angular acceleration for the same net torque.

Question 28: Worked example

A light rigid rod of length $L = 0.80\,\text{m}$ lies on a horizontal frictionless table and can rotate about a vertical axle. A small mass $m_1 = 0.50\,\text{kg}$ is attached at the left end, and a small mass $m_2 = 1.50\,\text{kg}$ is attached at the right end. First, the axle passes through the center of the rod, so each mass is at distance $r = 0.40\,\text{m}$ from the axis. A string pulls perpendicularly on the right end with force magnitude $F = 3.0\,\text{N}$.
Find:

(a) the moment of inertia I about the center axle,

(b) the torque magnitude τ due to the pull,

(c) the angular acceleration magnitude α, and

(d) the new moment of inertia and angular acceleration if the axle is moved to the left end of the rod while the same perpendicular force is still applied at the right end.

Solution: For parts (a)–(c), the axis passes through the center of the rod, so each mass is $r = 0.40\,\text{m}$ from the axis. Because the rod is light, treat its mass as negligible and include only the two point masses.

For part (a), the moment of inertia is

$$I = \sum m_i r_i^2 = m_1 r^2 + m_2 r^2.$$

Substitute the given values:

$$I = (0.50)(0.40)^2 + (1.50)(0.40)^2.$$

Since $(0.40)^2 = 0.16$,

$$I = (0.50)(0.16) + (1.50)(0.16) = 0.08 + 0.24 = 0.32\,\text{kg} \cdot \text{m}^2.$$

So the moment of inertia about the center axle is

$$I = 0.32\,\text{kg} \cdot \text{m}^2.$$

For part (b), the pull is perpendicular to the rod, so the torque magnitude is

$$\tau = rF.$$

Here the force is applied at the right end, which is $0.40\,\text{m}$ from the center axle. Therefore,

$$\tau = (0.40\,\text{m})(3.0\,\text{N}) = 1.2\,\text{N} \cdot \text{m}.$$

For part (c), use the rotational form of Newton's second law:

$$\tau = I\alpha.$$

Hence,

$$\alpha = \frac{\tau}{I} = \frac{1.2}{0.32} = 3.75\,\mathrm{rad/s^2}.$$

So for the center axle,

$$\alpha = 3.75\,\mathrm{rad/s^2}.$$

For part (d), move the axle to the left end. Then the left mass is on the axis, so its distance is $r_1 = 0$, and the right mass is $r_2 = L = 0.80\,\mathrm{m}$ from the axis. The new moment of inertia is

$$I' = m_1 r_1^2 + m_2 r_2^2.$$

Thus,

$$I' = (0.50)(0)^2 + (1.50)(0.80)^2.$$

Since $(0.80)^2 = 0.64$,

$$I' = 0 + (1.50)(0.64) = 0.96\,\mathrm{kg}\cdot\mathrm{m}^2.$$

Now the same perpendicular force is applied at the right end, which is $0.80\,\mathrm{m}$ from the new axis, so the torque magnitude is

$$\tau' = (0.80\,\mathrm{m})(3.0\,\mathrm{N}) = 2.4\,\mathrm{N}\cdot\mathrm{m}.$$

Then

$$\alpha' = \frac{\tau'}{I'} = \frac{2.4}{0.96} = 2.50\,\mathrm{rad/s^2}.$$

Therefore, with the axle at the left end,

$$I' = 0.96\,\mathrm{kg}\cdot\mathrm{m}^2, \qquad \alpha' = 2.50\,\mathrm{rad/s^2}.$$

Even though the torque becomes larger, the angular acceleration becomes smaller because much more of the mass is farther from the axis, so the moment of inertia increases substantially.

1.5.5 Rotational Equilibrium

This subsection treats the equilibrium case of fixed-axis rotational dynamics. In AP statics, a rigid body remains at rest only when both the translational and rotational balances are satisfied.

Definition 1.5.5: Rotational equilibrium and static equilibrium

Consider a rigid body about a chosen fixed axis with unit vector $\hat{k}$. Let $\vec{\alpha} = \alpha\hat{k}$ denote its angular acceleration, let $\vec{\tau}_{\text{net}} = \tau_{\text{net}}\hat{k}$ denote the net external torque about that axis, and let $\vec{F}_{\text{net}}$ denote the net external force on the body.
The body is in *rotational equilibrium* if

$$\vec{\alpha} = \vec{0},$$

so its angular velocity is constant. In the common AP statics case, the body is at rest and is also in *static equilibrium*, meaning that both its translational and rotational motion remain unchanged:

$$\vec{F}_{\text{net}} = \vec{0}, \qquad \vec{\alpha} = \vec{0}.$$

Thus rotational equilibrium is the $\alpha = 0$ special case of rotational dynamics, and static equilibrium is the at-rest case in which there is also no translational acceleration.

Theorem 1.5.4 Equilibrium conditions for a rigid body

Let $\vec{F}_{\text{net}}$ denote the net external force on a rigid body, and let $\vec{\tau}_{\text{net},O}$ denote the net external torque about a chosen point O. For static equilibrium in an inertial frame,

$$\sum \vec{F} = \vec{0}, \qquad \sum \vec{\tau}_O = \vec{0}.$$

For a planar fixed-axis problem, choose a positive sense of rotation about the axis. Then the equivalent scalar conditions are

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \sum \tau_O = 0,$$

where positive and negative torques are assigned by the declared sign convention. Force balance enforces translational equilibrium, and torque balance enforces rotational equilibrium.

Note:-

In equilibrium problems, the pivot can be chosen wherever it is most convenient. A smart choice is often a point through which one or more unknown forces act, because those forces then contribute zero torque about that point. After using $\sum \tau_O = 0$ to solve for a remaining unknown such as a tension, use $\sum \vec{F} = \vec{0}$ to find the support-force components. If $\sum \vec{F} = \vec{0}$, then torque balance about one point is equivalent to torque balance about any other point, so changing the pivot changes the algebra, not the physics. For fixed-axis AP problems, signed scalar torques are fully acceptable once a sign convention such as counterclockwise positive has been declared.

Short explanation from $\tau_{\text{net}} = I\alpha$: Let I denote the moment of inertia of the rigid body about the chosen fixed axis. Fixed-axis rotational dynamics gives

$$\tau_{\text{net}} = I\alpha.$$

If the body is in rotational equilibrium, then $\alpha = 0$, so

$$\tau_{\text{net}} = 0.$$

Now let m denote the total mass of the body, and let $\vec{a}_{\text{cm}}$ denote the acceleration of its center of mass. For statics the body also has no translational acceleration, so Newton's second law gives

$$\sum \vec{F} = m\vec{a}_{\text{cm}} = \vec{0}.$$

Therefore a rigid body at rest must satisfy both torque balance and force balance. Conversely, if both balances hold, the body has neither angular acceleration nor translational acceleration, so an initially resting body remains in static equilibrium. 😊

Question 29: Worked example

A uniform horizontal beam has length $L = 4.0\,\text{m}$ and weight $W_b = 200\,\text{N}$. It is hinged to a wall at its left end. A light cable is attached to the right end of the beam and makes an angle $\theta = 30°$ above the beam. A sign of weight $W_s = 300\,\text{N}$ hangs from the beam at a point $x_s = 3.0\,\text{m}$ from the hinge. Let T denote the cable tension. Let the hinge force on the beam be $\vec{H} = H_x\hat{\imath} + H_y\hat{\jmath}$, where $\hat{\imath}$ points horizontally to the right and $\hat{\jmath}$ points vertically upward. Choose counterclockwise torque as positive.
Find:

(a) the cable tension T,

(b) the horizontal component H_x, and

(c) the vertical component H_y of the hinge force.

Solution: Draw the beam's free-body diagram. The external forces on the beam are the hinge force $\vec{H}$ at the left end, the cable force $\vec{T}$ at the right end, the beam's weight $\vec{W}_b$ downward at its center, and the sign's weight $\vec{W}_s$ downward at $x_s = 3.0\,\text{m}$.

Choose the hinge as the pivot. Then the unknown hinge force produces zero torque about that point, which is why this pivot choice is efficient.

The beam's center is at

$$\frac{L}{2} = 2.0\,\mathrm{m}$$

from the hinge. The horizontal component of the cable force acts along the beam, so its line of action passes through the hinge and it produces zero torque about the hinge. Thus only the cable's vertical component contributes to the torque balance:

$$\sum \tau_{\text{hinge}} = 0.$$

Using counterclockwise as positive,

$$(T\sin\theta)L - W_b\left(\frac{L}{2}\right) - W_s x_s = 0.$$

Substitute the given values:

$$(T\sin 30^\circ)(4.0\,\mathrm{m}) - (200\,\mathrm{N})(2.0\,\mathrm{m}) - (300\,\mathrm{N})(3.0\,\mathrm{m}) = 0.$$

Since $\sin 30^\circ = 0.50$,

$$(0.50T)(4.0) - 400 - 900 = 0.$$

So

$$2.0T - 1300 = 0,$$

which gives

$$T = 650\,\mathrm{N}.$$

Now apply force balance in the horizontal direction:

$$\sum F_x = 0.$$

The cable pulls the beam toward the wall, so its horizontal component is to the left. Therefore,

$$H_x - T\cos 30^\circ = 0.$$

Hence

$$H_x = T\cos 30^\circ = (650\,\mathrm{N})(0.866) \approx 5.63 \times 10^2\,\mathrm{N}.$$

So the horizontal hinge-force component is

$$H_x \approx 563\,\mathrm{N}$$

to the right.

Now apply force balance in the vertical direction:

$$\sum F_y = 0.$$

Thus

$$H_y + T\sin 30^\circ - W_b - W_s = 0.$$

Substitute the values:

$$H_y + (650\,\mathrm{N})(0.50) - 200\,\mathrm{N} - 300\,\mathrm{N} = 0.$$

So

$$H_y + 325 - 500 = 0,$$

which gives

$$H_y = 175\,\mathrm{N}.$$

Thus the vertical hinge-force component is upward.

The final answers are

$$T = 650\,\mathrm{N}, \qquad H_x \approx 563\,\mathrm{N}\text{ to the right}, \qquad H_y = 175\,\mathrm{N}\text{ upward}.$$

This is the standard statics strategy: pair $\sum \vec{F} = \vec{0}$ with $\sum \tau = 0$, choose a pivot that eliminates unknown torque contributions, solve the torque equation first, and then use force balance to determine the support forces.

1.5.6 Newton's Second Law for Rotation

This subsection gives the fixed-axis rotational analog of $\sum F = ma$. In AP mechanics, the central idea is that the net external torque about a chosen axis determines the angular acceleration, with the moment of inertia setting how strongly the body resists that change in rotation.

Definition 1.5.6: Net torque about a fixed axis and the role of rotational inertia

Consider a rigid body that can rotate about a fixed axis with unit vector $\hat{k}$. Choose a positive sense of rotation by the right-hand rule. For an external force $\vec{F}_i$ applied at position $\vec{r}_i$ relative to a point on the axis, the signed scalar torque about the axis is

$$\tau_i = (\vec{r}_i \times \vec{F}_i) \cdot \hat{k}.$$

The net torque about the axis is the sum of the signed torques:

$$\sum \tau = \sum_i \tau_i.$$

Let I denote the moment of inertia of the body about that same axis, and let α denote the signed angular acceleration. The quantity I measures the rotational inertia of the body: for the same net torque, a larger I gives a smaller α. Thus I plays the rotational role that mass plays in translational motion.

Theorem 1.5.5 Newton's second law for fixed-axis rotation

Consider a rigid body rotating about a fixed axis with unit vector $\hat{k}$. Let I denote the moment of inertia about that axis, let α denote the signed angular acceleration, and let $\sum \tau$ denote the net external torque about the axis using the declared sign convention. Then

$$\sum \tau = I\alpha.$$

Equivalently, if $\vec{\tau}_{\text{net}}$ and $\vec{\alpha}$ both lie along the fixed axis, then

$$\vec{\tau}_{\text{net}} = I\vec{\alpha}.$$

In AP fixed-axis problems, the signed scalar form is usually the most convenient.

Short derivation from point-mass contributions: Model the rigid body as many particles. Let particle i have mass m_i and perpendicular distance $r_{\perp,i}$ from the axis. Because the body is rigid, every particle has the same angular acceleration α, so the tangential acceleration of particle i is

$$a_{t,i} = r_{\perp,i}\alpha.$$

Its tangential force component therefore satisfies

$$F_{t,i} = m_i a_{t,i} = m_i r_{\perp,i}\alpha.$$

Only this tangential component contributes to the torque about the axis, so the signed torque from particle i is

$$\tau_i = r_{\perp,i}F_{t,i} = m_i r_{\perp,i}^2 \alpha.$$

Summing over all particles gives

$$\sum \tau = \sum_i m_i r_{\perp,i}^2 \alpha = \left(\sum_i m_i r_{\perp,i}^2\right) \alpha.$$

Since

$$I = \sum_i m_i r_{\perp,i}^2,$$

it follows that

$$\sum \tau = I\alpha.$$

The same conclusion holds for a continuous rigid body by replacing the sum with an integral. 😄

Corollary 1.5.3 Constant net torque gives constant angular acceleration

If a rigid body rotates about a fixed axis with constant moment of inertia I and constant net external torque $\sum \tau$, then

$$\alpha = \frac{\sum \tau}{I}$$

is constant. Let ω_0 denote the angular velocity and let θ_0 denote the angular position at $t = 0$. Then the constant-angular-acceleration kinematic equations apply:

$$\omega = \omega_0 + \alpha t, \qquad \theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

So under a constant net torque, the angular velocity changes linearly in time and the angular position changes quadratically in time.

Question 30: Worked example

A wheel rotates about a fixed axle. Choose counterclockwise rotation as positive. The wheel has radius $R = 0.25\,\text{m}$ and moment of inertia $I = 0.50\,\text{kg}\cdot\text{m}^2$ about the axle. A tangential force of magnitude $F_1 = 18\,\text{N}$ is applied at the rim in the counterclockwise direction. At the same time, a second tangential force of magnitude $F_2 = 10\,\text{N}$ is applied at the rim in the clockwise direction. In addition, friction exerts a constant torque of magnitude $\tau_f = 0.50\,\text{N}\cdot\text{m}$ in the clockwise direction. The wheel starts from rest at $t = 0$.
Find:

(a) the net torque on the wheel,

(b) the angular acceleration α, and

(c) the angular speed and angular displacement after 4.0 s.

Solution: Use the declared sign convention: counterclockwise torques are positive and clockwise torques are negative.

The torque due to the first force is

$$\tau_1 = +RF_1 = (0.25\,\text{m})(18\,\text{N}) = +4.5\,\text{N}\cdot\text{m}.$$

The torque due to the second force is

$$\tau_2 = -RF_2 = (0.25\,\text{m})(10\,\text{N}) = -2.5\,\text{N}\cdot\text{m}.$$

The friction torque is clockwise, so

$$\tau_f = -0.50\,\text{N}\cdot\text{m}.$$

For part (a), the net torque is

$$\sum \tau = \tau_1 + \tau_2 + \tau_f.$$

Substitute the values:

$$\sum \tau = 4.5 - 2.5 - 0.50 = 1.5\,\text{N}\cdot\text{m}.$$

So the net torque is

$$\sum \tau = +1.5\,\text{N}\cdot\text{m}.$$

The positive sign means the wheel accelerates counterclockwise.

For part (b), apply Newton's second law for rotation:

$$\sum \tau = I\alpha.$$

Thus,

$$\alpha = \frac{\sum \tau}{I} = \frac{1.5\,\mathrm{N\cdot m}}{0.50\,\mathrm{kg\cdot m^2}} = 3.0\,\mathrm{rad/s^2}.$$

So the angular acceleration is

$$\alpha = +3.0\,\mathrm{rad/s^2}.$$

For part (c), the net torque is constant, so the angular acceleration is constant. Since the wheel starts from rest, $\omega_0 = 0$ and take $\theta_0 = 0$.

First find the angular speed after $t = 4.0\,\mathrm{s}$:

$$\omega = \omega_0 + \alpha t = 0 + (3.0\,\mathrm{rad/s^2})(4.0\,\mathrm{s}) = 12\,\mathrm{rad/s}.$$

Now find the angular displacement:

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2.$$

Substitute the values:

$$\theta = 0 + 0 + \tfrac{1}{2}(3.0\,\mathrm{rad/s^2})(4.0\,\mathrm{s})^2.$$

Since $(4.0)^2 = 16$,

$$\theta = \tfrac{1}{2}(3.0)(16) = 24\,\mathrm{rad}.$$

Therefore,

$$\sum \tau = +1.5\,\mathrm{N\cdot m}, \qquad \alpha = +3.0\,\mathrm{rad/s^2}, \qquad \omega = 12\,\mathrm{rad/s}, \qquad \theta = 24\,\mathrm{rad}.$$

This example shows the rotational analog of $\sum F = ma$: once the net torque and the moment of inertia are known, the angular acceleration follows directly.

1.6 Angular Momentum and Rolling Motion

This unit continues fixed-axis rotational mechanics by extending the work-energy ideas of Unit 5 to rotational kinetic energy and the rotational work-energy theorem. We then introduce angular momentum $\vec{L}$ and angular impulse, emphasizing how torque $\vec{\tau}$ changes rotational motion in the same way that force changes linear momentum.

We next focus on conservation of angular momentum for isolated systems, apply that reasoning to rolling without slipping, and conclude with the AP Physics treatment of circular Newtonian orbits, including orbit speed and energy. Throughout, the emphasis is on fixed-axis rotation, rolling motion, and standard circular orbit results within AP scope.

1.6.1 Rotational Kinetic Energy and Work by Torque

This subsection introduces the energy description of fixed-axis rotation. In AP mechanics, the key idea is that a net torque acting through an angular displacement changes a rigid body's rotational kinetic energy in the same way that a net force acting through a displacement changes translational kinetic energy.

Definition 1.6.1: Rotational kinetic energy and incremental work by torque

Consider a rigid body rotating about a fixed axis with unit vector $\hat{k}$. Let I denote the body's moment of inertia about that axis, let $\vec{\omega} = \omega\hat{k}$ denote its angular velocity, and let $\omega = |\vec{\omega}|$ denote the angular speed. The *rotational kinetic energy* of the rigid body is

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2.$$

Now let the body undergo an infinitesimal angular displacement $d\theta$ about the same axis, with the positive direction chosen consistently with $\hat{k}$. Let $\vec{\tau}_{\text{net}} = \tau_{\text{net}}\hat{k}$ denote the net external torque about that axis. Then the incremental work done by the net torque is

$$dW = \tau_{\text{net}}\, d\theta.$$

If τ_{net} is constant over a finite angular displacement $\Delta\theta$, then

$$W = \tau_{\text{net}}\Delta\theta.$$

The SI unit of both rotational kinetic energy and work is the joule, where $1\,\text{J} = 1\,\text{N}\cdot\text{m}$.

Theorem 1.6.1 Rotational work-energy relation for fixed-axis motion

Consider a rigid body rotating about a fixed axis with moment of inertia I. Let ω_i and ω_f denote the initial and final angular speeds, let θ_i and θ_f denote the corresponding angular positions, and let τ_{net} denote the signed net external torque about the axis. Then the net rotational work from the initial state to the final state is

$$W_{\text{net}} = \int_{\theta_i}^{\theta_f} \tau_{\text{net}}\, d\theta,$$

and the rotational work-energy theorem is

$$W_{\text{net}} = \Delta K_{\text{rot}} = K_{\text{rot},f} - K_{\text{rot},i} = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2.$$

Thus a positive net torque doing positive work increases rotational kinetic energy, while negative net work decreases it.

Note:-

This result is the rotational analog of the translational work-energy theorem $W_{\text{net}} = \Delta K$ with $K = \frac{1}{2}mv^2$. The correspondence is

$$\vec{F}_{\text{net}} \leftrightarrow \vec{\tau}_{\text{net}}, \qquad x \leftrightarrow \theta, \qquad v \leftrightarrow \omega, \qquad \tfrac{1}{2}mv^2 \leftrightarrow \tfrac{1}{2}I\omega^2.$$

In this subsection, assume a rigid body rotating about one fixed axis so that every part of the body shares the same angular displacement θ and angular speed ω, and so that I stays constant about that axis. The sign convention for τ_{net} must match the sign convention for θ.

Why $K_{\text{rot}} = \frac{1}{2}I\omega^2$ ***and*** $W_{\text{net}} = \Delta K_{\text{rot}}$***:*** Model the rigid body as particles labeled by an index i. Let particle i have mass m_i and perpendicular distance $r_{\perp,i}$ from the fixed axis. Because the body is rigid, each particle has speed

$$v_i = r_{\perp,i}\omega.$$

Therefore the total kinetic energy is

$$K_{\text{rot}} = \sum_i \tfrac{1}{2}m_i v_i^2 = \sum_i \tfrac{1}{2}m_i(r_{\perp,i}\omega)^2 = \tfrac{1}{2}\omega^2 \sum_i m_i r_{\perp,i}^2.$$

Since

$$I = \sum_i m_i r_{\perp,i}^2,$$

it follows that

$$K_{\text{rot}} = \tfrac{1}{2} I\omega^2.$$

For the work-energy relation, start with the fixed-axis rotational form of Newton's second law,

$$\tau_{\text{net}} = I\alpha,$$

where $\alpha = d\omega/dt$. Multiply both sides by $d\theta$:

$$\tau_{\text{net}}\, d\theta = I\alpha\, d\theta.$$

Using $\omega = d\theta/dt$, we have $d\theta = \omega\, dt$, so

$$I\alpha\, d\theta = I\frac{d\omega}{dt}(\omega\, dt) = I\omega\, d\omega.$$

Thus

$$dW = \tau_{\text{net}}\, d\theta = I\omega\, d\omega = d\left(\tfrac{1}{2} I\omega^2\right) = dK_{\text{rot}}.$$

Integrating from the initial state to the final state gives

$$W_{\text{net}} = \Delta K_{\text{rot}}.$$

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Question 31: Worked example

A wheel of radius $R = 0.25\,\text{m}$ rotates about a frictionless fixed axle. Its moment of inertia about the axle is $I = 1.0\,\text{kg}\cdot\text{m}^2$. A light string is wrapped around the rim, and a student pulls tangentially on the string with constant force magnitude $F = 12\,\text{N}$. The string does not slip, and a length $s = 2.0\,\text{m}$ of string unwinds. Initially the wheel is already spinning in the same direction as the applied torque with angular speed $\omega_i = 4.0\,\text{rad/s}$.
Find:

(a) the torque magnitude τ applied by the string,

(b) the angular displacement $\Delta\theta$ during the pull,

(c) the work done on the wheel by the torque, and

(d) the final angular speed ω_f.

Solution: Because the pull is tangential to the rim, the lever arm equals the radius R. Therefore the applied torque magnitude is

$$\tau = RF = (0.25\,\text{m})(12\,\text{N}) = 3.0\,\text{N}\cdot\text{m}.$$

Since the string does not slip, the unwound length equals the arc length at the rim:

$$s = R\Delta\theta.$$

Hence

$$\Delta\theta = \frac{s}{R} = \frac{2.0\,\text{m}}{0.25\,\text{m}} = 8.0\,\text{rad}.$$

The torque is constant and acts in the direction of rotation, so the work done on the wheel is

$$W = \tau\Delta\theta = (3.0\,\text{N}\cdot\text{m})(8.0\,\text{rad}) = 24\,\text{J}.$$

This also agrees with $W = Fs = (12\,\text{N})(2.0\,\text{m}) = 24\,\text{J}$.

Now apply the rotational work-energy theorem:

$$W = \Delta K_{\text{rot}} = \tfrac{1}{2} I\omega_f^2 - \tfrac{1}{2} I\omega_i^2.$$

Substitute the known values:

$$24 = \tfrac{1}{2}(1.0)\omega_f^2 - \tfrac{1}{2}(1.0)(4.0)^2.$$

Since

$$\tfrac{1}{2}(1.0)(4.0)^2 = 8,$$

we get

$$24 = \tfrac{1}{2}\omega_f^2 - 8.$$

Add 8 to both sides:

$$32 = \tfrac{1}{2}\omega_f^2.$$

Multiply by 2:

$$\omega_f^2 = 64.$$

Therefore,

$$\omega_f = 8.0\,\text{rad/s}.$$

So the results are

$$\tau = 3.0\,\text{N}\cdot\text{m}, \qquad \Delta\theta = 8.0\,\text{rad}, \qquad W = 24\,\text{J}, \qquad \omega_f = 8.0\,\text{rad/s}.$$

The key idea is that the applied torque does positive work, so the wheel's rotational kinetic energy increases.

1.6.2 Angular Momentum and Angular Impulse

This subsection connects torque to rotational motion in the same way that linear impulse connects force to linear momentum. The key AP idea is that a net external torque acting for a time interval changes angular momentum about a chosen point.

Definition 1.6.2: Angular momentum and angular impulse

Let O denote a chosen reference point. Let a particle of mass m have position vector $\vec{r}$ measured from O, velocity $\vec{v}$, and linear momentum $\vec{p} = m\vec{v}$. The angular momentum of the particle about O is

$$\vec{L}_O = \vec{r} \times \vec{p}.$$

For a system of particles, the total angular momentum about O is

$$\vec{L}_O = \sum_i \vec{r}_i \times \vec{p}_i.$$

Let $\vec{\tau}_{\text{ext},O}(t)$ denote the net external torque about the same point O over a time interval from t_i to t_f. The angular impulse about O over that interval is

$$\vec{J}_{\tau,O} = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\,dt.$$

Its SI unit is $\text{N}\cdot\text{m}\cdot\text{s}$, which is equivalent to $\text{kg}\cdot\text{m}^2/\text{s}$. For a rigid body rotating about a fixed axis with moment of inertia I about that axis and angular velocity $\vec{\omega}$, the angular momentum simplifies to

$$\vec{L} = I\vec{\omega}.$$

Theorem 1.6.2 Torque-angular momentum relation and angular impulse theorem

Let O denote a chosen reference point. Let $\vec{L}_O(t)$ be the total angular momentum of a particle or system about O, and let $\vec{\tau}_{\text{ext},O}(t)$ be the net external torque about O. Then

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_{\text{ext},O}.$$

Integrating from t_i to t_f gives

$$\Delta\vec{L}_O = \vec{L}_{O,f} - \vec{L}_{O,i} = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt = \vec{J}_{\tau,O}.$$

For fixed-axis rotation along a chosen axis with unit vector $\hat{k}$, let $\vec{L} = L\hat{k}$ and let $\vec{\tau}_{\text{ext}} = \tau_{\text{ext}}\hat{k}$. Then

$$\frac{dL}{dt} = \tau_{\text{ext}}, \qquad \Delta L = \int_{t_i}^{t_f} \tau_{\text{ext}}\, dt,$$

and if τ_{ext} is constant,

$$\Delta L = \tau_{\text{ext}}\Delta t.$$

Note:-

Angular momentum and torque depend on the chosen point O. The same object can have different $\vec{L}_O$ and $\vec{\tau}_{\text{ext},O}$ when a different origin is chosen, so use the same reference point consistently throughout a problem. The direction of $\vec{L}$ and $\vec{\tau}$ is set by the right-hand rule and is perpendicular to the plane of the relevant cross product. In many AP fixed-axis problems, this vector bookkeeping reduces to signed scalars along one axis: counterclockwise may be chosen positive and clockwise negative. The simplification $\vec{L} = I\vec{\omega}$ is valid for rigid rotation about that fixed axis with the stated moment of inertia about the same axis.

Short derivation from $\vec{L} = \vec{r} \times \vec{p}$***:*** For one particle about point O,

$$\vec{L}_O = \vec{r} \times \vec{p}.$$

Differentiate with respect to time:

$$\frac{d\vec{L}_O}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}_{\text{net}}.$$

Because $\vec{v} \times \vec{v} = \vec{0}$,

$$\frac{d\vec{L}_O}{dt} = \vec{r} \times \vec{F}_{\text{net}} = \vec{\tau}_{\text{net},O}.$$

For a system, summing over all particles gives the same form with the net external torque:

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_{\text{ext},O}.$$

Integrating from t_i to t_f yields

$$\Delta\vec{L}_O = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt.$$

For a rigid body rotating about a fixed axis with constant moment of inertia I,

$$\vec{L} = I\vec{\omega},$$

so along that axis the relation becomes

$$\vec{\tau}_{\text{ext}} = I\vec{\alpha}.$$

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Question 32: Worked example

A flywheel rotates about a frictionless fixed axle through its center. Choose counterclockwise as positive, so the axis direction is $\hat{k}$ by the right-hand rule. The flywheel has moment of inertia $I = 0.80\,\text{kg}\cdot\text{m}^2$ about the axle. Initially its angular velocity is

$$\vec{\omega}_i = 12\,\hat{k}\,\text{rad/s}.$$

From $t = 0$ to $t = 1.5\,\mathrm{s}$, a brake pad exerts a constant external torque

$$\vec{\tau}_{\text{ext}} = -3.2\,\hat{k}\,\mathrm{N\cdot m}.$$

Find:

(a) the angular impulse delivered by the brake,

(b) the flywheel's final angular momentum,

(c) the flywheel's final angular velocity, and

(d) whether the flywheel reverses direction during the interval.

Solution: First compute the initial angular momentum using $\vec{L} = I\vec{\omega}$:

$$\vec{L}_i = I\vec{\omega}_i = (0.80\,\mathrm{kg\cdot m^2})(12\,\hat{k}\,\mathrm{rad/s}) = 9.6\,\hat{k}\,\mathrm{kg\cdot m^2/s}.$$

For part (a), the torque is constant, so the angular impulse is

$$\vec{J}_\tau = \vec{\tau}_{\text{ext}}\Delta t.$$

Thus

$$\vec{J}_\tau = (-3.2\,\hat{k}\,\mathrm{N\cdot m})(1.5\,\mathrm{s}) = -4.8\,\hat{k}\,\mathrm{N\cdot m\cdot s}.$$

Using $1\,\mathrm{N\cdot m\cdot s} = 1\,\mathrm{kg\cdot m^2/s}$,

$$\vec{J}_\tau = -4.8\,\hat{k}\,\mathrm{kg\cdot m^2/s}.$$

For part (b), apply the angular impulse theorem:

$$\Delta\vec{L} = \vec{J}_\tau = \vec{L}_f - \vec{L}_i.$$

So

$$\vec{L}_f = \vec{L}_i + \vec{J}_\tau = (9.6 - 4.8)\,\hat{k}\,\mathrm{kg\cdot m^2/s} = 4.8\,\hat{k}\,\mathrm{kg\cdot m^2/s}.$$

For part (c), again use $\vec{L} = I\vec{\omega}$:

$$\vec{\omega}_f = \frac{\vec{L}_f}{I} = \frac{4.8\,\hat{k}\,\mathrm{kg\cdot m^2/s}}{0.80\,\mathrm{kg\cdot m^2}} = 6.0\,\hat{k}\,\mathrm{rad/s}.$$

For part (d), the final angular velocity still points in the $+\hat{k}$ direction, so the flywheel is still rotating counterclockwise. It slows down, but it does not reverse direction during the $1.5\,\mathrm{s}$ interval.

The results are

$$\vec{J}_\tau = -4.8\,\hat{k}\,\mathrm{N\cdot m\cdot s}, \qquad \vec{L}_f = 4.8\,\hat{k}\,\mathrm{kg\cdot m^2/s}, \qquad \vec{\omega}_f = 6.0\,\hat{k}\,\mathrm{rad/s}.$$

This example shows the key AP idea: a torque acting over time changes angular momentum directly, and the sign of the torque determines whether the wheel speeds up or slows down.

1.6.3 Conservation of Angular Momentum

This subsection gives the rotational conservation law that parallels conservation of linear momentum. In AP mechanics, the key question is whether the net *external* torque about a chosen origin or axis is zero over the interval of interest.

Definition 1.6.3: Angular-momentum-isolated system

Consider a system of particles labeled by an index $i = 1, 2, \ldots, N$. Choose an origin O. Let $\vec{r}_i$ denote the position vector of particle i relative to O, let m_i denote its mass, let $\vec{v}_i$ denote its velocity, and let $\vec{p}_i = m_i\vec{v}_i$ denote its linear momentum. The total angular momentum of the system about O is

$$\vec{L}_O = \sum_i \vec{r}_i \times \vec{p}_i.$$

Let $\vec{\tau}_{\text{ext},O}$ denote the net external torque about the same origin. The system is said to be *isolated for angular momentum about* O over a time interval if the external torque impulse about O is zero:

$$\int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt = \vec{0}.$$

In particular, if $\vec{\tau}_{\text{ext},O} = \vec{0}$ at every instant in the interval, then the system is angular-momentum-isolated about O. For a rigid body rotating about a fixed axis with unit vector $\hat{k}$, one often writes $\vec{L} = L\hat{k}$ and $\vec{\omega} = \omega\hat{k}$, so that in the fixed-axis case $L = I\omega$.

Theorem 1.6.3 Conservation law for angular momentum

Let $\vec{L}_O$ denote the total angular momentum of a system about a chosen origin O, and let $\vec{\tau}_{\text{ext},O}$ denote the net external torque about that same origin. Then

$$\Delta\vec{L}_O = \vec{L}_{O,f} - \vec{L}_{O,i} = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt.$$

Therefore, if the net external torque is zero throughout the interval, or more generally if the external torque impulse is zero, then

$$\vec{L}_{O,f} = \vec{L}_{O,i},$$

so total angular momentum about O is conserved. In the common fixed-axis AP case,

$$L_i = L_f \qquad \Rightarrow \qquad I_i\omega_i = I_f\omega_f$$

when the net external torque about that axis is zero.

Note:-

Angular momentum is conserved only about the origin or axis for which the net external torque is zero, so the choice of origin matters. Internal forces and internal torques can redistribute angular momentum among parts of the system and can change the moment of inertia I, but they do not change the system's *total* angular momentum about the chosen origin when $\vec{\tau}_{\text{ext},O} = \vec{0}$. Also, conservation of angular momentum does *not* imply conservation of kinetic energy: for example, a skater can pull in her arms, decrease I, increase ω, and increase rotational kinetic energy by doing internal work.

Short derivation from $d\vec{L}/dt = \vec{\tau}_{\text{ext}}$: Start with the angular-momentum form of Newton's second law about the chosen origin O:

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_{\text{ext},O}.$$

Integrate from t_i to t_f:

$$\int_{t_i}^{t_f} \frac{d\vec{L}_O}{dt}\, dt = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt.$$

This gives

$$\vec{L}_{O,f} - \vec{L}_{O,i} = \int_{t_i}^{t_f} \vec{\tau}_{\text{ext},O}\, dt.$$

If $\vec{\tau}_{\text{ext},O} = \vec{0}$ throughout the interval, or if the integral on the right is zero, then $\vec{L}_{O,f} - \vec{L}_{O,i} = \vec{0}$. Hence

$$\vec{L}_{O,f} = \vec{L}_{O,i},$$

which is the conservation of angular momentum. In a fixed-axis problem, this reduces to $I_i\omega_i = I_f\omega_f$. ☺

Question 33: Worked example

An ice skater spins about a vertical axis with unit vector $\hat{k}$. Assume the net external torque about that axis is negligible. With her arms extended, her moment of inertia is $I_i = 3.0\,\text{kg}\cdot\text{m}^2$ and her angular velocity is $\vec{\omega}_i = (2.0\,\text{rad/s})\hat{k}$. She then pulls her arms inward so that her final moment of inertia is $I_f = 1.2\,\text{kg}\cdot\text{m}^2$. Find:

(a) the final angular velocity vector $\vec{\omega}_f$,

(b) the initial and final angular momentum vectors, and

(c) the initial and final rotational kinetic energies.

Explain briefly why the energy result does not contradict conservation of angular momentum.

Solution: Because the net external torque about the vertical axis is negligible, angular momentum about that axis is conserved:

$$\vec{L}_i = \vec{L}_f.$$

For fixed-axis rotation, $\vec{L} = I\vec{\omega}$, so

$$I_i\vec{\omega}_i = I_f\vec{\omega}_f.$$

For part (a), solve for the final angular velocity:

$$\vec{\omega}_f = \frac{I_i}{I_f}\vec{\omega}_i = \frac{3.0}{1.2}(2.0\hat{k}) = (2.5)(2.0\hat{k}) = 5.0\hat{k}\,\text{rad/s}.$$

So,

$$\boxed{\vec{\omega}_f = (5.0\,\text{rad/s})\hat{k}}.$$

For part (b), compute the angular momentum before and after:

$$\vec{L}_i = I_i\vec{\omega}_i = (3.0)(2.0\hat{k}) = 6.0\hat{k}\,\text{kg}\cdot\text{m}^2/\text{s}.$$

Because angular momentum is conserved,

$$\vec{L}_f = \vec{L}_i = 6.0\hat{k}\,\text{kg}\cdot\text{m}^2/\text{s}.$$

Thus,

$$\boxed{\vec{L}_i = \vec{L}_f = (6.0\,\text{kg}\cdot\text{m}^2/\text{s})\hat{k}}.$$

For part (c), use $K_{\text{rot}} = \frac{1}{2}I\omega^2$.
Initially,

$$K_{\text{rot},i} = \tfrac{1}{2}I_i\omega_i^2 = \tfrac{1}{2}(3.0)(2.0)^2 = \tfrac{1}{2}(3.0)(4.0) = 6.0\,\text{J}.$$

Finally,

$$K_{\text{rot},f} = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}(1.2)(5.0)^2 = \tfrac{1}{2}(1.2)(25) = 15\,\text{J}.$$

So,

$$\boxed{K_{\text{rot},i} = 6.0\,\text{J}}, \qquad \boxed{K_{\text{rot},f} = 15\,\text{J}}.$$

The rotational kinetic energy increases even though angular momentum stays constant. This does not contradict conservation of angular momentum because the skater does internal work while pulling in her arms. That internal work increases K_{rot} while the external torque remains negligible, so $\vec{L}$ is still conserved.

1.6.4 Rolling Without Slipping

This subsection introduces pure rolling, where a rigid body both translates and rotates while the contact point does not slide relative to the surface.

Definition 1.6.4: Pure rolling and the rolling constraint

Consider a rigid body of mass M and radius R moving on a fixed surface. Let s_{cm} denote the distance traveled by its center of mass along the surface, let $v_{\text{cm}} = |\vec{v}_{\text{cm}}|$ denote the speed of the center of mass, and let $a_{t,\text{cm}}$ denote the component of $\vec{a}_{\text{cm}}$ tangent to the surface. Let θ denote the angular displacement of the body, let ω denote its angular speed, and let α denote its angular acceleration.

A body *rolls without slipping* if the point in contact with the surface is instantaneously at rest relative to the surface. For pure rolling,

$$s_{\text{cm}} = R\theta, \qquad v_{\text{cm}} = R\omega, \qquad a_{t,\text{cm}} = R\alpha.$$

These relations are called the *rolling constraint*. They apply only when there is no slipping at the contact.

Note:-

Let P denote the point on the rim that touches the ground at some instant. Its velocity relative to the ground is the vector sum of the center-of-mass velocity $\vec{v}_{\text{cm}}$ and the velocity of P relative to the center due to rotation. In pure rolling, these cancel exactly at the contact point, so P is instantaneously at rest even though the body is moving.

Also, the friction in rolling without slipping is *static* friction. Let $\vec{N}$ denote the normal force, let $N = |\vec{N}|$ denote its magnitude, and let f_s denote the magnitude of the static friction force. Static friction is not automatically equal to $\mu_s N$. Instead, its magnitude is whatever value is required to prevent slipping, provided that value satisfies $f_s \le \mu_s N$. On level ground at constant speed, the needed static friction can even be zero.

Example 1.6.1 (Illustrative example)

A wheel of radius $R = 0.30\,\text{m}$ rolls without slipping on level ground with angular speed $\omega = 8.0\,\text{rad/s}$. Find the speed of its center of mass and the speed of the top point of the wheel relative to the ground.

From the rolling constraint,

$$v_{\text{cm}} = R\omega = (0.30)(8.0) = 2.4\,\text{m/s}.$$

At the top of the wheel, the translational velocity and the rotational velocity point in the same direction, so the top-point speed is

$$v_{\text{top}} = v_{\text{cm}} + R\omega = 2v_{\text{cm}} = 4.8\,\text{m/s}.$$

Thus the wheel's center moves at $2.4\,\text{m/s}$, while the top point moves at $4.8\,\text{m/s}$ relative to the ground.

Proposition 1.6.1 Rolling kinematics, energy, and incline dynamics

Consider a rigid body of mass M, radius R, and moment of inertia I_{cm} about its center of mass. Let s_{cm}, v_{cm}, and $a_{t,\text{cm}}$ denote the center-of-mass distance, speed, and tangential acceleration, and let θ, ω, and α denote the corresponding angular variables. Let K_i and U_i denote the initial kinetic and potential energies, and let K_f and U_f denote the final kinetic and potential energies.

1. For pure rolling, $s_{\text{cm}} = R\theta$, $v_{\text{cm}} = R\omega$, $a_{t,\text{cm}} = R\alpha$.
2. The total kinetic energy is the sum of translational and rotational parts: $K = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$. Using $v_{\text{cm}} = R\omega$, this may also be written as $K = \frac{1}{2}\left(M + \frac{I_{\text{cm}}}{R^2}\right)v_{\text{cm}}^2$.
3. If no nonconservative force removes mechanical energy from the system, then for rolling without slipping, $K_i + U_i = K_f + U_f$. For a vertical drop of magnitude h from rest, $Mgh = \frac{1}{2}Mv_{\text{cm}}^2 + \frac{1}{2}I_{\text{cm}}\omega^2$.
4. For a body rolling without slipping down an incline of angle β, choose positive down the incline. Let a_{cm} denote the center-of-mass acceleration magnitude along the incline. If f_s denotes the magnitude of the static friction force, then $Mg\sin\beta - f_s = Ma_{\text{cm}}$, $f_sR = I_{\text{cm}}\alpha$, $a_{\text{cm}} = R\alpha$. Therefore, $a_{\text{cm}} = \frac{g\sin\beta}{1+I_{\text{cm}}/(MR^2)}$, $f_s = \frac{I_{\text{cm}}}{R^2}a_{\text{cm}}$. For an object accelerating down the incline, the static friction force on the object points up

the incline.

Question 34: Worked example

A solid cylinder of mass $M = 2.0\,\text{kg}$ and radius $R = 0.20\,\text{m}$ is released from rest on an incline that makes an angle $\beta = 30°$ with the horizontal. The cylinder rolls without slipping through a vertical drop $h = 0.75\,\text{m}$. Let $g = 9.8\,\text{m/s}^2$.
Find:

(a) the acceleration magnitude of the center of mass,

(b) the magnitude and direction of the static friction force,

(c) the speed of the center of mass after the drop, and

(d) the minimum coefficient of static friction required for rolling without slipping.

Solution: Let a_{cm} denote the acceleration magnitude of the center of mass down the incline, and let f_s denote the magnitude of the static friction force. For a solid cylinder,

$$I_{\text{cm}} = \tfrac{1}{2}MR^2.$$

Choose the positive axis down the incline. The forces parallel to the incline are the downslope component of the weight and the upslope static friction force, so Newton's second law for translation gives

$$Mg\sin\beta - f_s = Ma_{\text{cm}}.$$

The only torque about the center of mass is due to static friction, so

$$f_s R = I_{\text{cm}}\alpha.$$

Because the cylinder rolls without slipping,

$$a_{\text{cm}} = R\alpha.$$

Substitute $\alpha = a_{\text{cm}}/R$ and $I_{\text{cm}} = \frac{1}{2}MR^2$ into the torque equation:

$$f_s R = \left(\tfrac{1}{2}MR^2\right)\frac{a_{\text{cm}}}{R}.$$

Thus,

$$f_s = \tfrac{1}{2}Ma_{\text{cm}}.$$

Now substitute this into the translational equation:

$$Mg\sin\beta - \tfrac{1}{2}Ma_{\text{cm}} = Ma_{\text{cm}}.$$

So,

$$Mg\sin\beta = \tfrac{3}{2}Ma_{\text{cm}},$$

and therefore

$$a_{\text{cm}} = \frac{2}{3}g\sin\beta.$$

With $g = 9.8\,\text{m/s}^2$ and $\sin 30° = 0.50$,

$$a_{\text{cm}} = \frac{2}{3}(9.8)(0.50) = 3.27\,\text{m/s}^2.$$

This is the acceleration magnitude of the center of mass, directed down the incline.

For the static friction force,

$$f_s = \tfrac{1}{2}Ma_{\text{cm}} = \tfrac{1}{2}(2.0)(3.27) = 3.27\,\text{N}.$$

Its direction is up the incline, because it must provide the clockwise torque that increases the cylinder's rotation as the cylinder moves downward.

Now find the speed after the cylinder drops through height $h = 0.75\,\text{m}$. Since the cylinder rolls without slipping, static friction does no work at the contact point, so mechanical energy is conserved:

$$Mgh = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}I_{\text{cm}}\omega^2.$$

Using $I_{\text{cm}} = \tfrac{1}{2}MR^2$ and $v_{\text{cm}} = R\omega$,

$$Mgh = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v_{\text{cm}}}{R}\right)^2.$$

So,

$$Mgh = \tfrac{1}{2}Mv_{\text{cm}}^2 + \tfrac{1}{4}Mv_{\text{cm}}^2 = \tfrac{3}{4}Mv_{\text{cm}}^2.$$

Cancel M and solve for v_{cm}:

$$gh = \tfrac{3}{4}v_{\text{cm}}^2, \qquad v_{\text{cm}}^2 = \frac{4}{3}gh.$$

Substitute the numbers:

$$v_{\text{cm}}^2 = \frac{4}{3}(9.8)(0.75) = 9.8.$$

Hence,

$$v_{\text{cm}} = \sqrt{9.8} = 3.13\,\text{m/s}.$$

Finally, find the minimum coefficient of static friction. The normal force is

$$N = Mg\cos\beta = (2.0)(9.8)\cos 30^\circ = 17.0\,\text{N}$$

to three significant figures. For rolling without slipping, the needed static friction must satisfy

$$f_s \le \mu_s N.$$

Thus the minimum value is

$$\mu_{s,\min} = \frac{f_s}{N} = \frac{3.27}{17.0} = 0.192.$$

Therefore,

$$a_{\text{cm}} = 3.27\,\text{m/s}^2 \text{ down the incline}, \qquad f_s = 3.27\,\text{N up the incline},$$

$$v_{\text{cm}} = 3.13\,\text{m/s}, \qquad \mu_{s,\min} = 0.192.$$

The key ideas are the rolling constraint $v_{\text{cm}} = R\omega$, the split of kinetic energy into translational and rotational parts, and the fact that static friction adjusts to the amount needed for no slipping rather than automatically equaling $\mu_s N$.

1.6.5 Circular Orbits, Satellite Speed, and Orbital Energy

This subsection focuses on Newtonian circular orbits around a much more massive central body. The main AP results are the orbital-speed formula and the linked kinetic, potential, and total-energy relations for a satellite in a circular orbit.

Definition 1.6.5: Circular-orbit setup and gravitational potential-energy reference

Let a satellite of mass m move in a circular orbit around a spherically symmetric body of mass M. Let O denote the center of the central body. Let $\vec{r}$ denote the satellite's position vector from O, let $r = |\vec{r}|$ denote the constant orbital radius, let $\hat{r} = \vec{r}/r$ denote the outward radial unit vector, let $\vec{v}$ denote the satellite's velocity, and let $v = |\vec{v}|$ denote its speed. Let $\vec{L}_O = \vec{r} \times m\vec{v}$ denote the satellite's angular momentum about O. Let K denote kinetic energy, let U denote gravitational potential energy, and let $E = K + U$ denote total mechanical energy.
Choose the gravitational potential-energy reference so that $U = 0$ when the separation is infinite. Then for separation r,

$$U(r) = -\frac{GMm}{r}.$$

The gravitational force on the satellite is

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r}.$$

Theorem 1.6.4 Circular-orbit speed and energy relations

Let a satellite of mass m move in a circular orbit of radius r around a spherically symmetric body of mass M. Because $\vec{F}_g$ is parallel to $\vec{r}$,

$$\vec{\tau}_O = \vec{r} \times \vec{F}_g = \vec{0},$$

so the orbital angular momentum about the center is conserved.
For a circular orbit, gravity supplies the centripetal force:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}.$$

Therefore,

$$v = \sqrt{\frac{GM}{r}}.$$

The kinetic energy is then

$$K = \frac{1}{2}mv^2 = \frac{GMm}{2r}.$$

Using $U = -GMm/r$, the total mechanical energy is

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}.$$

Thus, for a circular Newtonian orbit,

$$v = \sqrt{\frac{GM}{r}}, \qquad K = \frac{GMm}{2r}, \qquad U = -\frac{GMm}{r}, \qquad E = -\frac{GMm}{2r}.$$

Example 1.6.2 (Illustrative example)

Two satellites of the same mass orbit the same planet in circular orbits. Satellite A has orbital radius r, and satellite B has orbital radius $4r$.
Since $v = \sqrt{GM/r}$,

$$v_B = \sqrt{\frac{GM}{4r}} = \frac{1}{2}\sqrt{\frac{GM}{r}} = \frac{1}{2}v_A.$$

Since $K = GMm/(2r)$,

$$K_B = \frac{GMm}{2(4r)} = \frac{1}{4}K_A.$$

Also,

$$U_B = -\frac{GMm}{4r} = \frac{1}{4}U_A, \qquad E_B = -\frac{GMm}{8r} = \frac{1}{4}E_A.$$

So a larger circular orbit has a lower speed and a less negative total energy.

Note:-

With the reference choice $U(\infty) = 0$, any bound gravitational orbit has negative total mechanical energy. For a circular orbit specifically, $E = -K = U/2 < 0$. Also, the formulas $v = \sqrt{GM/r}$, $K = GMm/(2r)$, and $E = -GMm/(2r)$ are for *circular* Newtonian orbits only. In a noncircular orbit, the speed is not constant, so these same expressions do not apply at every point of the motion.

Question 35: Worked example

An Earth satellite of mass $m = 850\,\text{kg}$ moves in a circular orbit at altitude $h = 4.00 \times 10^5$ m above Earth's surface. Let Earth's mass be $M_E = 5.97\times10^{24}$ kg, Earth's radius be $R_E = 6.37\times10^6$ m, and the gravitational constant be $G = 6.67 \times 10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$.
Find:

(a) why the satellite's angular momentum about Earth's center is conserved,

(b) the orbital speed v,

(c) the kinetic energy K,

(d) the gravitational potential energy U, and

(e) the total mechanical energy E.

Solution: Let r denote the orbital radius measured from Earth's center. First compute it from the given altitude:

$$r = R_E + h = 6.37 \times 10^6\,\text{m} + 4.00 \times 10^5\,\text{m} = 6.77 \times 10^6\,\text{m}.$$

For part (a), the only significant force on the satellite is Earth's gravitational force, which points along the line from Earth to the satellite. Therefore $\vec{F}_g$ is parallel to $\vec{r}$, so the torque about Earth's center is

$$\vec{\tau}_O = \vec{r} \times \vec{F}_g = \vec{0}.$$

Since

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_O,$$

it follows that

$$\frac{d\vec{L}_O}{dt} = \vec{0},$$

so the satellite's angular momentum about Earth's center is conserved.

For part (b), use the circular-orbit speed formula:

$$v = \sqrt{\frac{GM_E}{r}}.$$

Substitute the values:

$$v = \sqrt{\frac{(6.67 \times 10^{-11})\,(5.97 \times 10^{24})}{6.77 \times 10^6}}.$$

This gives

$$v \approx 7.67 \times 10^3\,\text{m/s}.$$

For part (c), the kinetic energy is

$$K = \frac{1}{2}mv^2.$$

Using $m = 850\,\text{kg}$ and the value of $v^2 = GM_E/r$,

$$K = \frac{1}{2}(850)\left(7.67 \times 10^3\right)^2 \text{ J} \approx 2.50 \times 10^{10} \text{ J}.$$

For part (d), the gravitational potential energy is

$$U = -\frac{GM_E m}{r}.$$

Substitute the numbers:

$$U = -\frac{\left(6.67 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)(850)}{6.77 \times 10^6} \text{ J} \approx -5.00 \times 10^{10} \text{ J}.$$

For part (e), the total mechanical energy is

$$E = K + U.$$

So,

$$E = (2.50 \times 10^{10}) + (-5.00 \times 10^{10}) \text{ J} \approx -2.50 \times 10^{10} \text{ J}.$$

This agrees with the circular-orbit relation

$$E = -\frac{GM_E m}{2r}.$$

Therefore,

$$v \approx 7.67 \times 10^3 \text{ m/s}, \qquad K \approx 2.50 \times 10^{10} \text{ J},$$
$$U \approx -5.00 \times 10^{10} \text{ J}, \qquad E \approx -2.50 \times 10^{10} \text{ J}.$$

The negative total energy shows that the satellite is in a bound orbit.

1.7 Oscillations

This unit develops the mechanics of oscillatory motion about stable equilibrium. We begin with the defining linear restoring-force model for simple harmonic motion (SHM), write its governing differential equation, and connect the motion to sinusoidal solutions, angular frequency, period, and frequency.

We then apply that framework to spring-mass oscillators, interpret SHM through energy, and finish with pendulum models. In AP Physics C: Mechanics, the pendulum results here are used in the small-angle regime, and the focus remains on undamped, unforced oscillations rather than broader circuit or driven-oscillation extensions.

1.7.1 Simple Harmonic Motion and Its Governing ODE

This subsection introduces simple harmonic motion as one-dimensional motion about a stable equilibrium under a linear restoring law.

Definition 1.7.1: Simple harmonic motion and the equilibrium coordinate

Let an object move along a line with fixed unit vector $\hat{u}$. Let

$$\vec{r}(t) = q(t)\hat{u}$$

denote the object's displacement from a stable equilibrium position, where $q(t)$ is the signed equilibrium coordinate. The motion is called *simple harmonic motion* (SHM) if the net restoring force is proportional to the displacement and points toward equilibrium:

$$\vec{F}_{\text{net}} = -kq\hat{u}$$

for some constant $k > 0$.
Equivalently, in scalar form along the chosen axis,

$$F_{\text{net}} = -kq.$$

The negative sign shows that when $q > 0$ the force is negative, and when $q < 0$ the force is positive, so the force always points back toward $q = 0$.

Theorem 1.7.1 SHM ODE, standard solution, and period relations

Let an object of mass m move in SHM with equilibrium coordinate $q(t)$ and restoring constant $k > 0$. Define

$$\omega = \sqrt{\frac{k}{m}}.$$

Then the governing differential equation is

$$q'' + \omega^2 q = 0.$$

Its standard solution may be written as

$$q(t) = C\cos(\omega t) + D\sin(\omega t),$$

where C and D are constants set by the initial conditions, or equivalently as

$$q(t) = A\cos(\omega t + \phi)$$

for amplitude $A \geq 0$ and phase constant ϕ.
The period T and frequency f are

$$T = \frac{2\pi}{\omega}, \qquad f = \frac{1}{T} = \frac{\omega}{2\pi}.$$

Short derivation from the linear restoring law: For SHM, the net force along the line of motion is

$$F_{\text{net}} = -kq.$$

Newton's second law gives

$$m\frac{d^2q}{dt^2} = -kq.$$

Divide by m:

$$\frac{d^2q}{dt^2} + \frac{k}{m}q = 0.$$

If we define

$$\omega^2 = \frac{k}{m},$$

then the equation becomes

$$q'' + \omega^2 q = 0.$$

The solutions of this constant-coefficient ODE are sinusoidal, so one may write

$$q(t) = C\cos(\omega t) + D\sin(\omega t).$$

Because sine and cosine repeat after an angle change of 2π, one full cycle takes time

$$T = \frac{2\pi}{\omega},$$

and therefore $f = 1/T = \omega/(2\pi)$. 😊

Example 1.7.1 (Illustrative example)

A particle's equilibrium coordinate satisfies

$$q'' + 25q = 0.$$

Identify ω, the period, and the frequency.
Compare this with the SHM form $q'' + \omega^2 q = 0$. Then

$$\omega^2 = 25 \qquad \Rightarrow \qquad \omega = 5.0\,\text{rad/s}.$$

So the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5.0}\,\text{s} = 1.26\,\text{s},$$

and the frequency is

$$f = \frac{1}{T} = \frac{5.0}{2\pi}\,\text{Hz} = 0.796\,\text{Hz}.$$

Thus this motion is SHM with angular frequency 5.0 rad/s, period 1.26 s, and frequency 0.796 Hz.

Question 36: Worked example

For one-dimensional SHM about equilibrium, let the equilibrium coordinate be

$$q(t) = (0.080\,\text{m})\cos\left(4\pi t - \tfrac{\pi}{3}\right)$$

with t in seconds.
Find:

(a) the amplitude,

(b) the angular frequency,

(c) the period and frequency,

(d) the displacement at $t = 0$, and

(e) the governing differential equation in the form $q'' + \omega^2 q = 0$.

Solution: From

$$q(t) = A\cos(\omega t + \phi),$$

we identify the amplitude as the coefficient of the cosine and the angular frequency as the coefficient of t inside the cosine.

For part (a),

$$A = 0.080\,\text{m}.$$

For part (b),

$$\omega = 4\pi\,\text{rad/s}.$$

For part (c), the period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4\pi} = 0.50\,\text{s}.$$

Therefore the frequency is

$$f = \frac{1}{T} = \frac{1}{0.50\,\text{s}} = 2.0\,\text{Hz}.$$

For part (d), substitute $t = 0$ into the position function:

$$q(0) = (0.080)\cos\left(-\frac{\pi}{3}\right)\ \text{m}.$$

Since $\cos(-\pi/3) = \cos(\pi/3) = 1/2$,

$$q(0) = (0.080)\left(\frac{1}{2}\right) = 0.040\,\text{m}.$$

For part (e), SHM always satisfies

$$q'' + \omega^2 q = 0.$$

Here $\omega = 4\pi\,\text{rad/s}$, so

$$\omega^2 = (4\pi)^2 = 16\pi^2.$$

Thus the governing ODE is

$$q'' + 16\pi^2 q = 0.$$

Therefore,

$$A = 0.080\,\text{m}, \qquad \omega = 4\pi\,\text{rad/s},$$

$$T = 0.50\,\text{s}, \qquad f = 2.0\,\text{Hz}, \qquad q(0) = 0.040\,\text{m},$$

and the motion is governed by

$$q'' + 16\pi^2 q = 0.$$

1.7.2 The Spring-Mass Oscillator

This subsection models a mass attached to an ideal spring, using displacement measured from equilibrium so that both horizontal motion and vertical motion about equilibrium take the same mathematical form.

Definition 1.7.2: Spring-mass oscillator and equilibrium coordinate

Let m denote the mass of an object and let $k_{\text{eff}} > 0$ denote the effective spring constant of the spring system attached to it. Let the motion occur along a line with positive direction given by the unit vector $\hat{u}$.

Define $x(t)$ to be the signed displacement of the mass from its equilibrium position, measured along that line. Then a *spring-mass oscillator* is a system for which the net restoring force is proportional to x and opposite its sign.

For a horizontal spring, x is measured directly from the equilibrium position on the track. For a vertical spring, let $y(t)$ denote the displacement measured from the spring's unstretched length and let y_{eq} denote the static equilibrium displacement. The equilibrium coordinate is then

$$x = y - y_{\text{eq}}.$$

Using x rather than y makes the vertical oscillator look exactly like the horizontal one.

Theorem 1.7.2 Governing equation and period of a spring-mass oscillator

Let m denote the mass, let k_{eff} denote the effective spring constant, and let $x(t)$ denote displacement from equilibrium. Then the motion satisfies

$$m\ddot{x} + k_{\text{eff}}x = 0.$$

Equivalently,

$$\ddot{x} + \omega^2 x = 0, \qquad \omega = \sqrt{\frac{k_{\text{eff}}}{m}}.$$

Thus the motion is simple harmonic. If T denotes the period and f denotes the frequency, then

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}}, \qquad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k_{\text{eff}}}{m}}.$$

One convenient position model is

$$x(t) = A\cos(\omega t + \phi),$$

where A is the amplitude and ϕ is a phase constant. Consequently,

$$v_{\max} = A\omega, \qquad a_{\max} = A\omega^2.$$

Why the equation is the same horizontally and vertically: For horizontal motion, let x denote displacement from equilibrium along $\hat{u}$. The spring force is

$$\vec{F}_s = -k_{\text{eff}}x\hat{u}.$$

By Newton's second law,

$$m\ddot{x} = -k_{\text{eff}}x,$$

so

$$m\ddot{x} + k_{\text{eff}}x = 0.$$

For vertical motion, choose downward as positive. Let y denote the downward displacement from the unstretched length, and let y_{eq} denote the equilibrium value. At equilibrium,

$$mg - k_{\text{eff}}y_{\text{eq}} = 0.$$

Now write the actual position as $y = y_{\text{eq}} + x$, where x is displacement from equilibrium. Then

$$m\ddot{y} = mg - k_{\text{eff}}y = mg - k_{\text{eff}}(y_{\text{eq}} + x).$$

Since $mg - k_{\text{eff}}y_{\text{eq}} = 0$ and $\ddot{y} = \ddot{x}$, this becomes

$$m\ddot{x} = -k_{\text{eff}}x.$$

So in either viewpoint,

$$m\ddot{x} + k_{\text{eff}}x = 0.$$

Comparing with $\ddot{x} + \omega^2 x = 0$ gives $\omega = \sqrt{k_{\text{eff}}/m}$, and then $T = 2\pi/\omega$ and $f = 1/T$. 😊

Example 1.7.2 (Illustrative example)

A block of mass $m = 0.40\,\text{kg}$ oscillates on a frictionless horizontal surface attached to a spring system with effective spring constant $k_{\text{eff}} = 100\,\text{N/m}$. Find the angular frequency, period, and frequency.
Use

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{100}{0.40}} = \sqrt{250} = 15.8\,\text{rad/s}.$$

Then

$$T = 2\pi\sqrt{\frac{m}{k_{\text{eff}}}} = 2\pi\sqrt{\frac{0.40}{100}} = 0.397\,\text{s},$$

and

$$f = \frac{1}{T} = \frac{1}{0.397} = 2.52\,\text{Hz}.$$

So the oscillator has angular frequency 15.8 rad/s, period 0.397 s, and frequency 2.52 Hz.

Question 37: Worked AP-style problem

A mass $m = 0.60\,\text{kg}$ hangs from a vertical spring with spring constant $k = 150\,\text{N/m}$. Choose downward as positive. Let y denote the mass's downward displacement from the spring's unstretched length, let y_{eq} denote the equilibrium displacement, and let

$$x = y - y_{\text{eq}}$$

denote displacement from equilibrium.
The mass is pulled downward 0.080 m from equilibrium and released from rest.
Find:

(a) the equilibrium displacement y_{eq},

(b) the differential equation for $x(t)$ together with the angular frequency and period, and

(c) the maximum speed of the mass.

Solution: At static equilibrium, the acceleration is zero, so the net force is zero:

$$mg - ky_{\text{eq}} = 0.$$

Therefore,

$$y_{\text{eq}} = \frac{mg}{k} = \frac{(0.60\,\text{kg})(9.8\,\text{m/s}^2)}{150\,\text{N/m}} = 0.0392\,\text{m}.$$

So the equilibrium position is 3.92×10^{-2} m below the unstretched length.

Now write the motion in terms of displacement from equilibrium:

$$x = y - y_{\text{eq}}.$$

The net force is

$$F_{\text{net}} = mg - ky = mg - k(y_{\text{eq}} + x).$$

Using $mg - ky_{\text{eq}} = 0$, this becomes

$$F_{\text{net}} = -kx.$$

Apply Newton's second law:

$$m\ddot{x} = -kx.$$

Hence the differential equation is

$$0.60\,\ddot{x} + 150x = 0,$$

or equivalently,

$$\ddot{x} + 250x = 0.$$

Therefore,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{150}{0.60}} = \sqrt{250} = 15.8\,\text{rad/s}.$$

The period is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{0.60}{150}} = 0.397\,\text{s}.$$

Because the mass is released from rest 0.080 m from equilibrium, the amplitude is

$$A = 0.080\,\text{m}.$$

For simple harmonic motion,

$$v_{\max} = A\omega.$$

So

$$v_{\max} = (0.080)(15.8) = 1.26\,\text{m/s}.$$

Thus,

$$y_{\text{eq}} = 0.0392\,\text{m}, \qquad \ddot{x} + 250x = 0, \qquad \omega = 15.8\,\text{rad/s}, \qquad T = 0.397\,\text{s},$$

and the maximum speed is

$$v_{\max} = 1.26\,\text{m/s}.$$

1.7.3 Energy in Simple Harmonic Motion

This subsection uses energy to describe how a frictionless spring-mass oscillator trades energy between motion and spring deformation.

Definition 1.7.3: Kinetic, potential, and total energy in spring SHM

Consider a block of mass m attached to an ideal spring of spring constant k and moving frictionlessly along the x-axis. Let x denote the signed displacement from equilibrium, let $\vec{v} = \dot{x}\hat{\imath}$ denote the block's velocity, let $v = |\vec{v}| = |\dot{x}|$ denote its speed, and let $A > 0$ denote the amplitude of the motion.
The kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\dot{x}^2.$$

The spring potential energy is

$$U_s = \tfrac{1}{2}kx^2.$$

The total mechanical energy is

$$E = K + U_s = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2.$$

Theorem 1.7.3 Conserved-energy relation for SHM

For the frictionless spring-mass oscillator above, the total mechanical energy is constant:

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2.$$

Therefore, at any displacement x,

$$\dot{x}^2 = \frac{k}{m}\left(A^2 - x^2\right), \qquad v = \sqrt{\frac{k}{m}\,(A^2 - x^2)}.$$

In particular, the maximum speed occurs at equilibrium $x = 0$:

$$v_{\text{max}} = \sqrt{\frac{k}{m}}\,A.$$

Note:-

At the turning points $x = \pm A$, the block reverses direction, so $v = 0$, $K = 0$, and all the mechanical energy is spring potential energy:

$$U_s = E = \tfrac{1}{2}kA^2.$$

At equilibrium $x = 0$, the spring is neither stretched nor compressed, so $U_s = 0$ and all the energy is kinetic:

$$K = E = \tfrac{1}{2}kA^2.$$

Thus SHM continually swaps energy between kinetic and potential forms. If $x(t) = A\cos(\omega t + \phi)$, then $U_s \propto \cos^2(\omega t + \phi)$ and $K \propto \sin^2(\omega t + \phi)$, so the two energy curves are out of phase and each repeats twice during one full oscillation.

Short derivation from conservation of mechanical energy: For a frictionless spring-mass system, the only horizontal interaction is the spring force

$$\vec{F}_s = -kx\hat{\imath},$$

which is conservative. Therefore the mechanical energy $E = K + U_s$ is constant. Using the spring potential-energy function,

$$U_s = \tfrac{1}{2}kx^2,$$

the total energy at any instant is

$$E = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2.$$

At a turning point, $x = \pm A$ and $\dot{x} = 0$, so

$$E = \tfrac{1}{2}kA^2.$$

Equating the two expressions for E gives

$$\tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \tfrac{1}{2}kA^2.$$

Solving for $\dot{x}^2$ yields

$$\dot{x}^2 = \frac{k}{m}\left(A^2 - x^2\right),$$

and taking the positive square root gives the speed formula for the magnitude $v = |\dot{x}|$. 😊

Question 38: Worked example

A block of mass $m = 0.40\,\mathrm{kg}$ is attached to an ideal horizontal spring of spring constant $k = 160\,\mathrm{N/m}$ and oscillates frictionlessly with amplitude $A = 0.10\,\mathrm{m}$. At one instant, the block is at displacement $x = +0.060\,\mathrm{m}$ from equilibrium.
Find:

(a) the total mechanical energy of the oscillator,

(b) the spring potential energy and kinetic energy at $x = +0.060\,\mathrm{m}$,

(c) the speed of the block at that displacement, and

(d) the maximum speed and where it occurs.

Solution: Let E denote the total mechanical energy. Because the motion is frictionless,

$$E = \tfrac{1}{2}kA^2.$$

Substitute $k = 160\,\mathrm{N/m}$ and $A = 0.10\,\mathrm{m}$:

$$E = \tfrac{1}{2}(160)(0.10)^2 = 80(0.010) = 0.80\,\mathrm{J}.$$

So the oscillator's total mechanical energy is

$$0.80\,\mathrm{J}.$$

At $x = +0.060\,\mathrm{m}$, the spring potential energy is

$$U_s = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(160)(0.060)^2.$$

Since $(0.060)^2 = 0.0036$,

$$U_s = 80(0.0036) = 0.288\,\mathrm{J}.$$

Then the kinetic energy is

$$K = E - U_s = 0.80 - 0.288 = 0.512\,\mathrm{J}.$$

Now use kinetic energy to find the speed:

$$K = \tfrac{1}{2}mv^2.$$

So

$$0.512 = \tfrac{1}{2}(0.40)v^2 = 0.20v^2.$$

Thus,

$$v^2 = \frac{0.512}{0.20} = 2.56, \qquad v = 1.60\,\mathrm{m/s}.$$

For the maximum speed, use the equilibrium position $x = 0$, where all the energy is kinetic:

$$\tfrac{1}{2}mv_{\max}^2 = E = 0.80\,\mathrm{J}.$$

Therefore,

$$0.20v_{\max}^2 = 0.80, \qquad v_{\max}^2 = 4.0, \qquad v_{\max} = 2.0\,\mathrm{m/s}.$$

Equivalently,

$$v_{\max} = \sqrt{\frac{k}{m}}A = \sqrt{\frac{160}{0.40}}(0.10) = 20(0.10) = 2.0\,\mathrm{m/s}.$$

Therefore,

$$E = 0.80\,\mathrm{J}, \qquad U_s = 0.288\,\mathrm{J}, \qquad K = 0.512\,\mathrm{J},$$

$$v = 1.60\,\mathrm{m/s}, \qquad v_{\max} = 2.0\,\mathrm{m/s} \text{ at } x = 0.$$

1.7.4 The Simple Pendulum

This subsection models a bob of mass on a light string, using angular displacement from the vertical as the natural coordinate.

Definition 1.7.4: Simple pendulum and angular coordinate

Let m denote the bob's mass, let $\ell > 0$ denote the string length, let g denote the magnitude of the gravitational field, and let $\theta(t)$ denote the angular displacement from the downward vertical, measured in radians and taken positive in the counterclockwise direction.
A *simple pendulum* is an idealized system consisting of a point mass m attached to a massless string of fixed length ℓ, swinging without friction in a uniform gravitational field. The bob moves along a circular arc of radius ℓ. If $s(t)$ denotes the arc displacement from equilibrium, then

$$s = \ell\theta.$$

The equilibrium position is $\theta = 0$.

Theorem 1.7.4 Exact pendulum equation and small-angle SHM model

For the simple pendulum above, the exact rotational equation of motion is

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

This equation is nonlinear, so the motion is not exactly simple harmonic for arbitrary amplitude.
If the oscillation remains at small angles so that $|\theta| \ll 1$ radian and $\sin\theta \approx \theta$, then the motion is approximated by

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0.$$

Thus the pendulum behaves approximately like SHM with angular frequency

$$\omega = \sqrt{\frac{g}{\ell}},$$

small-angle period

$$T = 2\pi\sqrt{\frac{\ell}{g}},$$

and small-angle frequency

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}.$$

Short derivation from torque and linearization: About the pivot, the gravitational torque on the bob is restoring, so

$$\tau = -mg\ell\sin\theta.$$

The bob acts like a point mass at distance ℓ, so its moment of inertia about the pivot is

$$I = m\ell^2.$$

Using rotational Newton's second law, $\sum \tau = I\ddot{\theta}$, gives

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta.$$

Divide by $m\ell^2$:

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

For small oscillations with $|\theta| \ll 1$ radian, use the small-angle approximation $\sin\theta \approx \theta$. Then

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0,$$

which is the standard SHM equation with $\omega^2 = g/\ell$. 😊

Example 1.7.3 (Illustrative example)

A pendulum oscillates through small angles. Its length is changed from $\ell_1 = 0.50\,\text{m}$ to $\ell_2 = 2.00\,\text{m}$. How do the period and frequency change?
For small-angle motion,

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

Therefore $T \propto \sqrt{\ell}$. Since

$$\frac{\ell_2}{\ell_1} = \frac{2.00}{0.50} = 4,$$

the new period is multiplied by

$$\sqrt{4} = 2.$$

So the period doubles. Because $f = 1/T$, the frequency is cut in half.

Question 39: Worked AP-style problem

A simple pendulum has length $\ell = 0.90\,\text{m}$. It is pulled aside to a maximum angle $\theta_{\text{max}} = 0.10\,\text{rad}$ and released from rest. Take $g = 9.8\,\text{m/s}^2$.
Assume the small-angle model is valid.
Find:

(a) the exact equation of motion and the small-angle approximate equation,

(b) the angular frequency, period, and frequency,

(c) the time required to move from maximum displacement to equilibrium, and

(d) the maximum linear speed of the bob.

Solution: Let $\theta(t)$ denote the angular displacement from the downward vertical.

For part (a), the exact pendulum equation is

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0.$$

Substitute $g = 9.8\,\text{m/s}^2$ and $\ell = 0.90\,\text{m}$:

$$\ddot{\theta} + \frac{9.8}{0.90}\sin\theta = 0.$$

Thus,

$$\ddot{\theta} + 10.9\sin\theta = 0$$

to three significant figures.

Under the small-angle approximation $\sin\theta \approx \theta$, the motion is modeled by

$$\ddot{\theta} + \frac{9.8}{0.90}\theta = 0,$$

or

$$\ddot{\theta} + 10.9\theta = 0.$$

For part (b), compare the small-angle equation with

$$\ddot{\theta} + \omega^2\theta = 0.$$

So

$$\omega = \sqrt{\frac{g}{\ell}} = \sqrt{\frac{9.8}{0.90}} = 3.30\,\text{rad/s}.$$

Then the period is

$$T = 2\pi\sqrt{\frac{\ell}{g}} = \frac{2\pi}{\omega} = \frac{2\pi}{3.30} = 1.90\,\text{s}.$$

The frequency is

$$f = \frac{1}{T} = \frac{1}{1.90} = 0.526\,\text{Hz}.$$

For part (c), a pendulum in SHM takes one-quarter of a cycle to move from an endpoint to equilibrium. Therefore,

$$t = \frac{T}{4} = \frac{1.90}{4} = 0.475\,\text{s}.$$

For part (d), the maximum angular speed in SHM is

$$\dot{\theta}_{\text{max}} = \omega\theta_{\text{max}}.$$

So

$$\dot{\theta}_{\text{max}} = (3.30)(0.10) = 0.330\,\text{rad/s}.$$

The bob's linear speed is related by $v = \ell\dot{\theta}$, so the maximum linear speed is

$$v_{\text{max}} = \ell\dot{\theta}_{\text{max}} = (0.90)(0.330) = 0.297\,\text{m/s}.$$

Therefore,

$$\ddot{\theta} + 10.9\sin\theta = 0, \qquad \ddot{\theta} + 10.9\theta = 0,$$

$$\omega = 3.30\,\text{rad/s}, \qquad T = 1.90\,\text{s}, \qquad f = 0.526\,\text{Hz},$$

and

$$t = 0.475\,\text{s}, \qquad v_{\text{max}} = 0.297\,\text{m/s}.$$

1.7.5 Physical Pendulum and Small-Angle Linearization

This subsection models the small oscillations of a rigid body that swings about a fixed pivot under gravity.

Definition 1.7.5: Physical pendulum, pivot-to-CM distance, and angular coordinate

Let a rigid body of mass m swing in a vertical plane about a fixed pivot point O. Let C denote the center of mass of the body, let

$$d = OC$$

denote the distance from the pivot to the center of mass, and let $\theta(t)$ denote the angular displacement from the stable vertical equilibrium position.

Such a system is called a *physical pendulum*. Unlike a simple pendulum, the body's mass is distributed throughout the rigid object, so its rotational inertia must be included in the dynamics.

Theorem 1.7.5 Exact torque equation and small-angle SHM model

Let m denote the mass of the rigid body, let d denote the distance from the pivot to the center of mass, let I denote the moment of inertia of the body about the pivot, let g denote the magnitude of the gravitational

field, and let $\theta(t)$ denote the angular displacement from stable equilibrium.
Then the exact rotational equation of motion is

$$I\ddot{\theta} = -mgd \sin\theta,$$

or equivalently,

$$I\ddot{\theta} + mgd \sin\theta = 0.$$

For small angular displacements, use the linearization $\sin\theta \approx \theta$ to obtain

$$I\ddot{\theta} + mgd\,\theta = 0.$$

Therefore the motion is approximately simple harmonic with angular frequency

$$\omega = \sqrt{\frac{mgd}{I}}$$

and period

$$T = 2\pi\sqrt{\frac{I}{mgd}}.$$

A simple pendulum is the special case in which all the mass is concentrated a distance L from the pivot, so $I = mL^2$ and $d = L$.

Short derivation from torque and linearization: The weight $m\vec{g}$ acts at the center of mass. When the body is displaced by angle θ, the gravitational torque about the pivot is restoring, so

$$\tau = -mgd \sin\theta.$$

For rotation about a fixed axis, Newton's second law for rotation gives

$$\sum \tau = I\ddot{\theta}.$$

Hence,

$$I\ddot{\theta} = -mgd \sin\theta,$$

which is the exact equation.

If the oscillations are small, then $\sin\theta \approx \theta$, so the equation becomes

$$I\ddot{\theta} + mgd\,\theta = 0.$$

Divide by I to get

$$\ddot{\theta} + \frac{mgd}{I}\theta = 0.$$

Comparing with the SHM form $q'' + \omega^2 q = 0$ shows that

$$\omega^2 = \frac{mgd}{I}, \qquad \omega = \sqrt{\frac{mgd}{I}}.$$

Therefore,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}.$$

😄

Example 1.7.4 (Illustrative example)

Show that the simple pendulum is a special case of the physical pendulum formula.

For a point mass m at distance L from the pivot,

$$I = mL^2, \qquad d = L.$$

Substitute into the physical-pendulum period formula:

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{mL^2}{mgL}} = 2\pi\sqrt{\frac{L}{g}}.$$

This is exactly the small-angle period of a simple pendulum.

Question 40: Worked AP-style problem

A uniform rod of mass $m = 1.50\,\text{kg}$ and length $L = 0.90\,\text{m}$ is pivoted about one end and allowed to swing in a vertical plane. Let $\theta(t)$ denote the angular displacement from the stable vertical equilibrium position. Assume the oscillations are small.
Find:

(a) the pivot-to-center-of-mass distance d and the rod's moment of inertia I about the pivot,

(b) the small-angle differential equation for $\theta(t)$, and

(c) the period of oscillation.

Solution: For a uniform rod pivoted about one end, the center of mass is at the midpoint, so

$$d = \frac{L}{2} = \frac{0.90\,\text{m}}{2} = 0.45\,\text{m}.$$

The moment of inertia of a uniform rod about one end is

$$I = \frac{1}{3}mL^2.$$

Substitute the given values:

$$I = \frac{1}{3}(1.50)(0.90)^2\,\text{kg}\cdot\text{m}^2.$$

Since $(0.90)^2 = 0.81$,

$$I = \frac{1}{3}(1.50)(0.81) = 0.405\,\text{kg}\cdot\text{m}^2.$$

For small oscillations, a physical pendulum satisfies

$$I\ddot{\theta} + mgd\,\theta = 0.$$

Now compute mgd:

$$mgd = (1.50)(9.8)(0.45) = 6.615.$$

So the differential equation is

$$0.405\,\ddot{\theta} + 6.615\,\theta = 0.$$

Divide by 0.405:

$$\ddot{\theta} + 16.3\,\theta = 0.$$

Thus,

$$\omega = \sqrt{16.3} = 4.04\,\text{rad/s}.$$

The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{4.04} = 1.56\,\text{s}.$$

Equivalently, using the period formula directly,

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{0.405}{6.615}} = 1.56\,\mathrm{s}.$$

Therefore,

$$d = 0.45\,\mathrm{m}, \qquad I = 0.405\,\mathrm{kg}\cdot\mathrm{m}^2,$$

and the small-angle motion is governed by

$$\ddot{\theta} + 16.3\,\theta = 0, \qquad T = 1.56\,\mathrm{s}.$$

Part II

Electricity & Magnetism

Chapter 2

Electricity & Magnetism

2.1 Electrostatics: Charge, Field, Flux

This unit develops the core ideas of electrostatics by starting with charge as a conserved quantity and then building the interaction model for stationary charges. In AP Physics C: Electricity and Magnetism, the emphasis is on careful charge bookkeeping, Coulomb's law, superposition, and the interpretation of the electric field vector $\vec{E}$ as a map of the force per unit charge that space assigns to a test charge.

From there, the unit extends point-charge reasoning to standard continuous charge distributions, then introduces electric flux as a measure of how much electric field passes through a surface. That flux viewpoint leads naturally to Gauss's law, especially for highly symmetric charge distributions and Gaussian surfaces that make the field easy to determine.

2.1.1 Charge Conservation and Charging Processes

This subsection treats electric charge as a conserved quantity and introduces the AP-level charging processes of friction, contact, and induction as charge-bookkeeping ideas.

Definition 2.1.1: Charge conservation and basic charging processes

Let q denote the net charge of an object or subsystem, measured in coulombs, and let

$$Q_{\text{tot}} = \sum_i q_i$$

denote the total charge of a chosen system.
Charge conservation states that for an isolated system,

$$Q_{\text{tot},f} = Q_{\text{tot},i}.$$

In ordinary AP charging problems, objects become charged because charge is redistributed:

1. *Charging by friction*: rubbing two materials can transfer electrons from one object to the other.
2. *Charging by contact*: when objects touch, charge can transfer between them before they separate.
3. *Charging by induction*: a nearby charged object causes charge separation in another object; with grounding, charge can enter or leave so the object may be left with a net charge after the process.

In each case, the total charge of the full isolated system remains constant.

Note:-

The sign of charge is bookkeeping. A positive net charge means an electron deficit, while a negative net charge means an electron excess. In ordinary friction, contact, and induction processes, charge is transferred from one place to another; it is not created from nothing. If one part of an isolated system gains $+\Delta q$, the rest of the system must change by $-\Delta q$. When grounding is involved, include the Earth in the system because it can supply or receive the transferred charge.

Proposition 2.1.1 Practical bookkeeping relations for isolated systems and simple sharing

Consider a chosen system of objects with initial charges $q_{1,i}, q_{2,i}, \ldots$ and final charges $q_{1,f}, q_{2,f}, \ldots$.

1. For any isolated system,

$$\sum_k q_{k,f} = \sum_k q_{k,i} \qquad \text{or} \qquad \Delta Q_{\text{tot}} = 0.$$

2. For charge transfer between two objects A and B within an isolated system,

$$\Delta q_A + \Delta q_B = 0,$$

so

$$q_{A,f} - q_{A,i} = -(q_{B,f} - q_{B,i}).$$

3. If two identical small conducting spheres with initial charges $q_{A,i}$ and $q_{B,i}$ are touched together and then separated, symmetry gives equal final charges:

$$q_{A,f} = q_{B,f} = \frac{q_{A,i} + q_{B,i}}{2}.$$

4. If a neutral object is charged by induction while connected to ground, then charge conservation must be applied to the combined object-Earth system. If the object starts neutral and ends with charge q_f, then the Earth changes by

$$\Delta q_{\text{Earth}} = -q_f.$$

Question 1: Worked AP-style problem

Three identical small conducting spheres A, B, and C are far apart initially. Let their initial charges be

$$q_{A,i} = +8.0\,\text{nC}, \qquad q_{B,i} = -2.0\,\text{nC}, \qquad q_{C,i} = 0.$$

First, sphere A is touched to sphere B and then separated. Next, sphere B is touched to sphere C and then separated.
Find:

(a) the charge on each sphere after the first contact,

(b) the final charge on each sphere after the second contact, and

(c) the number of electrons transferred during the second contact.

Take the elementary charge magnitude to be $e = 1.60 \times 10^{-19}\,\text{C}$.

Solution: Because the spheres are identical, whenever two of them touch and then separate, they share the total charge equally.

For the first contact, apply charge conservation to spheres A and B:

$$q_{A,i} + q_{B,i} = (+8.0\,\text{nC}) + (-2.0\,\text{nC}) = +6.0\,\text{nC}.$$

Since the spheres are identical, after they separate each has half of this total charge:

$$q_A = q_B = \frac{+6.0\,\text{nC}}{2} = +3.0\,\text{nC}.$$

So after the first contact,

$$q_A = +3.0\,\text{nC}, \qquad q_B = +3.0\,\text{nC}, \qquad q_C = 0.$$

Now sphere B touches sphere C. Just before this second contact, their total charge is

$$q_B + q_C = (+3.0\,\text{nC}) + 0 = +3.0\,\text{nC}.$$

Again they are identical, so after separation they share this total equally:

$$q_{B,f} = q_{C,f} = \frac{+3.0\,\text{nC}}{2} = +1.5\,\text{nC}.$$

Sphere A is not involved in the second contact, so its charge stays

$$q_{A,f} = +3.0\,\text{nC}.$$

Therefore the final charges are

$$q_{A,f} = +3.0\,\text{nC}, \qquad q_{B,f} = +1.5\,\text{nC}, \qquad q_{C,f} = +1.5\,\text{nC}.$$

To find the number of electrons transferred during the second contact, compute the magnitude of the charge change on either sphere. Sphere B changes from $+3.0\,\text{nC}$ to $+1.5\,\text{nC}$, so

$$\Delta q_B = (+1.5 - 3.0)\,\text{nC} = -1.5\,\text{nC}.$$

The negative change means sphere B gained electrons. Equivalently, sphere C changed from 0 to $+1.5\,\text{nC}$, so sphere C lost electrons. The magnitude of transferred charge is

$$|\Delta q| = 1.5 \times 10^{-9}\,\text{C}.$$

Let N denote the number of electrons transferred. Then

$$N = \frac{|\Delta q|}{e} = \frac{1.5 \times 10^{-9}}{1.60 \times 10^{-19}} = 9.375 \times 10^{9}.$$

To two significant figures,

$$N \approx 9.4 \times 10^{9} \text{ electrons.}$$

These electrons moved from sphere C to sphere B. This direction makes sense because sphere B became less positive while sphere C became more positive.

2.1.2 Coulomb's Law and Superposition

This subsection gives the electrostatic force between point charges and shows how forces from multiple source charges combine by vector addition.

Definition 2.1.2: Point charges, separation vector, and superposition

Let point charges $q_1, q_2, \ldots, q_N$ be located at position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ in an inertial frame. For two distinct charges q_i and q_j, define the separation vector from q_i to q_j by

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i,$$

let $r_{ij} = |\vec{r}_{ij}|$ be the separation distance, and let $\hat{r}_{ij} = \vec{r}_{ij}/r_{ij}$ be the corresponding unit vector.
A *point charge* is an idealized charged object whose size is negligible compared with the distances of interest. The *superposition principle* states that when several source charges act on a chosen charge, the net electric force is the vector sum of the individual forces exerted by each source charge separately.

Theorem 2.1.1 Coulomb's law in vector form and force superposition

Let $k = \dfrac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$. For two point charges q_i and q_j with separation vector $\vec{r}_{ij} \neq \vec{0}$, the electric force on q_j due to q_i is

$$\vec{F}_{i\to j} = k\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij} = k\frac{q_i q_j}{r_{ij}^3}\vec{r}_{ij}.$$

Its magnitude is

$$F_{i\to j} = k\frac{|q_i q_j|}{r_{ij}^2}.$$

Thus the force is proportional to the product of the charges, inversely proportional to the square of the separation distance, and directed along the line joining the charges. If N source charges act on q_j, then

$$\vec{F}_{\text{net},j} = \sum_{i\neq j}\vec{F}_{i\to j} = \sum_{i\neq j} k\frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|^3}(\vec{r}_j - \vec{r}_i).$$

Why the vector law has this form: For two point charges separated by distance r_{ij}, Coulomb's law gives the force magnitude

$$F_{i\to j} = k\frac{|q_i q_j|}{r_{ij}^2}.$$

The force must lie along the line connecting the charges, so its direction is either $+\hat{r}_{ij}$ or $-\hat{r}_{ij}$. If $q_i q_j > 0$, the charges have the same sign and repel, so the force on q_j points away from q_i, which is $+\hat{r}_{ij}$. If $q_i q_j < 0$, the charges have opposite signs and attract, so the force on q_j points toward q_i, which is $-\hat{r}_{ij}$. Writing the force as

$$\vec{F}_{i\to j} = k\frac{q_i q_j}{r_{ij}^2}\hat{r}_{ij}$$

captures both cases automatically through the sign of $q_i q_j$. Because force is a vector, multiple electric forces combine by ordinary vector addition, giving the superposition formula. 😊

Corollary 2.1.1 Collinear charges on the x-axis

Let fixed source charges $q_1, \ldots, q_N$ lie on the x-axis at coordinates $x_1, \ldots, x_N$. Let a test charge q be at coordinate x, with $x \neq x_i$ for all i. Then the net force on the test charge is purely along the x-axis:

$$\vec{F}_{\text{net}} = kq\left(\sum_{i=1}^{N} q_i \frac{x - x_i}{|x - x_i|^3}\right)\hat{\imath}.$$

So in one dimension, Coulomb superposition reduces to an algebraic sum of signed x-components. In particular, if two equal source charges $+Q$ are placed at $x = -a$ and $x = +a$, then at the midpoint $x = 0$ their forces cancel, so $\vec{F}_{\text{net}} = \vec{0}$ on any test charge placed there.

Question 2: Worked AP-style problem

In an xy-plane, let $q_1 = +4.0\,\mu\mathrm{C}$ be fixed at $\vec{r}_1 = \vec{0}$, let $q_2 = -2.0\,\mu\mathrm{C}$ be fixed at $\vec{r}_2 = (0.30\,\mathrm{m})\hat{\imath}$, and let $q_3 = +1.5\,\mu\mathrm{C}$ be located at $\vec{r}_3 = (0.40\,\mathrm{m})\hat{\jmath}$. Let $k = 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}$.
Find:

(a) the force $\vec{F}_{1\to 3}$ on q_3 due to q_1,

(b) the force $\vec{F}_{2\to 3}$ on q_3 due to q_2, and

(c) the net force $\vec{F}_{\text{net},3}$ on q_3, including its magnitude and direction measured counterclockwise from the positive x-axis.

Solution: First find the separation vectors to the charge q_3.

For the force due to q_1,

$$\vec{r}_{13} = \vec{r}_3 - \vec{r}_1 = (0.40\,\text{m})\hat{\jmath}, \qquad r_{13} = 0.40\,\text{m}.$$

For the force due to q_2,

$$\vec{r}_{23} = \vec{r}_3 - \vec{r}_2 = (-0.30\hat{\imath} + 0.40\hat{\jmath})\,\text{m}, \qquad r_{23} = \sqrt{(0.30)^2 + (0.40)^2}\,\text{m} = 0.50\,\text{m}.$$

For part (a), use Coulomb's law. Since q_1 and q_3 are both positive, the force on q_3 is repulsive and points away from q_1, which is in the $+\hat{\jmath}$ direction:

$$\vec{F}_{1\to3} = k\frac{q_1 q_3}{r_{13}^3}\vec{r}_{13}.$$

Its magnitude is

$$F_{1\to3} = k\frac{|q_1 q_3|}{r_{13}^2} = (8.99\times10^9)\frac{(4.0\times10^{-6})(1.5\times10^{-6})}{(0.40)^2}\,\text{N}.$$

So

$$F_{1\to3} = 0.337\,\text{N},$$

and therefore

$$\vec{F}_{1\to3} = (0.337\,\text{N})\hat{\jmath}.$$

For part (b), q_2 is negative and q_3 is positive, so the force on q_3 is attractive and points from q_3 toward q_2. Let $\hat{u}_{3\to2}$ denote the unit vector from q_3 to q_2. Then

$$\hat{u}_{3\to2} = \frac{(0.30\hat{\imath} - 0.40\hat{\jmath})\,\text{m}}{0.50\,\text{m}} = 0.60\hat{\imath} - 0.80\hat{\jmath}.$$

The magnitude is

$$F_{2\to3} = k\frac{|q_2 q_3|}{r_{23}^2} = (8.99\times10^9)\frac{(2.0\times10^{-6})(1.5\times10^{-6})}{(0.50)^2}\,\text{N}.$$

Thus

$$F_{2\to3} = 0.108\,\text{N}.$$

So the vector force is

$$\vec{F}_{2\to3} = F_{2\to3}\hat{u}_{3\to2} = (0.108)(0.60\hat{\imath} - 0.80\hat{\jmath})\,\text{N}.$$

Therefore,

$$\vec{F}_{2\to3} = (0.0647\hat{\imath} - 0.0863\hat{\jmath})\,\text{N}.$$

For part (c), add the forces componentwise:

$$\vec{F}_{\text{net},3} = \vec{F}_{1\to3} + \vec{F}_{2\to3}.$$

So

$$\vec{F}_{\text{net},3} = (0.0647\hat{\imath} + 0.2507\hat{\jmath})\,\text{N}.$$

Its magnitude is

$$|\vec{F}_{\text{net},3}| = \sqrt{(0.0647)^2 + (0.2507)^2}\,\text{N} = 0.259\,\text{N}.$$

Let θ denote the direction measured counterclockwise from the positive x-axis. Then

$$\tan\theta = \frac{0.2507}{0.0647} = 3.88,$$

so

$$\theta = \tan^{-1}(3.88) = 75.5^\circ.$$

Therefore,

$$\vec{F}_{1\to3} = (0.337\,\text{N})\hat{\jmath}, \qquad \vec{F}_{2\to3} = (0.0647\hat{\imath} - 0.0863\hat{\jmath})\,\text{N},$$

and the net force on q_3 is

$$\vec{F}_{\text{net},3} = (0.0647\hat{\imath} + 0.2507\hat{\jmath})\,\text{N},$$

with magnitude

$$|\vec{F}_{\text{net},3}| = 0.259\,\text{N}$$

at angle

$$\theta = 75.5^\circ$$

above the positive x-axis.

2.1.3 Electric Field as Force per Unit Charge

This subsection defines the electric field from source charges and shows how it determines the force on any charge placed at a point.

Definition 2.1.3: Electric field, source charges, and test charges

Let source charges create an electrostatic interaction in space. Let a field point have position vector $\vec{r}$, and let a small positive test charge q_0 be placed at that point. If the electric force on the test charge is $\vec{F}$, then the *electric field* at that point is defined by

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q_0}.$$

The source charges are the charges that produce the field. The test charge is a charge used only to probe the field at a location. Because the factor of q_0 divides out, the field depends on the source-charge configuration and the location $\vec{r}$, not on the particular test charge used to measure it. By convention, the direction of $\vec{E}$ is the direction of the force on a positive test charge. The SI units of electric field are N/C.

Theorem 2.1.2 Point-charge field law and force relation

Let a point source charge Q be fixed at position vector $\vec{r}_Q$. Let the field point have position vector $\vec{r}$, and define

$$\vec{R} = \vec{r} - \vec{r}_Q, \qquad R = |\vec{R}|, \qquad \hat{R} = \frac{\vec{R}}{R},$$

with $\vec{r} \neq \vec{r}_Q$. Then the electric field due to the point charge Q is

$$\vec{E}(\vec{r}) = k\frac{Q}{R^2}\hat{R} = k\frac{Q}{R^3}\vec{R},$$

where

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \, \mathrm{N\,m^2/C^2}.$$

If any charge q is placed at that field point, the electric force on that charge is

$$\vec{F} = q\vec{E}.$$

Derivation from Coulomb's law: Let a positive test charge q_0 be placed at the field point. Coulomb's law gives the force on the test charge due to the source charge Q as

$$\vec{F} = k\frac{Qq_0}{R^3}\vec{R}.$$

Now divide by q_0 and use the definition of electric field:

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q_0} = k\frac{Q}{R^3}\vec{R} = k\frac{Q}{R^2}\hat{R}.$$

This shows that the field is determined entirely by the source charge and geometry. Once $\vec{E}$ is known at a point, the force on any charge q placed there is obtained by multiplying by q, so $\vec{F} = q\vec{E}$. ☺

Corollary 2.1.2 Direction and superposition of electric fields

Let point source charges $Q_1, Q_2, \ldots, Q_N$ be fixed at position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. Let the field point have position vector $\vec{r}$, with $\vec{r} \neq \vec{r}_i$ for all i. Then the net electric field is the vector sum of the individual fields:

$$\vec{E}_{\text{net}}(\vec{r}) = \sum_{i=1}^{N} k\frac{Q_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i).$$

For a single source charge, the field points radially away from the charge if $Q_i > 0$ and radially toward the charge if $Q_i < 0$.

Question 3: Worked AP-style problem

Two point source charges lie on the x-axis. Let

$$q_1 = +3.0\,\mu\text{C} \qquad \text{at} \qquad \vec{r}_1 = (-0.20\,\text{m})\hat{\imath},$$

and let

$$q_2 = -2.0\,\mu\text{C} \qquad \text{at} \qquad \vec{r}_2 = (+0.30\,\text{m})\hat{\imath}.$$

Let point P be at the origin, so $\vec{r}_P = \vec{0}$. Take $k = 8.99 \times 10^9\,\text{N}\,\text{m}^2/\text{C}^2$.
Find:

(a) the electric field $\vec{E}_1$ at P due to q_1,

(b) the electric field $\vec{E}_2$ at P due to q_2,

(c) the net electric field $\vec{E}_{\text{net}}$ at P, and

(d) the electric force on a charge $q = -4.0\,\text{nC}$ placed at P.

Solution: First find the displacement vectors from each source charge to the field point P.

For q_1,

$$\vec{R}_1 = \vec{r}_P - \vec{r}_1 = (0.20\,\text{m})\hat{\imath}, \qquad R_1 = 0.20\,\text{m}.$$

For q_2,

$$\vec{R}_2 = \vec{r}_P - \vec{r}_2 = (-0.30\,\text{m})\hat{\imath}, \qquad R_2 = 0.30\,\text{m}.$$

For part (a), use the point-charge field law:

$$\vec{E}_1 = k\frac{q_1}{R_1^3}\vec{R}_1.$$

Because q_1 is positive, the field points away from q_1, which at the origin is in the $+\hat{\imath}$ direction. Its magnitude is

$$E_1 = k\frac{|q_1|}{R_1^2} = (8.99 \times 10^9)\frac{3.0 \times 10^{-6}}{(0.20)^2}\ \text{N/C}.$$

So

$$E_1 = 6.74 \times 10^5\ \text{N/C},$$

and therefore

$$\vec{E}_1 = (6.74 \times 10^5\ \text{N/C})\hat{\imath}.$$

For part (b),

$$\vec{E}_2 = k\frac{q_2}{R_2^3}\vec{R}_2.$$

Here q_2 is negative, so the field points toward q_2. Since q_2 is to the right of the origin, the field at the origin is again in the $+\hat{\imath}$ direction. Its magnitude is

$$E_2 = k\frac{|q_2|}{R_2^2} = (8.99 \times 10^9)\frac{2.0 \times 10^{-6}}{(0.30)^2}\ \text{N/C}.$$

Thus,

$$E_2 = 2.00 \times 10^5\ \text{N/C},$$

so

$$\vec{E}_2 = (2.00 \times 10^5\ \text{N/C})\hat{\imath}.$$

For part (c), add the fields as vectors:

$$\vec{E}_{\text{net}} = \vec{E}_1 + \vec{E}_2.$$

Since both fields point in the same direction,

$$\vec{E}_{\text{net}} = (6.74 \times 10^5 + 2.00 \times 10^5)\hat{\imath}\,\text{N/C}.$$

Therefore,

$$\vec{E}_{\text{net}} = (8.74 \times 10^5\,\text{N/C})\hat{\imath}.$$

For part (d), the force on a charge placed at P is

$$\vec{F} = q\vec{E}_{\text{net}}.$$

Substitute $q = -4.0 \times 10^{-9}$ C:

$$\vec{F} = (-4.0 \times 10^{-9})(8.74 \times 10^5)\hat{\imath}\,\text{N}.$$

So

$$\vec{F} = (-3.50 \times 10^{-3}\,\text{N})\hat{\imath}.$$

The negative sign means the force points in the $-\hat{\imath}$ direction. Its magnitude is 3.50×10^{-3} N.

Therefore,

$$\vec{E}_1 = (6.74 \times 10^5\,\text{N/C})\hat{\imath}, \qquad \vec{E}_2 = (2.00 \times 10^5\,\text{N/C})\hat{\imath},$$

$$\vec{E}_{\text{net}} = (8.74 \times 10^5\,\text{N/C})\hat{\imath}, \qquad \vec{F} = (-3.50 \times 10^{-3}\,\text{N})\hat{\imath}.$$

2.1.4 Fields of Continuous Charge Distributions

This subsection extends electric-field superposition from point charges to rods, arcs, rings, surfaces, and volumes by replacing discrete sums with integrals over charge elements.

Definition 2.1.4: Charge densities and differential field contribution

Let the field point have position vector $\vec{r}$. Let a small source element at position vector $\vec{r}\,'$ carry charge dq. Define

$$\vec{R} = \vec{r} - \vec{r}\,', \qquad R = |\vec{R}|, \qquad \hat{R} = \frac{\vec{R}}{R}.$$

For a continuous distribution, the charge element is written as

$$dq = \lambda\,dl \quad \text{(line charge)}, \qquad dq = \sigma\,dA \quad \text{(surface charge)}, \qquad dq = \rho\,dV \quad \text{(volume charge)},$$

where λ is linear charge density in C/m, σ is surface charge density in C/m^2, and ρ is volume charge density in C/m^3. The electric-field contribution of the source element at the field point is

$$d\vec{E} = k\frac{dq}{R^2}\hat{R} = k\frac{dq}{R^3}\vec{R},$$

where

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\text{N m}^2/\text{C}^2.$$

Theorem 2.1.3 Continuous superposition integral for electric field

For a static continuous charge distribution, the net electric field at the field point $\vec{r}$ is the vector integral

$$\vec{E}(\vec{r}) = \int d\vec{E} = k\int \frac{1}{R^3}\vec{R}\,dq.$$

Equivalently,

$$\vec{E}(\vec{r}) = k\int \frac{1}{R^3}\vec{R}\,\lambda\, dl, \qquad \vec{E}(\vec{r}) = k\int \frac{1}{R^3}\vec{R}\,\sigma\, dA, \qquad \vec{E}(\vec{r}) = k\int \frac{1}{R^3}\vec{R}\,\rho\, dV,$$

depending on the geometry. In practice, write $d\vec{E}$ for one source element, resolve it into components, use symmetry to identify any canceling components, and integrate only the nonzero component(s).

Example 2.1.1 (Illustrative example)

A uniformly charged semicircular arc of radius R lies above the x-axis and is centered at the origin. Let its total charge be $Q > 0$. Find the electric field at the center.
The arc length is πR, so the linear charge density is

$$\lambda = \frac{Q}{\pi R}.$$

Let θ denote the polar angle of a source element, measured from the positive x-axis, with $0 \le \theta \le \pi$. Then

$$dq = \lambda R\, d\theta.$$

Each source element is distance R from the center, so

$$dE = k\frac{dq}{R^2}.$$

By symmetry, the x-components cancel. The y-components all point downward, so

$$dE_y = -dE \sin\theta = -k\frac{dq}{R^2}\sin\theta.$$

Substitute $dq = \lambda R\, d\theta$:

$$dE_y = -k\frac{\lambda}{R}\sin\theta\, d\theta.$$

Integrate from 0 to π:

$$E_y = -k\frac{\lambda}{R}\int_0^{\pi}\sin\theta\, d\theta = -k\frac{\lambda}{R}(2).$$

Therefore,

$$\vec{E} = -\frac{2kQ}{\pi R^2}\hat{\jmath}.$$

The field points downward because the positive charges on the upper arc repel a positive test charge at the center.

Note:-

For continuous distributions, the hardest step is usually not the integral but the geometry. Start with $d\vec{E}$ from one source element, then ask which components cancel by symmetry. On a ring, sideways components cancel and only the axial component survives. On a symmetric finite line, horizontal components cancel at the perpendicular bisector and only the perpendicular component survives. Also choose dq to match the object's dimension: use $dq = \lambda\, dl$ for rods and arcs, $dq = \sigma\, dA$ for sheets, and $dq = \rho\, dV$ for three-dimensional charge distributions.

Question 4: Worked AP-style problem

A thin ring of radius a is centered at the origin and lies in the yz-plane. The ring carries total charge Q distributed uniformly around its circumference. Let point P lie on the ring's axis at position

$$\vec{r}_P = x\hat{\imath},$$

where $x > 0$. Let

$$k = \frac{1}{4\pi\varepsilon_0}.$$

Find the electric field $\vec{E}$ at point P in terms of Q, a, x, and k.

Solution: Let λ denote the ring's linear charge density. Since the ring circumference is $2\pi a$,

$$\lambda = \frac{Q}{2\pi a}.$$

Choose a small source element dq on the ring. Let $\vec{R}$ denote the displacement vector from that source element to point P. Every source element on the ring is the same distance from P, so

$$R = \sqrt{x^2 + a^2}.$$

The magnitude of the field due to dq is therefore

$$dE = k\frac{dq}{R^2} = k\frac{dq}{x^2 + a^2}.$$

Now use symmetry. For each source element on the ring, there is an opposite element whose field contribution has the same magnitude. Their components in the y- and z-directions cancel, while their components along the axis add. Therefore the net field must point along $\hat{\imath}$, so we only need the x-component of $d\vec{E}$.

Let ϕ denote the angle between $\vec{R}$ and the positive x-axis. Then

$$\cos\phi = \frac{x}{R} = \frac{x}{\sqrt{x^2 + a^2}}.$$

So the axial component of the differential field is

$$dE_x = dE\cos\phi = \left(k\frac{dq}{R^2}\right)\left(\frac{x}{R}\right) = k\frac{x\,dq}{R^3}.$$

Since $R = \sqrt{x^2 + a^2}$ is constant over the ring,

$$dE_x = k\frac{x\,dq}{(x^2 + a^2)^{3/2}}.$$

Integrate all the way around the ring:

$$E_x = \int dE_x = k\frac{x}{(x^2 + a^2)^{3/2}}\int dq.$$

But

$$\int dq = Q,$$

so

$$E_x = k\frac{Qx}{(x^2 + a^2)^{3/2}}.$$

Therefore,

$$\vec{E} = k\frac{Qx}{(x^2 + a^2)^{3/2}}\hat{\imath}.$$

If $Q > 0$, the field points in the $+\hat{\imath}$ direction, and if $Q < 0$, it points in the $-\hat{\imath}$ direction.

This result also passes two quick checks. If $x = 0$, then

$$\vec{E} = \vec{0},$$

which matches the symmetry at the ring's center. If $x \gg a$, then $x^2 + a^2 \approx x^2$, so

$$\vec{E} \approx k\frac{Q}{x^2}\hat{\imath},$$

which is the field of a point charge Q far away from the ring.

2.1.5 Electric Flux

This subsection introduces electric flux as a signed measure of how much electric field passes through an oriented surface.

Definition 2.1.5: Area vector and electric flux

Let S be a surface broken into small area elements of scalar area dA. Let $\hat{n}$ denote a chosen unit normal to a surface element. The corresponding *area vector element* is

$$d\vec{A} = \hat{n}\, dA.$$

For an open surface, either choice of normal may be used, but the choice must be kept consistent across the surface. For a closed surface, the standard choice is the outward normal.
Let $\vec{E}$ denote the electric field at each point of the surface. The *electric flux* through the oriented surface is the scalar

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{A}.$$

Its SI units are $\mathrm{N \cdot m^2/C}$.

Theorem 2.1.4 Surface-integral form and uniform-field special case

Let S be an oriented surface with area vector element $d\vec{A} = \hat{n}\, dA$, and let $\vec{E}$ be the electric field on that surface. Then the electric flux through S is

$$\Phi_E = \iint_S \vec{E} \cdot d\vec{A}.$$

This integral adds the component of $\vec{E}$ perpendicular to the surface over the entire surface.
If the surface is flat with area A, the field is uniform over it, and $\vec{A} = A\hat{n}$ denotes the surface's area vector, then

$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta,$$

where $E = |\vec{E}|$, θ is the angle between $\vec{E}$ and $\vec{A}$, and $A = |\vec{A}|$. For a closed surface, the same formula is applied piece by piece using outward area vectors on all patches of the surface.

Example 2.1.2 (Illustrative example)

Let a uniform electric field be

$$\vec{E} = (300\,\mathrm{N/C})\hat{\imath}.$$

Let a flat surface have area

$$A = 0.20\,\mathrm{m}^2,$$

and let its area vector make an angle $\theta = 60^\circ$ with $\vec{E}$.
Then the flux is

$$\Phi_E = EA\cos\theta = (300)(0.20)\cos 60^\circ\ \mathrm{N \cdot m^2/C}.$$

So

$$\Phi_E = 30\,\mathrm{N \cdot m^2/C}.$$

Because the angle is acute, the flux is positive.

Note:-

Electric flux is not the same thing as electric field magnitude. Flux depends on both the field and the oriented surface. Reversing the chosen normal reverses the sign of Φ_E. A positive flux means the field points generally in the same direction as the chosen area vector, while a negative flux means it points generally opposite that direction. If the field is parallel to the surface, then it is perpendicular to $d\vec{A}$ and the flux is zero even if $|\vec{E}|$ is large.

Question 5: Worked AP-style problem

A cube of side length $L = 0.20\,\text{m}$ is placed in a uniform electric field

$$\vec{E} = (500\,\text{N/C})\hat{\imath}.$$

Let the cube's faces be aligned with the coordinate axes. Let the outward area vector of the right face be in the $+\hat{\imath}$ direction, and let the outward area vector of the left face be in the $-\hat{\imath}$ direction.
Find:

(a) the electric flux through the right face,

(b) the electric flux through the left face,

(c) the electric flux through any one of the four remaining faces, and

(d) the net electric flux through the entire closed cube.

Solution: Let the area of one face be A. Since each face is a square of side length L,

$$A = L^2 = (0.20\,\text{m})^2 = 0.040\,\text{m}^2.$$

For each face of the cube, use

$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta,$$

where θ is the angle between the electric field and that face's outward area vector.

For part (a), the right face has outward area vector in the $+\hat{\imath}$ direction, the same direction as $\vec{E}$. Thus

$$\theta = 0^\circ.$$

So

$$\Phi_{E,\text{right}} = EA\cos 0^\circ = (500)(0.040)(1)\,\text{N}\cdot\text{m}^2/\text{C}.$$

Therefore,

$$\Phi_{E,\text{right}} = 20\,\text{N}\cdot\text{m}^2/\text{C}.$$

For part (b), the left face has outward area vector in the $-\hat{\imath}$ direction, opposite the field. Thus

$$\theta = 180^\circ.$$

So

$$\Phi_{E,\text{left}} = EA\cos 180^\circ = (500)(0.040)(-1)\,\text{N}\cdot\text{m}^2/\text{C}.$$

Therefore,

$$\Phi_{E,\text{left}} = -20\,\text{N}\cdot\text{m}^2/\text{C}.$$

For part (c), on any of the other four faces, the outward area vector is perpendicular to $\vec{E}$. Thus

$$\theta = 90^\circ,$$

so

$$\Phi_E = EA\cos 90^\circ = 0.$$

Therefore, the flux through each of those four faces is

$$0\,\text{N}\cdot\text{m}^2/\text{C}.$$

For part (d), add the fluxes from all six faces:

$$\Phi_{E,\text{net}} = \Phi_{E,\text{right}} + \Phi_{E,\text{left}} + 4(0).$$

So

$$\Phi_{E,\text{net}} = 20 + (-20) = 0\,\text{N}\cdot\text{m}^2/\text{C}.$$

Therefore,

$$\Phi_{E,\text{right}} = 20\,\text{N}\cdot\text{m}^2/\text{C}, \qquad \Phi_{E,\text{left}} = -20\,\text{N}\cdot\text{m}^2/\text{C},$$

$$\Phi_E = 0\,\text{N}\cdot\text{m}^2/\text{C} \text{ for each of the other four faces},$$

and the net flux through the closed cube is

$$\Phi_{E,\text{net}} = 0.$$

2.1.6 Gauss's Law and Symmetry Reduction

This subsection states Gauss's law and shows how symmetry can reduce a difficult flux integral to simple algebra when the charge distribution is highly symmetric.

Definition 2.1.6: Gaussian surface and enclosed charge

Let S be any closed imaginary surface in space, and let $d\vec{A}$ denote an outward-pointing area element on that surface. The surface S is called a *Gaussian surface*. The *enclosed charge* q_{enc} is the algebraic sum of all charges contained inside S. Charges outside S can affect the electric field on the surface, but they do not contribute to q_{enc}.

Theorem 2.1.5 Gauss's law and when symmetry makes it useful

Let S be any closed surface with outward area element $d\vec{A}$, and let q_{enc} be the net charge enclosed by S. Then Gauss's law states

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}.$$

This law is always true. It becomes a practical method for solving for the electric field when the charge distribution has enough symmetry that one can choose a Gaussian surface for which the magnitude $E = |\vec{E}|$ is constant on each flux-contributing part of the surface and the angle between $\vec{E}$ and $d\vec{A}$ is everywhere $0°$, $180°$, or $90°$. Then the flux integral reduces to algebraic terms such as EA, $-EA$, or 0. Common useful cases are spherical, cylindrical, and planar symmetry.

Note:-

Gauss's law is always true, but it is not always useful for finding $\vec{E}$. In a general asymmetric charge distribution, knowing only the total flux through a closed surface does not tell you the field at each point on that surface. Also, zero net enclosed charge implies zero *net flux*, not necessarily zero field everywhere. The main strategy is therefore: first identify strong symmetry, then choose a Gaussian surface matched to that symmetry.

Why symmetry reduces the integral: Let a point charge Q be at the center of a spherical Gaussian surface of radius r. By spherical symmetry, the electric field is radial and has the same magnitude $E(r)$ at every point on the sphere. The outward area element $d\vec{A}$ is also radial, so

$$\vec{E}\cdot d\vec{A} = E(r)\,dA$$

everywhere on the surface. Therefore,

$$\oint_S \vec{E}\cdot d\vec{A} = E(r)\oint_S dA = E(r)(4\pi r^2).$$

Since the enclosed charge is $q_{\text{enc}} = Q$, Gauss's law gives

$$E(r)(4\pi r^2) = \frac{Q}{\varepsilon_0}, \qquad E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}.$$

The law itself is general, but the symmetry is what allowed $E(r)$ to be pulled outside the integral. 😊

Question 6: Worked AP-style problem

A very long straight wire carries a uniform positive linear charge density

$$\lambda = 3.0 \times 10^{-6}\,\mathrm{C/m}.$$

Let point P be at perpendicular distance

$$r = 0.20\,\mathrm{m}$$

from the wire. Choose a cylindrical Gaussian surface of radius r and length L coaxial with the wire. Find:

(a) the enclosed charge q_{enc} for that Gaussian surface,

(b) the electric flux through the curved side and through the two flat end caps, and

(c) the magnitude and direction of the electric field at P.

Solution: Let the cylinder have radius r and length L. Because the wire has uniform linear charge density λ, the charge enclosed by the Gaussian surface is

$$q_{\text{enc}} = \lambda L.$$

By cylindrical symmetry, the electric field due to the long wire points radially outward from the wire and has the same magnitude $E(r)$ everywhere on the curved side of the Gaussian cylinder. Let $\hat{s}$ denote the outward radial unit vector from the wire.

On the curved side, the area element $d\vec{A}$ also points radially outward, so $\vec{E}$ is parallel to $d\vec{A}$. Thus,

$$\vec{E} \cdot d\vec{A} = E(r)\,dA$$

on the curved surface.

On each flat end cap, the area element $d\vec{A}$ points along the axis of the wire, while $\vec{E}$ points perpendicular to that axis. Therefore,

$$\vec{E} \cdot d\vec{A} = 0$$

on both end caps, so the flux through each end cap is zero.

The total flux is therefore entirely through the curved side:

$$\oint_S \vec{E} \cdot d\vec{A} = E(r)\big(2\pi r L\big).$$

Apply Gauss's law:

$$E(r)(2\pi r L) = \frac{\lambda L}{\varepsilon_0}.$$

Cancel L:

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

Now substitute $\lambda = 3.0 \times 10^{-6}\,\mathrm{C/m}$, $r = 0.20\,\mathrm{m}$, and $\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/(N\,m^2)}$:

$$E(r) = \frac{3.0 \times 10^{-6}}{2\pi(8.85 \times 10^{-12})(0.20)}\,\mathrm{N/C}.$$

This gives

$$E(r) = 2.7 \times 10^{5}\,\mathrm{N/C}.$$

So the fluxes are

$$\Phi_{\text{curved}} = E(r)(2\pi r L) = \frac{\lambda L}{\varepsilon_0}, \qquad \Phi_{\text{cap 1}} = 0, \qquad \Phi_{\text{cap 2}} = 0,$$

and the electric field at P is

$$\vec{E}(P) = (2.7 \times 10^{5}\,\mathrm{N/C})\hat{s},$$

where $\hat{s}$ points radially away from the positively charged wire.

2.2 Electric Potential and Energy

This unit develops the energy viewpoint for electrostatics. In AP Physics C: Electricity and Magnetism, the central idea is that electric interactions can be described not only with the vector field $\vec{E}$ but also with the scalar quantities electric potential energy U and electric potential V. That scalar viewpoint often makes multi-charge systems and energy changes easier to analyze.

The flow begins with electric potential energy for charge configurations, then defines electric potential and voltage, connects potential to the electric field, and finishes with equipotentials and the energy changes of moving charges. The scope here is electrostatics only, so the field remains conservative and no induction-related electric fields are considered yet.

2.2.1 Electric Potential Energy

This subsection introduces electric potential energy as an energy of a charge configuration and relates it to work done by electric forces.

Definition 2.2.1: Electric potential energy of a system

Let a system contain interacting charges. Let U denote the *electric potential energy* of the system, measured in joules. Electric potential energy is a property of the *configuration of the system*, not of one charge by itself. For any two configurations,

$$\Delta U = U_f - U_i,$$

and if the electric force does work W_{elec} on the system, then

$$W_{\text{elec}} = -\Delta U.$$

Thus, when the electric force does positive work, the system loses electric potential energy, and when an external agent slowly assembles a configuration against the electric force, the system gains electric potential energy.

Theorem 2.2.1 Point-charge pair formula and work relations

Let two point charges q_1 and q_2 be separated by distance r, and choose the reference value

$$U(\infty) = 0.$$

Then the electric potential energy of the two-charge system is

$$U(r) = k\frac{q_1 q_2}{r},$$

where

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9\,\mathrm{N\,m^2/C^2}.$$

If the separation changes from r_i to r_f, then

$$\Delta U = U_f - U_i = kq_1q_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right).$$

The work done by the electric force is

$$W_{\text{elec}} = -\Delta U,$$

and for a slow external rearrangement of the charges,

$$W_{\text{ext}} = \Delta U.$$

Note:-

Because $U = kq_1q_2/r$, the sign of U depends on the signs of the two charges. If $q_1q_2 > 0$, then $U > 0$ and positive external work is required to bring the like charges closer together. If $q_1q_2 < 0$, then $U < 0$ and the electric force itself tends to pull the unlike charges together. The important viewpoint is that U belongs to the pair of charges as a system. It is not correct to say that a single isolated charge "has" electric potential energy by itself.

Derivation from quasistatic assembly: Let charge q_1 be fixed, and let charge q_2 be brought slowly from infinity to a final separation r. Let x denote the instantaneous separation during the motion, with outward radial unit vector $\hat{r}$. The electric force on q_2 is

$$\vec{F}_{\text{elec}} = k\frac{q_1q_2}{x^2}\hat{r}.$$

For a quasistatic move, the external force balances the electric force, so

$$\vec{F}_{\text{ext}} = -\vec{F}_{\text{elec}}.$$

The external work done in assembling the pair is the increase in potential energy:

$$U(r) - U(\infty) = W_{\text{ext}} = \int_{\infty}^{r} \vec{F}_{\text{ext}} \cdot d\vec{r}.$$

Since $d\vec{r} = \hat{r}\,dx$,

$$U(r) - 0 = -\int_{\infty}^{r} k\frac{q_1q_2}{x^2}\,dx = -kq_1q_2\left[-\frac{1}{x}\right]_{\infty}^{r} = k\frac{q_1q_2}{r}.$$

This also gives

$$\Delta U = kq_1q_2\left(\frac{1}{r_f} - \frac{1}{r_i}\right),$$

and because electric potential energy is defined for a conservative force, the electric-force work satisfies $W_{\text{elec}} = -\Delta U$. 😄

Question 7: Worked AP-style problem

Two point charges form a system. Let

$$q_1 = +2.0\,\mu\text{C}, \qquad q_2 = -3.0\,\mu\text{C}.$$

Initially the charges are separated by

$$r_i = 0.50\,\text{m},$$

and they are moved slowly until their final separation is

$$r_f = 0.20\,\text{m}.$$

Take

$$k = 8.99 \times 10^9\,\text{N}\,\text{m}^2/\text{C}^2.$$

Find:

(a) the initial electric potential energy U_i,

(b) the final electric potential energy U_f,

(c) the change in electric potential energy ΔU, and

(d) the work done by the external agent and by the electric force during the slow move.

Solution: Use

$$U = k\frac{q_1q_2}{r}.$$

Because the charges have opposite signs, the product $q_1 q_2$ is negative:

$$q_1 q_2 = (2.0 \times 10^{-6}\,\mathrm{C})(-3.0 \times 10^{-6}\,\mathrm{C}) = -6.0 \times 10^{-12}\,\mathrm{C}^2.$$

For part (a), the initial potential energy is

$$U_i = k\frac{q_1 q_2}{r_i} = (8.99 \times 10^9)\frac{-6.0 \times 10^{-12}}{0.50}\,\mathrm{J}.$$

So

$$U_i = -1.08 \times 10^{-1}\,\mathrm{J} = -0.108\,\mathrm{J}.$$

For part (b), the final potential energy is

$$U_f = k\frac{q_1 q_2}{r_f} = (8.99 \times 10^9)\frac{-6.0 \times 10^{-12}}{0.20}\,\mathrm{J}.$$

Thus,

$$U_f = -2.70 \times 10^{-1}\,\mathrm{J} = -0.270\,\mathrm{J}.$$

For part (c),

$$\Delta U = U_f - U_i = (-0.270) - (-0.108)\,\mathrm{J}.$$

Therefore,

$$\Delta U = -0.162\,\mathrm{J}.$$

For part (d), because the move is slow,

$$W_{\mathrm{ext}} = \Delta U = -0.162\,\mathrm{J}.$$

The negative sign means the external agent removes energy from the system rather than supplying it. The electric force does the opposite amount of work:

$$W_{\mathrm{elec}} = -\Delta U = +0.162\,\mathrm{J}.$$

This result makes physical sense. Unlike charges attract, so when they are brought closer together, the system energy becomes more negative.

Therefore,

$$U_i = -0.108\,\mathrm{J}, \qquad U_f = -0.270\,\mathrm{J},$$
$$\Delta U = -0.162\,\mathrm{J}, \qquad W_{\mathrm{ext}} = -0.162\,\mathrm{J}, \qquad W_{\mathrm{elec}} = +0.162\,\mathrm{J}.$$

2.2.2 Electric Potential and Voltage

This subsection defines electric potential as electric potential energy per unit charge, interprets voltage as a potential difference, and emphasizes that potentials from multiple source charges add as scalars.

Definition 2.2.2: Electric potential and voltage

Let U denote the electric potential energy of a system containing a chosen test charge $q \neq 0$ at a specified point in an electrostatic field. The *electric potential* V at that point is defined by

$$V = \frac{U}{q}.$$

If points A and B have potentials V_A and V_B, then the *potential difference* from A to B is

$$\Delta V = V_B - V_A = \frac{\Delta U}{q},$$

where

$$\Delta U = U_B - U_A.$$

In common AP usage, *voltage* usually means this potential difference between two points.

Note:-

Electric potential is a scalar, not a vector, so it has magnitude and sign but no direction. A positive source charge produces positive potential, and a negative source charge produces negative potential, when the reference value is chosen as $V = 0$ at infinity. Only potential differences are directly physical, so the zero of potential is a convenient reference choice rather than an absolute requirement.

Proposition 2.2.1 Point-charge potential and scalar superposition

Let fixed point charges $Q_1, Q_2, \ldots, Q_N$ be located at position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. Let point P have position vector $\vec{r}$, with $\vec{r} \neq \vec{r}_i$ for all i. For each source charge, define the separation distance from Q_i to P by

$$R_i = |\vec{r} - \vec{r}_i|.$$

Let

$$k = \frac{1}{4\pi\varepsilon_0} = 8.99 \times 10^9 \,\mathrm{N\,m^2/C^2}.$$

Choosing the reference value $V(\infty) = 0$, the electric potential at P due to one point charge Q_i is

$$V_i(P) = k\frac{Q_i}{R_i}.$$

Because potential is a scalar, the total potential at P due to all the source charges is

$$V(P) = \sum_{i=1}^{N} V_i(P) = k\sum_{i=1}^{N} \frac{Q_i}{R_i}.$$

If points A and B have potentials V_A and V_B, then the voltage between them is

$$\Delta V_{A\to B} = V_B - V_A.$$

For a charge q moved quasistatically from A to B by an external agent,

$$\Delta U = q\Delta V, \qquad W_{\text{ext}} = \Delta U = q\Delta V.$$

Question 8: Worked AP-style problem

Three fixed point charges lie in an xy-plane. Let

$$Q_1 = +4.0\,\mu\mathrm{C} \quad \text{at} \quad \vec{r}_1 = \vec{0},$$

$$Q_2 = -2.0\,\mu\mathrm{C} \quad \text{at} \quad \vec{r}_2 = (0.30\,\mathrm{m})\hat{\imath},$$

and

$$Q_3 = +3.0\,\mu\mathrm{C} \quad \text{at} \quad \vec{r}_3 = (0.40\,\mathrm{m})\hat{\jmath}.$$

Let point P be at

$$\vec{r}_P = (0.30\hat{\imath} + 0.40\hat{\jmath})\,\mathrm{m}.$$

Take

$$k = 8.99 \times 10^9 \,\mathrm{N\,m^2/C^2}.$$

Find:

(a) the electric potential $V(P)$ relative to infinity, and

(b) the external work required to bring a test charge

$$q_0 = +2.0\,\mathrm{nC}$$

from infinity to point P.

Solution: First find the distances from each source charge to point P.

From Q_1 at the origin to P,

$$R_1 = |\vec{r}_P - \vec{r}_1| = \sqrt{(0.30)^2 + (0.40)^2}\,\text{m} = 0.50\,\text{m}.$$

From Q_2 at $(0.30\,\text{m})\hat{\imath}$ to P,

$$R_2 = 0.40\,\text{m}.$$

From Q_3 at $(0.40\,\text{m})\hat{\jmath}$ to P,

$$R_3 = 0.30\,\text{m}.$$

Because electric potential is a scalar, add the contributions algebraically:

$$V(P) = k\left(\frac{Q_1}{R_1} + \frac{Q_2}{R_2} + \frac{Q_3}{R_3}\right).$$

Substitute the given values:

$$V(P) = 8.99\times 10^9\left(\frac{4.0\times 10^{-6}}{0.50} + \frac{-2.0\times 10^{-6}}{0.40} + \frac{3.0\times 10^{-6}}{0.30}\right)\text{V}.$$

Evaluate each term inside the parentheses:

$$\frac{4.0\times 10^{-6}}{0.50} = 8.0\times 10^{-6}, \qquad \frac{-2.0\times 10^{-6}}{0.40} = -5.0\times 10^{-6}, \qquad \frac{3.0\times 10^{-6}}{0.30} = 1.0\times 10^{-5}.$$

So

$$V(P) = 8.99\times 10^9\left(1.3\times 10^{-5}\right)\text{V}.$$

Therefore,

$$V(P) = 1.17\times 10^5\,\text{V}.$$

For part (b), the potential at infinity is the chosen reference value,

$$V(\infty) = 0,$$

so the potential difference from infinity to P is

$$\Delta V = V(P) - V(\infty) = 1.17\times 10^5\,\text{V}.$$

The external work required to bring the test charge slowly from infinity to P is the change in electric potential energy:

$$W_{\text{ext}} = \Delta U = q_0 \Delta V.$$

Substitute $q_0 = 2.0\times 10^{-9}\,\text{C}$:

$$W_{\text{ext}} = (2.0\times 10^{-9})(1.17\times 10^5)\,\text{J}.$$

Thus,

$$W_{\text{ext}} = 2.34\times 10^{-4}\,\text{J}.$$

So the electric potential at point P is

$$V(P) = 1.17\times 10^5\,\text{V},$$

and the required external work to bring the positive test charge from infinity to P is

$$W_{\text{ext}} = 2.34\times 10^{-4}\,\text{J}.$$

2.2.3 The Field-Potential Relation

This subsection connects electric field and electric potential, both globally through a line integral and locally through the slope or gradient of the potential.

Definition 2.2.3: Potential difference from the electric field

Let A and B be two points in an electrostatic region, let C be any path from A to B, and let $d\vec{\ell}$ denote an infinitesimal displacement along that path. If the electric field is $\vec{E}$, then the infinitesimal potential change is

$$dV = -\vec{E} \cdot d\vec{\ell}.$$

Integrating from A to B gives the potential difference

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell}.$$

For electrostatics, this value is independent of the path because the electric field is conservative.

Theorem 2.2.2 Global and local field-potential relations

Let $V(\vec{r})$ denote the electric potential at position vector $\vec{r}$, and let $\vec{E}(\vec{r})$ denote the electric field there. Then in electrostatics,

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{\ell}.$$

Locally,

$$dV = -\vec{E} \cdot d\vec{\ell}.$$

Since also

$$dV = \nabla V \cdot d\vec{\ell},$$

comparison gives the vector relation

$$\vec{E} = -\nabla V.$$

For one-dimensional motion along the x-axis,

$$E_x = -\frac{dV}{dx}.$$

Thus the x-component of the electric field is the negative slope of the potential graph.

Derivation from work per unit charge: Let a charge q move through an infinitesimal displacement $d\vec{\ell}$ in an electric field $\vec{E}$. The electric force is

$$\vec{F} = q\vec{E},$$

so the infinitesimal work done by the field is

$$dW = \vec{F} \cdot d\vec{\ell} = q\vec{E} \cdot d\vec{\ell}.$$

Electric potential difference is potential-energy change per unit charge, so

$$dV = \frac{dU}{q}.$$

Because the work done by the electric field decreases electric potential energy,

$$dU = -dW.$$

Therefore,

$$dV = \frac{-dW}{q} = -\vec{E} \cdot d\vec{\ell}.$$

Integrating between two points gives

$$\Delta V = -\int \vec{E} \cdot d\vec{\ell}.$$

Comparing this with the differential identity $dV = \nabla V \cdot d\vec{\ell}$ yields $\vec{E} = -\nabla V$. 😊

Corollary 2.2.1 Useful special cases

Let x denote position along the x-axis.

(a) In one dimension,

$$E_x = -\frac{dV}{dx}.$$

So where a graph of V versus x slopes downward, E_x is positive, and where it slopes upward, E_x is negative.

(b) If the electric field is uniform and parallel to the displacement, so $\vec{E} = E\hat{u}$ and $\Delta\vec{\ell} = \Delta s\hat{u}$, then

$$\Delta V = -E\Delta s.$$

In particular, between parallel plates with nearly uniform field magnitude E and separation d measured in the field direction,

$$|\Delta V| = Ed.$$

Question 9: Worked AP-style problem

Along the x-axis, the electric potential is

$$V(x) = 120 - 40x + 5x^2,$$

where V is in volts and x is in meters.
Find:

(a) the electric-field component $E_x(x)$,

(b) the electric field at $x = 2.0\,\text{m}$,

(c) the potential difference $\Delta V = V(4.0\,\text{m}) - V(1.0\,\text{m})$, and

(d) the work done by the electric field on a charge $q = +2.0\,\mu\text{C}$ moving from $x = 1.0\,\text{m}$ to $x = 4.0\,\text{m}$.

Solution: For part (a), use the one-dimensional field-potential relation:

$$E_x = -\frac{dV}{dx}.$$

Differentiate the given potential function:

$$\frac{dV}{dx} = \frac{d}{dx}(120 - 40x + 5x^2) = -40 + 10x.$$

Therefore,

$$E_x = -(-40 + 10x) = 40 - 10x.$$

So the field as a function of position is

$$E_x(x) = 40 - 10x$$

in units of N/C.

For part (b), substitute $x = 2.0\,\text{m}$:

$$E_x(2.0) = 40 - 10(2.0) = 20\,\text{N/C}.$$

Since this value is positive, the electric field points in the $+x$ direction:

$$\vec{E}(2.0\,\mathrm{m}) = (20\,\mathrm{N/C})\hat{\imath}.$$

For part (c), first evaluate the potential at each position:

$$V(4.0) = 120 - 40(4.0) + 5(4.0)^2 = 120 - 160 + 80 = 40\,\mathrm{V},$$

and

$$V(1.0) = 120 - 40(1.0) + 5(1.0)^2 = 120 - 40 + 5 = 85\,\mathrm{V}.$$

Thus,

$$\Delta V = V(4.0) - V(1.0) = 40 - 85 = -45\,\mathrm{V}.$$

For part (d), the work done by the electric field is related to potential difference by

$$W_{\text{field}} = -q\Delta V.$$

Substitute $q = +2.0 \times 10^{-6}$ C and $\Delta V = -45$ V:

$$W_{\text{field}} = -(2.0 \times 10^{-6})(-45)\,\mathrm{J} = 9.0 \times 10^{-5}\,\mathrm{J}.$$

So the field does positive work:

$$W_{\text{field}} = 9.0 \times 10^{-5}\,\mathrm{J} = 90\,\mu\mathrm{J}.$$

Therefore,

$$E_x(x) = 40 - 10x, \qquad \vec{E}(2.0\,\mathrm{m}) = (20\,\mathrm{N/C})\hat{\imath},$$
$$\Delta V = -45\,\mathrm{V}, \qquad W_{\text{field}} = 9.0 \times 10^{-5}\,\mathrm{J}.$$

2.2.4 Equipotentials and Energy Conservation for Moving Charges

This subsection explains how equipotential curves or surfaces encode the direction of $\vec{E}$ and how potential differences determine changes in kinetic and electric potential energy for moving charges.

Definition 2.2.4: Equipotentials and the energy-change relation

Let $V(\vec{r})$ denote the electric potential at position vector $\vec{r}$. An *equipotential* is a curve in two dimensions or a surface in three dimensions on which the potential has one constant value:

$$V(\vec{r}) = \text{constant}.$$

If a charge q moves from point A to point B, define

$$\Delta V = V_B - V_A$$

and

$$\Delta U = U_B - U_A.$$

Then the change in electric potential energy is

$$\Delta U = q\Delta V.$$

Thus potential difference tells how the electric potential energy of a chosen charge changes between two points.

Theorem 2.2.3 Equipotentials, no work, and kinetic-energy change

Let $d\vec{\ell}$ denote an infinitesimal displacement in an electrostatic field $\vec{E}$. Then

$$dV = -\vec{E} \cdot d\vec{\ell}.$$

If the displacement is along an equipotential, then $dV = 0$, so

$$\vec{E} \cdot d\vec{\ell} = 0.$$

Therefore the electric field is perpendicular to an equipotential, and the electric force does no work on a charge moved along an equipotential:

$$W_{\text{elec}} = q \int \vec{E} \cdot d\vec{\ell} = 0.$$

For any motion of a charge q from A to B in electrostatics,

$$\Delta U = q\Delta V.$$

If only the electric force does work, conservation of mechanical energy gives

$$K_i + U_i = K_f + U_f,$$

so

$$\Delta K = K_f - K_i = -\Delta U = -q\Delta V.$$

Hence a charge speeds up when its electric potential energy decreases.

Example 2.2.1 (Illustrative example)

Points A and B lie on the same equipotential,

$$V_A = V_B = 120\,\text{V}.$$

Let a proton of charge

$$q = +e = +1.60 \times 10^{-19}\,\text{C}$$

move from A to B.
Because the two points are on the same equipotential,

$$\Delta V = V_B - V_A = 0.$$

So the change in electric potential energy is

$$\Delta U = q\Delta V = 0,$$

and the work done by the electric field is also zero. The field may still be present, but along that displacement it is perpendicular to the motion.

Note:-

Be careful to distinguish *lower potential* from *lower potential energy*. Since $\Delta U = q\Delta V$, a positive charge has lower potential energy at lower potential, but a negative charge has lower potential energy at *higher* potential. Thus a positive charge released from rest tends to speed up toward lower V, whereas an electron released from rest tends to speed up toward higher V. In both cases the rule is the same: the charge moves spontaneously in the direction that makes U decrease and K increase.

Question 10: Worked AP-style problem

Two large parallel plates create a uniform electrostatic region. Let point A be near the negative plate and point B be near the positive plate. The potentials are

$$V_A = 100\,\text{V}, \qquad V_B = 400\,\text{V}.$$

An electron is released from rest at point A and moves to point B. Let the electron charge be

$$q = -1.60 \times 10^{-19}\,\mathrm{C}$$

and the electron mass be

$$m_e = 9.11 \times 10^{-31}\,\mathrm{kg}.$$

Find:

(a) the potential difference $\Delta V = V_B - V_A$,

(b) the change in electric potential energy ΔU,

(c) the change in kinetic energy ΔK, and

(d) the electron's speed at point B.

Solution: First compute the potential difference:

$$\Delta V = V_B - V_A = 400\,\mathrm{V} - 100\,\mathrm{V} = 300\,\mathrm{V}.$$

For part (b), use the relation

$$\Delta U = q\Delta V.$$

Substitute the electron charge and the potential difference:

$$\Delta U = (-1.60 \times 10^{-19}\,\mathrm{C})(300\,\mathrm{V}).$$

Since $1\,\mathrm{V} = 1\,\mathrm{J/C}$,

$$\Delta U = -4.80 \times 10^{-17}\,\mathrm{J}.$$

For part (c), only the electric force does work, so

$$\Delta K = -\Delta U.$$

Therefore,

$$\Delta K = +4.80 \times 10^{-17}\,\mathrm{J}.$$

This sign makes sense. The electron moves toward higher potential, but because its charge is negative, that motion lowers its electric potential energy and increases its kinetic energy.

For part (d), the electron starts from rest, so

$$K_i = 0.$$

Thus

$$K_f = \Delta K = 4.80 \times 10^{-17}\,\mathrm{J}.$$

Use the kinetic-energy formula

$$K_f = \frac{1}{2} m_e v^2.$$

Solve for the speed v:

$$v = \sqrt{\frac{2K_f}{m_e}}.$$

Substitute the values:

$$v = \sqrt{\frac{2(4.80 \times 10^{-17}\,\mathrm{J})}{9.11 \times 10^{-31}\,\mathrm{kg}}}.$$

This gives

$$v = 1.03 \times 10^{7}\,\mathrm{m/s}.$$

Therefore,

$$\Delta V = 300\,\mathrm{V}, \qquad \Delta U = -4.80 \times 10^{-17}\,\mathrm{J},$$
$$\Delta K = +4.80 \times 10^{-17}\,\mathrm{J}, \qquad v = 1.03 \times 10^{7}\,\mathrm{m/s}.$$

2.3 Capacitance, Dielectrics, and Energy Storage

This unit shifts the focus from electric fields and potentials produced by isolated charge distributions to systems designed to store electric energy: capacitors. The central idea is that a conductor (or pair of conductors) can hold charge and associated electric potential energy in a controlled geometry, and that the ability to store charge at a given voltage is quantified by the capacitance C.

The flow begins with conductors in electrostatic equilibrium, where charge resides entirely on surfaces and the interior field vanishes. It then covers charge redistribution through contact, induction, and grounding. Next, capacitance and the standard capacitor geometries (parallel-plate, spherical, cylindrical) are introduced, followed by the energy stored in capacitor electric fields. The unit concludes with dielectrics, showing how inserting an insulating material increases capacitance through polarization.

2.3.1 Conductors in Electrostatic Equilibrium

This subsection summarizes the standard properties of conductors in electrostatics and shows how to apply them in AP-style reasoning.

Definition 2.3.1: Electrostatic equilibrium in a conductor

Let a conductor contain mobile charges, and let $\vec{E}$ denote the electric field inside the conducting material. The conductor is in *electrostatic equilibrium* when the free charges have redistributed so that there is no sustained macroscopic charge motion. In an ideal conductor, this requires

$$\vec{E} = \vec{0}$$

everywhere inside the conducting material. Otherwise a nonzero electric field would exert a force on the free charges and they would continue to move.

Note:-

Free charges in a conductor move in response to any interior electric field. Electrons drift until their rearranged surface charges create an induced field that cancels the original field inside the metal. Equilibrium therefore requires no tangential field along the surface and no field anywhere inside the conducting material. If either existed, the charges would keep moving, so the situation would not be electrostatic.

Example 2.3.1 (Illustrative example)

A neutral metal sphere is placed in an initially uniform external electric field directed to the right. At first the sphere's free electrons move to the left, leaving an induced positive region on the right surface and an induced negative region on the left surface. This redistribution continues until the induced field cancels the external field everywhere inside the metal. In the final equilibrium state,

$$\vec{E} = \vec{0}$$

inside the sphere, and the electric field just outside the surface is perpendicular to the surface.

Proposition 2.3.1 Standard properties of conductors in electrostatic equilibrium

Let a conductor be in electrostatic equilibrium. Let A and B be any two points in the conducting material. Let $\hat{n}$ denote the outward unit normal just outside the surface, and let σ denote the surface charge density at that point on the surface. Then:

(a) The electric field inside the conducting material is zero:

$$\vec{E}_{\text{inside}} = \vec{0}.$$

(b) The conductor is an equipotential, so

$$V_A = V_B.$$

In particular, the entire surface of a connected conductor is at one potential.

(c) Any excess charge placed on an isolated conductor resides on its surface rather than in the interior bulk material.

(d) The electric field just outside the surface is perpendicular to the surface. Its tangential component is zero; otherwise surface charges would move.

(e) For an ideal conductor, the field just outside satisfies

$$\vec{E}_{\text{outside}} = \frac{\sigma}{\varepsilon_0}\hat{n},$$

because the field just inside is zero.

Question 11: Worked AP-style problem

An isolated solid conducting sphere has radius

$$R = 0.20\,\text{m}$$

and net charge

$$Q = +6.0 \times 10^{-9}\,\text{C}.$$

Let point A be at the center of the sphere, let point B be just inside the metal surface, and let point C be just outside the surface. Let $\hat{n}$ denote the outward unit normal at point C. Take

$$k = 8.99 \times 10^{9}\,\text{N}\,\text{m}^2/\text{C}^2$$

and proton charge

$$q_p = +1.60 \times 10^{-19}\,\text{C}.$$

Find:

(a) the electric field at A and at B,

(b) the magnitude and direction of the electric field at C,

(c) the potential difference $V_A - V_B$, and

(d) the magnitude and direction of the electric force on a proton placed at C.

Solution: Because the sphere is a conductor in electrostatic equilibrium, the electric field everywhere inside the conducting material is zero. Therefore,

$$\vec{E}(A) = \vec{0}, \qquad \vec{E}(B) = \vec{0}.$$

For a charged conducting sphere, the external field is the same as that of a point charge Q located at the center. Point C is just outside the surface, so its distance from the center is essentially

$$r = R = 0.20\,\text{m}.$$

Thus the field magnitude there is

$$E(C) = k\frac{Q}{R^2}.$$

Substitute the given values:

$$E(C) = \left(8.99 \times 10^{9}\right)\frac{6.0 \times 10^{-9}}{(0.20)^2}\,\text{N/C}.$$

Since

$$(0.20)^2 = 0.040,$$

we get

$$E(C) = \frac{53.94}{0.040}\,\text{N/C} = 1.35 \times 10^{3}\,\text{N/C}.$$

Because Q is positive, the field points radially outward, perpendicular to the surface. So

$$\vec{E}(C) = (1.35 \times 10^3\,\mathrm{N/C})\hat{n},$$

where $\hat{n}$ is the outward normal.

Next, the entire conductor is an equipotential in electrostatic equilibrium. Since both A and B lie in the conductor,

$$V_A = V_B.$$

Therefore,

$$V_A - V_B = 0\,\mathrm{V}.$$

Finally, the electric force on a proton at C has magnitude

$$F = q_p E(C).$$

Substitute the values:

$$F = \left(1.60 \times 10^{-19}\,\mathrm{C}\right)\left(1.35 \times 10^3\,\mathrm{N/C}\right).$$

This gives

$$F = 2.16 \times 10^{-16}\,\mathrm{N}.$$

Because the proton has positive charge, its force is in the same direction as $\vec{E}$, so the force is radially outward:

$$\vec{F} = (2.16 \times 10^{-16}\,\mathrm{N})\hat{n}.$$

Therefore,

$$\vec{E}(A) = \vec{0}, \qquad \vec{E}(B) = \vec{0},$$
$$\vec{E}(C) = (1.35 \times 10^3\,\mathrm{N/C})\hat{n},$$
$$V_A - V_B = 0\,\mathrm{V}, \qquad \vec{F} = (2.16 \times 10^{-16}\,\mathrm{N})\hat{n}.$$

2.3.2 Charge Redistribution, Contact, Induction, and Grounding

This subsection explains how charge moves on conductors, why touching conductors exchange charge until they reach one potential, and how induction plus grounding can leave an object with a net charge without direct contact.

Definition 2.3.2: Charge redistribution, contact, induction, and grounding

Let $q_A, q_B, \ldots$ denote the net charges on conductors $A, B, \ldots$, and let $V_A, V_B, \ldots$ denote their electric potentials.

1. *Charge redistribution*: mobile charge in a conductor moves through the material until electrostatic equilibrium is reached.
2. *Charging by contact*: if conductors touch or are connected by a wire, charge can flow between them until the connected conductors reach one common potential.
3. *Charging by induction*: a nearby charged object causes charges in another object to separate without direct contact.
4. *Grounding*: connecting an object to Earth allows charge to flow between the object and Earth, which acts as a very large charge reservoir.

Note:-

Charge redistribution in a conductor continues until electrostatic equilibrium is reached. In electrostatics, that means the electric field inside the conductor is zero and all parts of one connected conductor are at the same potential. Therefore contact problems are usually solved with charge conservation plus an equal-potential idea. For identical small conducting spheres, symmetry makes equal potential equivalent to equal final charge, but for conductors of different size or shape equal potential does not generally mean equal charge.

Example 2.3.2 (Illustrative example)

Two identical small conducting spheres A and B are far apart initially. Let their initial charges be

$$q_{A,i} = +6.0\,\text{nC}, \qquad q_{B,i} = 0.$$

If the spheres are touched together and then separated, charge is conserved and the identical spheres must finish with equal charge. The total charge is

$$q_{\text{tot}} = q_{A,i} + q_{B,i} = +6.0\,\text{nC},$$

so each sphere ends with

$$q_{A,f} = q_{B,f} = \frac{q_{\text{tot}}}{2} = +3.0\,\text{nC}.$$

This is a contact example: the charge does not disappear or appear; it redistributes until the connected conductors reach electrostatic equilibrium.

Proposition 2.3.2 Practical relations and qualitative rules

Let q_{tot} denote the total charge of a chosen isolated system, let $q_{A,i}, q_{B,i}$ and $q_{A,f}, q_{B,f}$ denote initial and final charges, let $q_{\text{object},f}$ denote the final charge of a grounded object, and let Δq_{Earth} denote the charge change of Earth.

1. For any isolated system,

$$q_{\text{tot},f} = q_{\text{tot},i}.$$

2. If two identical small conducting spheres touch and then separate,

$$q_{A,f} = q_{B,f} = \frac{q_{A,i} + q_{B,i}}{2}.$$

3. Induction without grounding redistributes charge but does not change the net charge of the induced object. For an initially neutral conductor, the near side becomes opposite in sign to the external object, the far side becomes the same sign, and the net charge remains zero.

4. Grounding is charge exchange with Earth. If a positive object is nearby, electrons can flow from Earth onto the grounded conductor. If a negative object is nearby, electrons can flow from the conductor to Earth.

5. In the usual induction-charging sequence, the ground connection is removed first and the external charged object is removed second. Then the conductor is left with a net charge opposite in sign to the inducing object. If the object-Earth system is initially neutral, charge conservation for that system gives

$$q_{\text{object},f} + \Delta q_{\text{Earth}} = 0.$$

Question 12: Worked AP-style problem

A neutral metal sphere S is mounted on an insulating stand. A negatively charged rod is brought near the left side of the sphere but does not touch it. While the rod remains in place, the sphere is briefly connected to Earth by a wire. The grounding wire is then removed, and finally the rod is taken away. Find:

(a) the signs of the induced charges on the left and right sides of the sphere before the grounding wire is attached,

(b) the direction of electron flow while the sphere is grounded,

(c) the net charge left on the sphere after the full sequence, and

(d) whether charge conservation is violated by the sphere ending with a net charge.

Solution: Before the sphere is grounded, the rod is negative, so it repels mobile electrons in the metal sphere. Those electrons shift toward the right side, farther from the rod.

Therefore, before grounding,

- the left side of the sphere is induced to be positive, and
- the right side of the sphere is induced to be negative.

Even though the charges have separated, the sphere is still overall neutral at this stage because no charge has entered or left the sphere.

Now the sphere is connected to Earth while the negative rod remains nearby. The excess electrons on the sphere are repelled by the negative rod, and the grounding wire gives those electrons a path to leave. Thus electrons flow

from the sphere to Earth.

Next, the grounding wire is removed while the rod is still present. Because the sphere is no longer connected to Earth, the electrons that left cannot return. The sphere has lost some electrons, so it now has a net positive charge.

Finally, the rod is taken away. With the rod gone, the remaining positive charge is no longer pulled to one side, so it redistributes over the outer surface of the sphere. The final result is that the sphere is left

positively charged.

Charge conservation is not violated. The correct isolated system is the sphere together with Earth. During grounding, electrons moved from the sphere to Earth, so the sphere became positive and Earth gained an equal amount of negative charge. If the final charge on the sphere is $q_{S,f} > 0$, then

$$q_{S,f} + \Delta q_{\text{Earth}} = 0.$$

So the total charge of the combined system is unchanged; the process is charge transfer, not charge creation.

2.3.3 Capacitance and Capacitor Geometries

This subsection defines capacitance, shows how capacitor geometry controls it, and derives the parallel-plate result from Gauss's law and the field-potential relation.

Definition 2.3.3: Capacitor and capacitance

A *capacitor* is a system of two conductors that can hold equal and opposite charges. Let the conductors carry charges $+Q$ and $-Q$, and let

$$\Delta V = V_{\text{high}} - V_{\text{low}}$$

denote the magnitude of the potential difference between them. The *capacitance* C of the system is defined by

$$C = \frac{Q}{\Delta V}.$$

Capacitance measures how much charge is stored per unit potential difference. Its SI unit is the farad:

$$1\,\text{F} = 1\,\text{C/V}.$$

Theorem 2.3.1 Capacitance formula and common geometries

Let a capacitor carry plate charges $\pm Q$, and let ΔV be the magnitude of the potential difference between its conductors. Then

$$C = \frac{Q}{\Delta V}.$$

For a vacuum parallel-plate capacitor with plate area A and plate separation d, define the surface charge density by

$$\sigma = \frac{Q}{A}.$$

Ignoring edge effects, Gauss's law gives the nearly uniform electric field between the plates:

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\,\hat{n}, \qquad E = \frac{\sigma}{\varepsilon_0}.$$

Using the potential-drop relation with a path across the gap,

$$\Delta V = \left| -\int \vec{E} \cdot d\vec{\ell} \right| = Ed,$$

so

$$\Delta V = \frac{\sigma d}{\varepsilon_0} = \frac{Qd}{\varepsilon_0 A}.$$

Substitute into $C = Q/\Delta V$:

$$C = \frac{Q}{Qd/(\varepsilon_0 A)} = \varepsilon_0 \frac{A}{d}.$$

Thus, for an ideal vacuum parallel-plate capacitor,

$$C = \varepsilon_0 \frac{A}{d}.$$

Two other useful vacuum results are

$$C_{\text{spherical}} = 4\pi\varepsilon_0 \frac{ab}{b-a}$$

for concentric spheres of radii a and b with $b > a$, and

$$C_{\text{cylindrical}} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}$$

for coaxial cylinders of length L and radii a and b with $b > a$. In every case, capacitance is determined by geometry and the material between the conductors.

Example 2.3.3 (Illustrative example)

A vacuum parallel-plate capacitor has plate area

$$A = 3A_0$$

and separation

$$d = 2d_0.$$

If a reference capacitor with area A_0 and separation d_0 has capacitance C_0, then

$$C_0 = \varepsilon_0 \frac{A_0}{d_0}.$$

For the new capacitor,

$$C = \varepsilon_0 \frac{3A_0}{2d_0} = \frac{3}{2} C_0.$$

So increasing plate area increases capacitance, while increasing plate separation decreases it.

Note:-

For an ideal capacitor, C is a property of the physical setup, not of the momentary charge or battery setting. In a vacuum parallel-plate capacitor, changing A or d changes C, and inserting a dielectric would also change C. But if the same capacitor is connected to a larger battery, then ΔV increases and Q increases proportionally, so the ratio $Q/\Delta V$ stays the same.

Question 13: Worked AP-style problem

A vacuum parallel-plate capacitor has plate area

$$A = 2.0 \times 10^{-2}\,\mathrm{m}^2$$

and plate separation

$$d = 1.5 \times 10^{-3}\,\mathrm{m}.$$

It is connected to a battery that maintains a potential difference

$$\Delta V = 12.0\,\mathrm{V}.$$

Take

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{F/m}.$$

Find:

(a) the capacitance C,

(b) the charge magnitude Q on each plate,

(c) the electric field magnitude E between the plates, and

(d) the surface charge density σ on either plate.

Solution: For part (a), use the parallel-plate formula

$$C = \varepsilon_0 \frac{A}{d}.$$

Substitute the given values:

$$C = (8.85 \times 10^{-12}) \frac{2.0 \times 10^{-2}}{1.5 \times 10^{-3}}\,\mathrm{F}.$$

First evaluate the geometry factor:

$$\frac{2.0 \times 10^{-2}}{1.5 \times 10^{-3}} = 13.3.$$

So

$$C = (8.85 \times 10^{-12})(13.3)\,\mathrm{F} = 1.18 \times 10^{-10}\,\mathrm{F}.$$

Therefore,

$$C = 1.18 \times 10^{-10}\,\mathrm{F}.$$

For part (b), use the definition of capacitance:

$$Q = C\Delta V.$$

Substitute the capacitance and the battery voltage:

$$Q = (1.18 \times 10^{-10}\,\mathrm{F})(12.0\,\mathrm{V}).$$

This gives

$$Q = 1.42 \times 10^{-9}\,\mathrm{C}.$$

So the plates carry charges $+Q$ and $-Q$ with magnitude

$$Q = 1.42 \times 10^{-9}\,\mathrm{C}.$$

For part (c), the field between ideal parallel plates is approximately uniform, so the potential difference satisfies

$$\Delta V = Ed.$$

Solve for E:

$$E = \frac{\Delta V}{d}.$$

Substitute the values:

$$E = \frac{12.0\,\mathrm{V}}{1.5 \times 10^{-3}\,\mathrm{m}} = 8.0 \times 10^{3}\,\mathrm{V/m}.$$

Since $1\,\mathrm{V/m} = 1\,\mathrm{N/C}$,

$$E = 8.0 \times 10^{3}\,\mathrm{N/C}.$$

For part (d), use

$$\sigma = \frac{Q}{A}.$$

Substitute the charge and area:

$$\sigma = \frac{1.42 \times 10^{-9}\,\mathrm{C}}{2.0 \times 10^{-2}\,\mathrm{m}^2}.$$

This gives

$$\sigma = 7.1 \times 10^{-8}\,\mathrm{C/m^2}.$$

As a check, the field relation for parallel plates predicts

$$E = \frac{\sigma}{\varepsilon_0} = \frac{7.1 \times 10^{-8}}{8.85 \times 10^{-12}}\,\mathrm{N/C} \approx 8.0 \times 10^{3}\,\mathrm{N/C},$$

which agrees with part (c).

Therefore,

$$C = 1.18 \times 10^{-10}\,\mathrm{F}, \qquad Q = 1.42 \times 10^{-9}\,\mathrm{C},$$
$$E = 8.0 \times 10^{3}\,\mathrm{N/C}, \qquad \sigma = 7.1 \times 10^{-8}\,\mathrm{C/m^2}.$$

2.3.4 Energy Stored in Capacitors and Fields

This subsection develops the standard formulas for capacitor energy and connects them to the electric-field energy density in vacuum.

Definition 2.3.4: Capacitor energy and the field-energy viewpoint

Let a capacitor have capacitance C, let the magnitude of the charge on either conductor be Q, and let

$$\Delta V = V_{\mathrm{high}} - V_{\mathrm{low}}$$

denote the magnitude of the potential difference between the conductors. The *energy stored in the capacitor*, denoted U_C, is the electric potential energy gained while the capacitor is charged from 0 to Q.
In the field viewpoint, let u_E denote the *electric-field energy density*, measured in joules per cubic meter. For a vacuum region with electric-field magnitude E, the stored energy can be regarded as distributed through the field-filled volume.

Theorem 2.3.2 Equivalent capacitor-energy formulas and vacuum field-energy density

Let a capacitor have capacitance C, plate charges $\pm Q$, and potential-difference magnitude ΔV. Then the stored energy can be written in any of the equivalent forms

$$U_C = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2 = \frac{Q^2}{2C}.$$

For a vacuum region in which the electric-field magnitude is E, the electric-field energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2.$$

For an ideal vacuum parallel-plate capacitor of plate area A and separation d, the field is approximately uniform, so

$$U_C = u_E(Ad).$$

Note:-

The energy grows quadratically with charge or voltage because charging is cumulative. For a capacitor with fixed C, the potential difference is not constant while it charges: it rises in proportion to the accumulated charge, since $\Delta V = Q/C$. That means later bits of charge are harder to add than earlier ones. The average potential difference during charging is therefore half the final value, which is why $U_C = \frac{1}{2}Q\Delta V$ and why doubling Q or ΔV makes the stored energy four times as large for the same capacitor.

Short derivation from charging work: Let q denote the instantaneous charge on the capacitor during a slow charging process from $q = 0$ to $q = Q$. At that instant, the potential difference is

$$\Delta V(q) = \frac{q}{C}.$$

To move an additional small charge dq onto the capacitor, the external work required is

$$dU = \Delta V(q)\, dq = \frac{q}{C}\, dq.$$

Integrate from 0 to Q:

$$U_C = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q = \frac{Q^2}{2C}.$$

Using

$$Q = C\Delta V,$$

this becomes

$$U_C = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2.$$

For an ideal vacuum parallel-plate capacitor,

$$C = \varepsilon_0\frac{A}{d} \qquad \text{and} \qquad E = \frac{\Delta V}{d}.$$

Substituting into $U_C = \frac{1}{2}C(\Delta V)^2$ gives

$$U_C = \frac{1}{2}\left(\varepsilon_0\frac{A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2(Ad).$$

Dividing by the volume Ad yields

$$u_E = \frac{U_C}{Ad} = \frac{1}{2}\varepsilon_0 E^2.$$

😃

Question 14: Worked AP-style problem

A vacuum parallel-plate capacitor has plate area

$$A = 2.0 \times 10^{-2}\,\text{m}^2$$

and plate separation

$$d = 1.0 \times 10^{-3}\,\text{m}.$$

It is connected to a battery that maintains a potential difference

$$\Delta V = 20.0\,\text{V}.$$

Take

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}.$$

Find:

(a) the capacitance C,

(b) the charge magnitude Q on each plate,

(c) the stored energy U_C, and

(d) the electric-field energy density u_E between the plates, then verify that $U_C = u_E(Ad)$.

Solution: For part (a), use the parallel-plate formula

$$C = \varepsilon_0 \frac{A}{d}.$$

Substitute the given values:

$$C = (8.85 \times 10^{-12}) \frac{2.0 \times 10^{-2}}{1.0 \times 10^{-3}}\,\mathrm{F}.$$

The geometry factor is

$$\frac{2.0 \times 10^{-2}}{1.0 \times 10^{-3}} = 20.0,$$

so

$$C = (8.85 \times 10^{-12})(20.0)\,\mathrm{F} = 1.77 \times 10^{-10}\,\mathrm{F}.$$

For part (b), use

$$Q = C\Delta V.$$

Then

$$Q = (1.77 \times 10^{-10}\,\mathrm{F})(20.0\,\mathrm{V}) = 3.54 \times 10^{-9}\,\mathrm{C}.$$

For part (c), use any equivalent energy formula. Using $U_C = \frac{1}{2}C(\Delta V)^2$,

$$U_C = \frac{1}{2}(1.77 \times 10^{-10})(20.0)^2\,\mathrm{J}.$$

Since

$$(20.0)^2 = 400,$$

we obtain

$$U_C = \frac{1}{2}(1.77 \times 10^{-10})(400)\,\mathrm{J} = 3.54 \times 10^{-8}\,\mathrm{J}.$$

As a check,

$$U_C = \frac{1}{2}Q\Delta V = \frac{1}{2}(3.54 \times 10^{-9})(20.0)\,\mathrm{J} = 3.54 \times 10^{-8}\,\mathrm{J},$$

which agrees.

For part (d), first find the field magnitude between the plates:

$$E = \frac{\Delta V}{d} = \frac{20.0\,\mathrm{V}}{1.0 \times 10^{-3}\,\mathrm{m}} = 2.0 \times 10^{4}\,\mathrm{V/m}.$$

Now use the vacuum field-energy density formula:

$$u_E = \frac{1}{2}\varepsilon_0 E^2.$$

Substitute the values:

$$u_E = \frac{1}{2}(8.85 \times 10^{-12})(2.0 \times 10^{4})^2\,\mathrm{J/m^3}.$$

Because

$$(2.0 \times 10^{4})^2 = 4.0 \times 10^{8},$$

we get

$$u_E = \frac{1}{2}(8.85 \times 10^{-12})(4.0 \times 10^{8})\,\mathrm{J/m^3} = 1.77 \times 10^{-3}\,\mathrm{J/m^3}.$$

To verify the field viewpoint, compute the volume between the plates:

$$Ad = (2.0 \times 10^{-2})(1.0 \times 10^{-3})\,\mathrm{m^3} = 2.0 \times 10^{-5}\,\mathrm{m^3}.$$

Then

$$u_E(Ad) = (1.77 \times 10^{-3})(2.0 \times 10^{-5})\,\mathrm{J} = 3.54 \times 10^{-8}\,\mathrm{J}.$$

This matches the capacitor-energy result.

Therefore,

$$C = 1.77 \times 10^{-10}\,\mathrm{F}, \qquad Q = 3.54 \times 10^{-9}\,\mathrm{C},$$
$$U_C = 3.54 \times 10^{-8}\,\mathrm{J}, \qquad u_E = 1.77 \times 10^{-3}\,\mathrm{J/m^3}.$$

2.3.5 Dielectrics and Polarization

This subsection explains how dielectric materials polarize in an electric field and how that polarization changes capacitor behavior in the two common AP settings: fixed charge and fixed voltage.

Definition 2.3.5: Dielectric, polarization, and dielectric constant

A *dielectric* is an insulating material placed between capacitor plates. Its charges are not free to flow through the material as they do in a conductor, but the positive and negative charges within its atoms or molecules can shift slightly.

This internal charge separation is called *polarization*. In a polarized dielectric, the side nearer the positive plate becomes slightly negative and the side nearer the negative plate becomes slightly positive, so the dielectric produces an induced electric field that opposes the original field between the plates.

Let C_0 denote the capacitance of a capacitor when the gap is vacuum or air, and let C denote the capacitance when a dielectric fully fills the gap. The *dielectric constant* κ of the material is the factor by which the capacitance increases:

$$C = \kappa C_0, \qquad \kappa > 1.$$

For an ideal parallel-plate capacitor of plate area A and separation d,

$$C_0 = \varepsilon_0 \frac{A}{d} \qquad \text{and therefore} \qquad C = \kappa \varepsilon_0 \frac{A}{d}.$$

Note:-

The dielectric does not cancel the capacitor's field completely. Instead, polarization creates *bound* charges on the dielectric surfaces, and those bound charges produce a field opposite the field from the capacitor plates. That opposition reduces the net interior field.

The key AP distinction is what stays fixed during insertion.

- If the capacitor is isolated after charging, then the free charge on the plates cannot change, so this is a fixed-Q situation. The dielectric lowers the field and voltage.
- If the capacitor remains connected to a battery, then the battery keeps the potential difference fixed, so this is a fixed-V situation. The dielectric still tends to reduce the field, but the battery pushes additional charge onto the plates until the original voltage is restored.

Example 2.3.4 (Illustrative example)

An isolated capacitor has initial capacitance

$$C_0 = 10.0\,\mathrm{pF}$$

and is charged to

$$V_0 = 30.0\,\mathrm{V}.$$

The battery is disconnected, and a dielectric with

$$\kappa = 2.5$$

is inserted so that it fully fills the gap.
The new capacitance is

$$C = \kappa C_0 = (2.5)(10.0\,\text{pF}) = 25.0\,\text{pF}.$$

Because the capacitor is isolated, the charge stays constant:

$$Q = Q_0 = C_0 V_0 = (10.0\,\text{pF})(30.0\,\text{V}) = 300\,\text{pC}.$$

The new voltage is therefore

$$V = \frac{Q}{C} = \frac{300\,\text{pC}}{25.0\,\text{pF}} = 12.0\,\text{V}.$$

So inserting the dielectric increases the capacitance while reducing the voltage for the same stored charge.

Proposition 2.3.3 Main capacitor relations for a dielectric that completely fills the gap

Let C_0, Q_0, V_0, E_0, and U_0 denote the capacitance, plate-charge magnitude, potential difference, electric-field magnitude, and stored energy before insertion. Let C, Q, V, E, and U denote the corresponding quantities after a dielectric of constant κ fully fills the gap.
For an ideal parallel-plate capacitor,

$$C = \kappa C_0 = \kappa \varepsilon_0 \frac{A}{d}.$$

If the capacitor is *isolated* after charging (fixed Q), then

$$Q = Q_0, \qquad V = \frac{V_0}{\kappa}, \qquad E = \frac{E_0}{\kappa}, \qquad U = \frac{U_0}{\kappa}.$$

If the capacitor remains *connected to a battery* (fixed V), then

$$V = V_0, \qquad Q = \kappa Q_0, \qquad E = E_0, \qquad U = \kappa U_0.$$

In both cases, the dielectric increases capacitance by the factor κ. What changes is which other quantity the circuit constraint forces to stay constant.

Question 15: Worked AP-style problem

A vacuum parallel-plate capacitor has plate area

$$A = 1.50 \times 10^{-2}\,\text{m}^2$$

and plate separation

$$d = 1.00 \times 10^{-3}\,\text{m}.$$

It remains connected to a battery that maintains a potential difference

$$V_0 = 12.0\,\text{V}.$$

A dielectric with dielectric constant

$$\kappa = 3.00$$

is inserted so that it completely fills the gap. Take

$$\varepsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}.$$

Find:

(a) the initial capacitance C_0,

(b) the final capacitance C,

(c) the initial and final plate-charge magnitudes Q_0 and Q,

(d) the electric-field magnitude between the plates before and after insertion, and

(e) the initial and final stored energies U_0 and U.

Solution: Because the dielectric fully fills the gap, the capacitance increases by the factor κ:

$$C = \kappa C_0.$$

Because the battery remains connected, the voltage stays fixed at

$$V = V_0 = 12.0\,\text{V}.$$

For part (a), first find the initial vacuum capacitance:

$$C_0 = \varepsilon_0 \frac{A}{d}.$$

Substitute the given values:

$$C_0 = (8.85 \times 10^{-12}) \frac{1.50 \times 10^{-2}}{1.00 \times 10^{-3}}\,\text{F}.$$

The geometry factor is

$$\frac{1.50 \times 10^{-2}}{1.00 \times 10^{-3}} = 15.0,$$

so

$$C_0 = (8.85 \times 10^{-12})(15.0)\,\text{F} = 1.33 \times 10^{-10}\,\text{F}.$$

For part (b), multiply by $\kappa = 3.00$:

$$C = \kappa C_0 = (3.00)(1.33 \times 10^{-10}\,\text{F}) = 3.98 \times 10^{-10}\,\text{F}.$$

For part (c), before insertion the plate-charge magnitude is

$$Q_0 = C_0 V_0.$$

Thus,

$$Q_0 = (1.33 \times 10^{-10}\,\text{F})(12.0\,\text{V}) = 1.59 \times 10^{-9}\,\text{C}.$$

After insertion, the voltage is unchanged but the capacitance is larger, so

$$Q = CV = (3.98 \times 10^{-10}\,\text{F})(12.0\,\text{V}) = 4.78 \times 10^{-9}\,\text{C}.$$

This is also consistent with

$$Q = \kappa Q_0 = (3.00)(1.59 \times 10^{-9}\,\text{C}) = 4.78 \times 10^{-9}\,\text{C}.$$

For part (d), the electric-field magnitude in an ideal parallel-plate capacitor is related to the maintained potential difference by

$$E = \frac{V}{d}.$$

Before insertion,

$$E_0 = \frac{V_0}{d} = \frac{12.0\,\text{V}}{1.00 \times 10^{-3}\,\text{m}} = 1.20 \times 10^{4}\,\text{V/m}.$$

Because the battery keeps the same voltage across the same plate spacing, the final field magnitude is the same:

$$E = \frac{V}{d} = \frac{12.0\,\text{V}}{1.00 \times 10^{-3}\,\text{m}} = 1.20 \times 10^{4}\,\text{V/m}.$$

For part (e), the initial stored energy is

$$U_0 = \frac{1}{2}C_0 V_0^2.$$

Substitute the values:

$$U_0 = \frac{1}{2}(1.33 \times 10^{-10})(12.0)^2 \text{ J}.$$

Since

$$(12.0)^2 = 144,$$

we get

$$U_0 = \frac{1}{2}(1.33 \times 10^{-10})(144) \text{ J} = 9.56 \times 10^{-9} \text{ J}.$$

After insertion,

$$U = \frac{1}{2}CV^2 = \frac{1}{2}(3.98 \times 10^{-10})(12.0)^2 \text{ J} = 2.87 \times 10^{-8} \text{ J}.$$

Equivalently,

$$U = \kappa U_0 = (3.00)(9.56 \times 10^{-9} \text{ J}) = 2.87 \times 10^{-8} \text{ J}.$$

Therefore,

$$C_0 = 1.33 \times 10^{-10} \text{ F}, \qquad C = 3.98 \times 10^{-10} \text{ F},$$
$$Q_0 = 1.59 \times 10^{-9} \text{ C}, \qquad Q = 4.78 \times 10^{-9} \text{ C},$$
$$E_0 = E = 1.20 \times 10^4 \text{ V/m},$$

and

$$U_0 = 9.56 \times 10^{-9} \text{ J}, \qquad U = 2.87 \times 10^{-8} \text{ J}.$$

2.4 Direct-Current Circuits

This unit develops the circuit viewpoint for electric charge motion. In AP Physics C: Electricity and Magnetism, the central idea is that steady currents through resistive elements obey precise relationships between current, voltage, and resistance. The treatment is restricted to direct-current (DC) circuits — configurations where sources are ideal constant-voltage batteries and fields have settled so that charge flows at a constant rate.

The flow begins with the microscopic description of current, including drift velocity and current density, then introduces resistance, resistivity, and Ohm's law at both macroscopic and microscopic levels. It continues with electric power dissipation in resistors, then builds up to multi-resistor circuits through equivalent resistance for series and parallel combinations. Kirchhoff's junction and loop rules are introduced as the systematic tool for analyzing complex networks. RC transients with a time constant and internal resistance, which affects real batteries and measurement devices, round out the unit.

2.4.1 Current, Drift Velocity, and Current Density

This subsection connects the macroscopic current I in a wire to the microscopic motion of charge carriers through the current density $\vec{J}$ and the drift velocity $\vec{v}_d$.

Definition 2.4.1: Current, drift velocity, and current density

Let Δq denote the net charge that crosses a chosen surface in a time interval Δt. The *electric current* through that surface is

$$I = \frac{\Delta q}{\Delta t}$$

in the limit of very small time intervals.

Let $\vec{v}_d$ denote the average drift velocity of the charge carriers, let n denote the number of carriers per unit volume, and let q denote the charge of each carrier. The *current density* $\vec{J}$ is the vector that describes current per unit area, with direction defined by the motion of positive charge. For an oriented surface element $d\vec{A}$,

$$dI = \vec{J} \cdot d\vec{A}.$$

Thus $\vec{J}$ links the local flow of charge to the total current through a cross section.

Note:-

Conventional current is defined to point in the direction that positive charges would move. Therefore $\vec{J}$ points in the conventional-current direction. In a metal wire, the mobile charge carriers are electrons, so $q = -e$ and the electron drift velocity $\vec{v}_d$ points opposite to $\vec{J}$. If the carriers were positive instead, then $\vec{v}_d$ and $\vec{J}$ would point in the same direction.

Proposition 2.4.1 Microscopic-macroscopic current relations

Let n denote the carrier number density, let q denote the charge of each carrier, let $\vec{v}_d$ denote the drift velocity, and let S be a surface with oriented area element $d\vec{A}$. Then the current density is

$$\vec{J} = nq\vec{v}_d.$$

The total current through S is

$$I = \iint_S \vec{J} \cdot d\vec{A}.$$

If $\vec{J}$ is uniform across a flat cross section of area A and parallel to the area normal, then

$$I = JA \qquad \text{and} \qquad J = \frac{I}{A}.$$

For a straight wire with uniform carrier density, the corresponding magnitude relation is

$$I = n|q|Av_d,$$

where $v_d = |\vec{v}_d|$.

Question 16: Worked AP-style problem

A long copper wire carries a steady current

$$I = 3.0\,\text{A}$$

to the right. The wire has radius

$$r = 0.80\,\text{mm} = 8.0 \times 10^{-4}\,\text{m},$$

so its cross-sectional area is $A = \pi r^2$. Assume the conduction-electron number density is

$$n = 8.5 \times 10^{28}\,\text{m}^{-3},$$

and the charge of each electron is

$$q_e = -1.60 \times 10^{-19}\,\text{C}.$$

Let $+\hat{\imath}$ point to the right.
Find:

(a) the current density magnitude J,

(b) the current density vector $\vec{J}$, and

(c) the electron drift velocity vector $\vec{v}_d$.

Solution: First compute the wire's cross-sectional area:

$$A = \pi r^2 = \pi(8.0 \times 10^{-4}\,\mathrm{m})^2.$$

Since

$$(8.0 \times 10^{-4})^2 = 6.4 \times 10^{-7},$$

we get

$$A = \pi(6.4 \times 10^{-7})\,\mathrm{m}^2 = 2.01 \times 10^{-6}\,\mathrm{m}^2.$$

For part (a), use

$$J = \frac{I}{A}.$$

Then

$$J = \frac{3.0\,\mathrm{A}}{2.01 \times 10^{-6}\,\mathrm{m}^2} = 1.49 \times 10^6\,\mathrm{A/m^2}.$$

For part (b), the current is to the right, so conventional current and $\vec{J}$ point to the right:

$$\vec{J} = (1.49 \times 10^6\,\mathrm{A/m^2})\hat{\imath}.$$

For part (c), use the microscopic relation

$$\vec{J} = nq_e\vec{v}_d.$$

Solve for the drift velocity:

$$\vec{v}_d = \frac{\vec{J}}{nq_e}.$$

Because q_e is negative, $\vec{v}_d$ must point opposite to $\vec{J}$, so it points left. Its magnitude is

$$v_d = \frac{J}{n|q_e|}.$$

Substitute the values:

$$v_d = \frac{1.49 \times 10^6}{(8.5 \times 10^{28})(1.60 \times 10^{-19})}\,\mathrm{m/s}.$$

The denominator is

$$(8.5 \times 10^{28})(1.60 \times 10^{-19}) = 1.36 \times 10^{10},$$

so

$$v_d = \frac{1.49 \times 10^6}{1.36 \times 10^{10}}\,\mathrm{m/s} = 1.10 \times 10^{-4}\,\mathrm{m/s}.$$

Therefore the drift velocity vector is

$$\vec{v}_d = -(1.10 \times 10^{-4}\,\mathrm{m/s})\hat{\imath}.$$

Therefore,

$$J = 1.49 \times 10^6\,\mathrm{A/m^2}, \qquad \vec{J} = (1.49 \times 10^6\,\mathrm{A/m^2})\hat{\imath},$$
$$\vec{v}_d = -(1.10 \times 10^{-4}\,\mathrm{m/s})\hat{\imath}.$$

2.4.2 Resistance, Resistivity, and Ohm's Law

This subsection introduces resistance and resistivity, connects the macroscopic Ohm's law $V = IR$ to the microscopic relation $\vec{J} = \sigma\vec{E}$, and derives the geometric expression $R = \rho L/A$ for a uniform conductor.

Definition 2.4.2: Resistance

Let a conducting element have a potential difference V across its ends and carry a steady current I. The *resistance* R of the element is

$$R = \frac{V}{I}.$$

The SI unit of resistance is the *ohm*, denoted Ω, where $1\,\Omega = 1\,\mathrm{V/A}$.

Definition 2.4.3: Resistivity, conductivity

The *resistivity* ρ of a material is an intrinsic property that quantifies how strongly that material opposes the flow of electric current. The *conductivity* σ is the reciprocal of resistivity:

$$\sigma = \frac{1}{\rho}.$$

The SI unit of resistivity is the ohm-meter ($\Omega\cdot$m). The SI unit of conductivity is $(\Omega\cdot\mathrm{m})^{-1}$, also called siemens per meter (S/m).

Note:-

A perfect conductor has $\rho = 0$ and $\sigma \to \infty$. A perfect insulator has $\rho \to \infty$ and $\sigma \to 0$. Metals have very low resistivity (typically $10^{-8}\,\Omega\cdot$m); good insulators like glass have resistivity on the order of $10^{10}\,\Omega\cdot$m or higher.

Theorem 2.4.1 Microscopic Ohm's law

Let $\vec{E}$ denote the electric field inside a conducting material and let $\vec{J}$ denote the resulting current density at the same point. If the material is an *ohmic conductor* – one for which ρ is independent of the magnitude of $\vec{E}$ – then at every point inside the material,

$$\vec{J} = \sigma\vec{E} \qquad \text{or equivalently} \qquad \vec{E} = \rho\vec{J},$$

where $\sigma = 1/\rho$ is the conductivity.

Microscopic Ohm's law from the macroscopic form: Consider a straight wire of uniform cross-sectional area A and length L, made of a material with resistivity ρ. Suppose a potential difference V is applied across the ends, producing a uniform field $\vec{E}$ along the wire and a uniform current density $\vec{J}$.

From the macroscopic definition of resistance, $R = V/I$. For a uniform wire, the field and current density are related to the macroscopic quantities by $E = V/L$ and $J = I/A$. Substituting these into the resistance formula gives

$$R = \frac{EL}{JA}.$$

The resistance of a uniform wire is also known from experiment and geometry to be $R = \rho L/A$. Equating the two expressions for R yields

$$\frac{\rho L}{A} = \frac{EL}{JA},$$

which simplifies to $\rho J = E$, or $\vec{E} = \rho\vec{J}$. This is the microscopic form of Ohm's law. The same relation holds pointwise even if the field and current density vary spatially, because resistivity is a local material property. 😄

Corollary 2.4.1 Ohm's law is empirical

Not all materials obey Ohm's law. Semiconductors, diodes, and superconductors are non-ohmic: their I-V characteristic is not linear. The microscopic relation $\vec{J} = \sigma\vec{E}$ holds only for ohmic conductors where σ is independent of $\vec{E}$.

Proposition 2.4.2 Resistance of a uniform conductor

A straight wire of length L, uniform cross-sectional area A, and resistivity ρ has resistance

$$R = \rho\,\frac{L}{A}.$$

This relation follows from combining $V = IR$ with the microscopic Ohm's law $\vec{E} = \rho\vec{J}$ applied to a geometry where $\vec{E}$ and $\vec{J}$ are uniform and parallel to the wire axis.

Note:-

The resistance of a uniform wire increases linearly with length and decreases inversely with cross-sectional area. This is the electrical analogue of fluid flow through a pipe: a longer pipe gives more resistance, and a wider pipe gives less.

Theorem 2.4.2 Temperature dependence of resistivity

For many materials, over a limited temperature range, the resistivity varies approximately as

$$\rho = \rho_0\left[1 + \alpha(T - T_0)\right],$$

where ρ_0 is the resistivity at a reference temperature T_0, and α is the *temperature coefficient of resistivity* for that material. The SI unit of α is K^{-1} (or equivalently $^\circ\mathrm{C}^{-1}$). For most metals, $\alpha > 0$, so resistivity increases with temperature.

Example 2.4.1 (Illustrative example)

A copper wire and an aluminum wire of the same length and cross-sectional area are connected to the same potential difference. Since copper has a lower resistivity than aluminum, the copper wire will carry more current and dissipate less power. The ratio of their resistivities at room temperature is approximately $\rho_{\mathrm{Cu}}/\rho_{\mathrm{Al}} \approx 1.7\times10^{-8}/2.8\times10^{-8} \approx 0.61$.

Question 17: Worked example

A cylindrical copper wire of length

$$L = 50\,\mathrm{m}$$

and radius

$$r = 1.0\,\mathrm{mm} = 1.0\times10^{-3}\,\mathrm{m}$$

has a potential difference

$$V = 10\,\mathrm{V}$$

applied across its ends. The resistivity of copper at room temperature is

$$\rho = 1.7\times10^{-8}\,\Omega\cdot\mathrm{m}.$$

Assume the conduction-electron number density of copper is

$$n = 8.5\times10^{28}\,\mathrm{m}^{-3},$$

the elementary charge is $e = 1.60\times10^{-19}\,\mathrm{C}$, and the wire is uniform. Let the current flow from the high-potential end toward the low-potential end, and let $+\hat{\imath}$ point in the direction of the current.
Find:

(a) the resistance R of the wire,

(b) the current I through the wire,

(c) the current density magnitude J and vector $\vec{J}$,

(d) the drift velocity magnitude v_d of the electrons, and

(e) the power dissipated in the wire.

Solution: **Part (a).** The cross-sectional area of the cylindrical wire is

$$A = \pi r^2 = \pi(1.0 \times 10^{-3}\,\mathrm{m})^2 = \pi \times 1.0 \times 10^{-6}\,\mathrm{m}^2 = 3.14 \times 10^{-6}\,\mathrm{m}^2.$$

The resistance is

$$R = \rho \frac{L}{A}.$$

Substitute the values:

$$R = \left(1.7 \times 10^{-8}\,\Omega{\cdot}\mathrm{m}\right) \frac{50\,\mathrm{m}}{3.14 \times 10^{-6}\,\mathrm{m}^2}.$$

Compute the numerator:

$$\left(1.7 \times 10^{-8}\right)(50) = 8.5 \times 10^{-7}\,\Omega{\cdot}\mathrm{m}^2.$$

Then

$$R = \frac{8.5 \times 10^{-7}}{3.14 \times 10^{-6}}\,\Omega = 0.271\,\Omega.$$

Part (b). Ohm's law gives the current:

$$I = \frac{V}{R} = \frac{10\,\mathrm{V}}{0.271\,\Omega} = 36.9\,\mathrm{A}.$$

Part (c). The current density magnitude is

$$J = \frac{I}{A} = \frac{36.9\,\mathrm{A}}{3.14 \times 10^{-6}\,\mathrm{m}^2} = 1.18 \times 10^7\,\mathrm{A/m}^2.$$

The current flows in the $+\hat{\imath}$ direction (from high to low potential), so the current density vector is

$$\vec{J} = \left(1.18 \times 10^7\,\mathrm{A/m}^2\right)\hat{\imath}.$$

Part (d). The drift velocity of electrons relates to the current density by

$$\vec{J} = n q_e \vec{v}_d,$$

where $q_e = -e$ is the charge of an electron. Since the electrons are negatively charged, their drift velocity is opposite to the current direction. The magnitude is

$$v_d = \frac{J}{ne} = \frac{1.18 \times 10^7\,\mathrm{A/m}^2}{(8.5 \times 10^{28}\,\mathrm{m}^{-3})(1.60 \times 10^{-19}\,\mathrm{C})}.$$

The denominator is

$$\left(8.5 \times 10^{28}\right)\left(1.60 \times 10^{-19}\right) = 1.36 \times 10^{10}\,\mathrm{C/m}^3,$$

so

$$v_d = \frac{1.18 \times 10^7}{1.36 \times 10^{10}}\,\mathrm{m/s} = 8.69 \times 10^{-4}\,\mathrm{m/s}.$$

Electrons drift opposite to $\vec{J}$, so $\vec{v}_d = -\left(8.69 \times 10^{-4}\,\mathrm{m/s}\right)\hat{\imath}$.

Part (e). The power dissipated in the wire is

$$P = IV = (36.9\,\mathrm{A})(10\,\mathrm{V}) = 369\,\mathrm{W}.$$

Equivalently, $P = I^2R = (36.9\,\text{A})^2(0.271\,\Omega) = 369\,\text{W}$, or $P = V^2/R = (10\,\text{V})^2/(0.271\,\Omega) = 369\,\text{W}$. All three give the same result.

Therefore,

$$R = 0.271\,\Omega, \qquad I = 36.9\,\text{A}, \qquad J = 1.18 \times 10^7\,\text{A/m}^2,$$

$$\vec{J} = \left(1.18 \times 10^7\,\text{A/m}^2\right)\hat{\imath}, \qquad v_d = 8.69 \times 10^{-4}\,\text{m/s}, \qquad P = 369\,\text{W}.$$

2.4.3 Electric Power and Dissipation

Electric power quantifies the rate at which electrical energy is transferred or dissipated in a circuit element. When a charge q moves through a potential difference ΔV, its electric potential energy changes by $\Delta U = q\,\Delta V$. The power delivered to (or dissipated by) a circuit element is the rate of this energy transfer.

Definition 2.4.4: Electric power

Let a circuit element have a potential difference ΔV across it and carry a current I through it. The *electric power* delivered to the element is

$$P = I\,\Delta V.$$

The SI unit of power is the watt ($1\,\text{W} = 1\,\text{J/s} = 1\,\text{A}\cdot\text{V}$). When the element is a resistor, electrical energy is dissipated as thermal energy (Joule heating).

Note:-

Power is the rate of energy transfer. A battery *supplies* power to the circuit, while resistors *dissipate* it. In a steady circuit, the total power supplied equals the total power dissipated.

Example 2.4.2 (Illustrative example)

A 60 W incandescent lightbulb is connected to a 120 V household outlet. Find (a) the current through the bulb and (b) the resistance of its filament.

(a) From $P = I\,\Delta V$,

$$I = \frac{P}{\Delta V} = \frac{60\,\text{W}}{120\,\text{V}} = 0.50\,\text{A}.$$

(b) From Ohm's law,

$$R = \frac{\Delta V}{I} = \frac{120\,\text{V}}{0.50\,\text{A}} = 240\,\Omega.$$

Note:-

The I^2R form shows why transmission lines use very high voltages: for a fixed power $P = IV$, raising the voltage lowers the current, and since resistive losses scale as I^2R, the dissipation drops dramatically.

Theorem 2.4.3 Microscopic power density

In a continuous medium, the local rate of energy dissipation per unit volume is the dot product of the current density $\vec{J}$ and the electric field $\vec{E}$ at that point:

$$p = \vec{J} \cdot \vec{E}.$$

The total power delivered to a volume $\mathcal{V}$ is the integral

$$P = \iiint_{\mathcal{V}} \vec{J} \cdot \vec{E}\, d\tau.$$

Microscopic to macroscopic: Consider a straight wire of length L and cross-sectional area A, with a uniform electric field $\vec{E}$ along its axis and a uniform current I. The field is related to the potential difference by $\Delta V = E\,L$. The current density is $J = I/A$, and by Ohm's law $E = \rho\,J$ (where ρ is the resistivity). The total power dissipated is

$$P = \iiint_{\mathcal{V}} \vec{J} \cdot \vec{E}\, d\tau = J\, E\, (A\, L)$$

since $\vec{J}$ and $\vec{E}$ are parallel and uniform. Substituting $J = I/A$ and $E = \Delta V/L$,

$$P = \left(\frac{I}{A}\right)\left(\frac{\Delta V}{L}\right)(A\, L) = I\, \Delta V,$$

recovering the macroscopic expression. Alternatively, using $E = \rho\, J = \rho\,(I/A)$ and $R = \rho\, L/A$,

$$P = J\, E\, (A\, L) = \frac{I^2}{A^2} \cdot \rho \frac{I}{A} \cdot A\, L = I^2 \frac{\rho\, L}{A} = I^2 R.$$ 😆

Question 18: Worked example

A resistor is connected across a 12.0 V battery. The current through the resistor is measured to be 2.00 A. Find:

(a) the power dissipated by the resistor,

(b) the resistance of the resistor, and

(c) the total energy dissipated as heat over a time interval of 5.00 min.

Solution: **Part (a).** The power dissipated by the resistor is given by

$$P = I\, \Delta V.$$

Substitute the given values:

$$P = (2.00\,\text{A})(12.0\,\text{V}) = 24.0\,\text{W}.$$

Part (b). By Ohm's law,

$$R = \frac{\Delta V}{I}.$$

Substitute the values:

$$R = \frac{12.0\,\text{V}}{2.00\,\text{A}} = 6.00\,\Omega.$$

As a check, verify using $P = I^2 R$:

$$P = (2.00\,\text{A})^2(6.00\,\Omega) = 4.00 \times 6.00 = 24.0\,\text{W},$$

which agrees with part (a).

Part (c). The total energy dissipated is power multiplied by time:

$$E = P\, t.$$

Convert the time to seconds:

$$t = 5.00\,\text{min} \times 60.0\,\text{s/min} = 300\,\text{s}.$$

Then

$$E = (24.0\,\text{W})(300\,\text{s}) = 7200\,\text{J}.$$

Alternatively, using $E = I\, \Delta V\, t$ directly:

$$E = (2.00\,\text{A})(12.0\,\text{V})(300\,\text{s}) = 7200\,\text{J}.$$

Note:-

Energy can also be written as $E = I^2 R t$ or $E = (\Delta V)^2 t/R$. All three are equivalent and follow from $E = P\, t$ together with Ohm's law.

2.4.4 Equivalent Resistance of Series and Parallel Circuits

This subsection defines equivalent resistance for resistive networks, derives the series and parallel combination rules from Kirchhoff's laws and Ohm's law, and shows how to reduce compound circuits step by step to a single equivalent resistor.

Definition 2.4.5: Equivalent resistance

Consider a network of resistors connected to an ideal battery of emf $\mathcal{E}$. The *equivalent resistance* R_{eq} of the network is defined by

$$R_{eq} = \frac{\mathcal{E}}{I_{total}},$$

where I_{total} is the total current delivered by the battery. The equivalent resistance is the resistance of a single resistor that would draw the same current from the same battery as the entire network.

Note:-

Equivalent resistance is a bookkeeping device: it replaces an entire resistive sub-network by a single resistor whose effect on the rest of the circuit is identical. The replacement is always done between two terminals, and it preserves the I–V relationship at those terminals.

Definition 2.4.6: Series combination

Two (or more) resistors are in *series* when they share exactly one common node and no other element is connected to that node. Equivalently, the same current I flows through each resistor in a series chain, and the total voltage is the sum of individual voltage drops:

$$V = V_1 + V_2 + \cdots + V_n.$$

The equivalent resistance of n resistors in series is

$$R_{eq} = R_1 + R_2 + \cdots + R_n = \sum_{i=1}^{n} R_i.$$

Theorem 2.4.4 Voltage divider rule

Consider two resistors R_1 and R_2 in series connected to a potential difference V_{in}. The voltage across R_2 alone is

$$V_{out} = V_{R_2} = V_{in}\frac{R_2}{R_1 + R_2}.$$

This relation follows from Ohm's law and the fact that the same current $I = V_{in}/(R_1 + R_2)$ flows through both resistors.

Note:-

The voltage divider distributes the input voltage proportionally to each resistance. A larger resistance drops more voltage. This rule is the electrical analogue of the weighted average: V_{R_i} is the fraction of V_{in} carried by R_i.

Definition 2.4.7: Parallel combination

Two (or more) resistors are in *parallel* when they are connected between the same two nodes, so the potential difference V across each is identical. The total current splits among the branches:

$$I_{\text{total}} = I_1 + I_2 + \cdots + I_n.$$

The equivalent resistance of n resistors in parallel is

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

For two resistors in parallel, this simplifies to

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}.$$

Theorem 2.4.5 Current divider rule

Consider two resistors R_1 and R_2 in parallel connected to a total current I_{total}. The current through R_1 is

$$I_1 = I_{\text{total}} \frac{R_2}{R_1 + R_2}.$$

This follows from KCL and Ohm's law: the common voltage is $V = I_{\text{total}}/G_{\text{eq}}$ where $G_{\text{eq}} = 1/R_1 + 1/R_2$, and $I_1 = V/R_1$.

Note:-

Current divides inversely to resistance: the smaller resistor carries more current. In the limit $R_1 \ll R_2$, essentially all current flows through R_1; we say R_1 *shunts* R_2.

Proposition 2.4.3 Series and parallel equivalent resistance

For n resistors, the equivalent resistance satisfies

$$\text{Series:} \quad R_{\text{eq}} = \sum_{i=1}^{n} R_i \qquad \text{and} \qquad \text{Parallel:} \quad \frac{1}{R_{\text{eq}}} = \sum_{i=1}^{n} \frac{1}{R_i}.$$

The current through and voltage across each individual resistor are obtained by applying Ohm's law $V_i = I_i R_i$ once I_i or V_i is known from the reduction analysis.

Series combination: Kirchhoff's voltage law (the loop rule) states that the sum of potential differences around any closed loop is zero. For two resistors in series with a battery of emf $\mathcal{E}$,

$$\mathcal{E} - V_1 - V_2 = 0,$$

so $V_1 + V_2 = \mathcal{E}$. Since the same current I flows through both resistors, $V_1 = IR_1$ and $V_2 = IR_2$. Substituting gives $I(R_1 + R_2) = \mathcal{E}$, or $R_{\text{eq}} = \mathcal{E}/I = R_1 + R_2$. The result extends immediately to n resistors by induction. 😊

Parallel combination: Kirchhoff's current law (the node rule) states that the sum of currents entering a node equals the sum of currents leaving. At the top node of the parallel connection, $I_{\text{total}} = I_1 + I_2 + \cdots + I_n$. Since each branch has the same voltage V, we have $I_i = V/R_i$. Thus

$$I_{\text{total}} = V \sum_{i=1}^{n} \frac{1}{R_i}.$$

But $I_{\text{total}} = V/R_{\text{eq}}$, so $1/R_{\text{eq}} = \sum 1/R_i$. 😊

Corollary 2.4.2 Bounds on equivalent resistance

For any combination of series and parallel resistors, the equivalent resistance of a parallel block is always less than the smallest resistance in that block:

$$R_{\text{parallel}} < \min_i(R_i).$$

Conversely, the equivalent resistance of a series chain is always greater than the largest individual resistance. These bounds follow from the positivity of all resistances and provide a useful sanity check on computed results.

Example 2.4.3 (Illustrative example)

When resistors are identical, the formulas simplify. n identical resistors of resistance R in series give $R_{\text{eq}} = nR$. In parallel, $R_{\text{eq}} = R/n$. Thus three $12\,\Omega$ resistors in parallel give $R_{\text{eq}} = 4\,\Omega$, while in series they give $36\,\Omega$.

Note:-

The strategy for analysing a mixed (compound) circuit is: (1) identify the innermost series or parallel groups. (2) Replace each group by its equivalent resistance. (3) Repeat until the entire network is reduced to a single resistor R_{eq}. (4) Use $I_{\text{total}} = \mathcal{E}/R_{\text{eq}}$ to find the battery current, then work backwards through the reduction steps to find individual branch currents and voltages.

Note:-

On the AP Physics C E&M exam, equivalent resistance problems test your ability to correctly identify series versus parallel connections and to apply Kirchhoff's laws systematically. The most common errors are misidentifying which elements share the same current (series) and which share the same voltage (parallel). Always trace the current paths and label the nodes to verify your classification.

Question 19: Worked example

An ideal battery of emf

$$\mathcal{E} = 24\,\text{V}$$

is connected to a resistive network consisting of four resistors arranged as follows. Resistors $R_1 = 2.0\,\Omega$ and $R_2 = 4.0\,\Omega$ are connected in series. This series combination is connected in parallel with resistor $R_3 = 6.0\,\Omega$. Finally, resistor $R_4 = 3.0\,\Omega$ is connected in series with this entire parallel block, and the combination is connected to the battery.
Find:

(a) the total equivalent resistance R_{eq} of the network,

(b) the total current I_{total} delivered by the battery,

(c) the voltage drop across each resistor, and

(d) the current through each resistor.

Solution: **Part (a).** Begin by reducing the network from the inside out. Resistors R_1 and R_2 are in series, so their combined resistance is

$$R_{12} = R_1 + R_2 = 2.0\,\Omega + 4.0\,\Omega = 6.0\,\Omega.$$

This series combination is in parallel with $R_3 = 6.0\,\Omega$. The equivalent resistance of the parallel block is

$$\frac{1}{R_p} = \frac{1}{R_{12}} + \frac{1}{R_3} = \frac{1}{6.0\,\Omega} + \frac{1}{6.0\,\Omega} = \frac{2}{6.0\,\Omega} = \frac{1}{3.0\,\Omega}.$$

Hence $R_p = 3.0\,\Omega$.

The parallel block is in series with R_4, so the total equivalent resistance is

$$R_{\text{eq}} = R_p + R_4 = 3.0\,\Omega + 3.0\,\Omega = 6.0\,\Omega.$$

Part (b). Ohm's law gives the total current from the battery:

$$I_{\text{total}} = \frac{\mathcal{E}}{R_{\text{eq}}} = \frac{24\,\text{V}}{6.0\,\Omega} = 4.0\,\text{A}.$$

Parts (c) and (d). Now work backwards through the reduction. The current through R_4 equals the total current, since R_4 is in series with the battery:

$$I_4 = I_{\text{total}} = 4.0\,\text{A}.$$

The voltage drop across R_4 is

$$V_4 = I_4 R_4 = (4.0\,\text{A})(3.0\,\Omega) = 12\,\text{V}.$$

The remaining voltage appears across the parallel block:

$$V_p = \mathcal{E} - V_4 = 24\,\text{V} - 12\,\text{V} = 12\,\text{V}.$$

Since the resistors R_1, R_2, and R_3 are all connected to the parallel block, the voltage across each is $V_p = 12\,\text{V}$ (for R_3 directly) and $V_p = 12\,\text{V}$ (for the series pair R_1–R_2).

The current through R_3 is

$$I_3 = \frac{V_p}{R_3} = \frac{12\,\text{V}}{6.0\,\Omega} = 2.0\,\text{A}.$$

The current through the R_1–R_2 branch is

$$I_{12} = \frac{V_p}{R_{12}} = \frac{12\,\text{V}}{6.0\,\Omega} = 2.0\,\text{A}.$$

Since R_1 and R_2 are in series, the same current flows through both:

$$I_1 = I_2 = I_{12} = 2.0\,\text{A}.$$

The individual voltage drops are

$$V_1 = I_1 R_1 = (2.0\,\text{A})(2.0\,\Omega) = 4.0\,\text{V},$$
$$V_2 = I_2 R_2 = (2.0\,\text{A})(4.0\,\Omega) = 8.0\,\text{V}.$$

Checks:

- KVL on the left loop: $V_1 + V_2 = 4.0\,\text{V} + 8.0\,\text{V} = 12\,\text{V} = V_p$.
- KCL at the junction: $I_{12} + I_3 = 2.0\,\text{A} + 2.0\,\text{A} = 4.0\,\text{A} = I_{\text{total}}$.
- KVL on the full loop: $V_p + V_4 = 12\,\text{V} + 12\,\text{V} = 24\,\text{V} = \mathcal{E}$.

Therefore,

$$R_{\text{eq}} = 6.0\,\Omega,$$
$$I_{\text{total}} = 4.0\,\text{A},$$
$$(V_1, V_2, V_3, V_4) = (4.0\,\text{V}, 8.0\,\text{V}, 12\,\text{V}, 12\,\text{V}),$$
$$(I_1, I_2, I_3, I_4) = (2.0\,\text{A}, 2.0\,\text{A}, 2.0\,\text{A}, 4.0\,\text{A}).$$

2.4.5 Kirchhoff's Junction and Loop Rules

This subsection introduces Kirchhoff's two rules for analyzing electric circuits. The junction rule (Kirchhoff's current law) expresses charge conservation at circuit nodes. The loop rule (Kirchhoff's voltage law) expresses energy conservation around any closed loop. Together they provide a systematic method for finding unknown currents in multi-loop circuits.

Definition 2.4.8: Electric circuit junction (node)

A *junction* (or *node*) is a point in a circuit where three or more conductors meet. The current in each conductor leading to the junction is called a *branch current*.

Note:-

In a steady-state dc circuit the charge at any junction is constant: no charge accumulates at a node. This is the physical reason the junction rule holds.

Theorem 2.4.6 Kirchhoff's junction rule (KCL)

At any junction in a steady-state dc circuit, the sum of currents entering the junction equals the sum of currents leaving the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}. \tag{2.1}$$

Equivalently, the algebraic sum of all currents at a junction is zero:

$$\sum I = 0, \tag{2.2}$$

where currents entering the junction are taken as positive and currents leaving are taken as negative.

Calc 3 connection: The junction rule is the circuit analogue of the steady-state continuity equation $\nabla \cdot \vec{J} = 0$ (charge conservation with no time-varying charge density). Integrating $\nabla \cdot \vec{J} = 0$ over a small volume enclosing the junction gives $\oiint \vec{J} \cdot d\vec{A} = 0$, which is exactly $\sum I = 0$.

Definition 2.4.9: Electric circuit loop

A *loop* is any closed path through a circuit, traversing a sequence of circuit elements and returning to the starting point.

Definition 2.4.10: Sign convention for traversing circuit elements

When applying the loop rule, traverse the loop in a chosen direction and assign potential changes as follows:

- *Resistor:* Traversing in the *same* direction as the assumed current gives a potential *drop* of IR (contribute $-IR$). Traversing *opposite* to the assumed current gives a potential *rise* of IR (contribute $+IR$).
- *Battery (emf):* Traversing from the *negative* terminal to the *positive* terminal gives a potential *rise* of $\mathcal{E}$ (contribute $+\mathcal{E}$). Traversing from *positive* to *negative* gives a potential *drop* of $\mathcal{E}$ (contribute $-\mathcal{E}$).

The assumed direction of each branch current must be declared before writing equations. If the solved value of a current is negative, the actual current flows opposite to the assumed direction.

Theorem 2.4.7 Kirchhoff's loop rule (KVL)

For any closed loop in a circuit, the algebraic sum of the potential differences across all elements in the loop is zero:

$$\sum \Delta V = 0. \tag{2.3}$$

Calc 3 connection: The loop rule follows from energy conservation: a test charge q that moves around a closed path and returns to its starting point must have zero net change in potential energy, so $\oint \vec{E} \cdot d\vec{\ell} = 0$. In electrostatics this is a consequence of $\vec{E}$ being a conservative field. More generally, in quasi-static circuits with no changing magnetic flux through the loop, Faraday's law gives $\nabla \times \vec{E} = -\partial \vec{B}/\partial t = 0$, so the integral over any closed loop vanishes.

Note:-

The loop rule holds for *any* closed loop, not just the obvious "mesh" loops of a circuit diagram. Any closed path through the elements counts. In practice one typically chooses the minimal loops (meshes) because they lead to the most economical system of equations.

Note:-

The sign of a solved current encodes direction. A negative result does *not* mean the current is unphysical — it simply means the actual current is opposite to the assumed direction. Always state the assumed direction and report the final direction clearly.

Proposition 2.4.4 Algorithm for solving multi-loop circuits

Given a multi-loop circuit with unknown branch currents:

1. Label every branch with a current variable and an assumed direction.
2. Identify all junctions. For N junctions, write $N - 1$ independent junction equations.
3. Choose enough independent loops (typically meshes) so that the total number of equations (junction + loop) equals the number of unknown currents.
4. Apply the loop rule to each chosen loop, using the sign conventions above.
5. Solve the resulting system of linear equations.
6. Check: any current with a negative value flows opposite to the assumed direction. Verify that all junction equations are satisfied.

The number of independent equations needed equals the number of unknown branch currents.

Example 2.4.4 (Illustrative example)

Consider a junction with three wires meeting. Current $I_1 = 3.0\,\text{A}$ enters the junction and current $I_2 = 1.5\,\text{A}$ leaves. The third branch carries current I_3. By the junction rule, $I_1 = I_2 + I_3$, so $I_3 = 1.5\,\text{A}$ leaves the junction.

Note:-

For circuits with a single battery and resistors in simple series or parallel, the equivalent-resistance method is faster. Kirchhoff's rules are needed whenever the circuit cannot be reduced to simple series/parallel combinations — for instance, when there are two or more batteries arranged in different branches, forming multiple loops.

Question 20: Worked example

Consider the two-loop circuit shown in the diagram below. The circuit consists of a left loop and a right loop sharing a common middle branch.

- The *left branch* contains a battery of emf $\mathcal{E}_1 = 12\,\text{V}$ (positive terminal up) in series with a resistor $R_1 = 4.0\,\Omega$.
- The *middle branch* contains a resistor $R_3 = 3.0\,\Omega$.
- The *right branch* contains a battery of emf $\mathcal{E}_2 = 6.0\,\text{V}$ (positive terminal up) in series with a resistor $R_2 = 6.0\,\Omega$.

Diagram description: Two rectangular loops share a vertical middle branch. The left vertical branch has the $12\,\text{V}$ battery (positive up) and $4.0\,\Omega$ resistor. The middle vertical branch has the $3.0\,\Omega$ resistor. The right vertical branch has the $6.0\,\text{V}$ battery (positive up) and $6.0\,\Omega$ resistor. All three vertical branches connect at top and bottom horizontal wires (ideal conductors with zero resistance).

Let the top junction be A and the bottom junction be B. Define three branch currents:

- I_1 flows *upward* through the left branch (from B to A).
- I_2 flows *upward* through the right branch (from B to A).

- I_3 flows *downward* through the middle branch (from A to B).

Assume these directions when applying Kirchhoff's rules.
Find:

(a) the junction equation at node A,

(b) the two independent loop equations (left loop and right loop),

(c) the values of all three currents I_1, I_2, and I_3, and

(d) the direction of each current (consistent with the assumed direction).

Solution: **Part (a). Junction equation.** At junction A, the currents I_1 and I_2 both enter (they flow upward from B to A in their respective branches). The current I_3 leaves A (it flows downward from A to B). By the junction rule:

$$I_1 + I_2 = I_3.$$

This is our first equation.

Part (b). Loop equations.

Left loop (traverse clockwise starting from junction B):

- Go up through the left branch, in the same direction as I_1: the battery $\mathcal{E}_1$ is traversed from $-$ to $+$, contributing $+\mathcal{E}_1 = +12\,\text{V}$. The resistor R_1 is traversed in the same direction as I_1, contributing $-I_1R_1 = -4.0\,I_1$.
- Go across the top wire from the left branch to the middle branch (ideal wire, $\Delta V = 0$).
- Go down through the middle branch, in the same direction as I_3: resistor R_3 is traversed in the direction of I_3, contributing $-I_3R_3 = -3.0\,I_3$.
- Go across the bottom wire back to B (ideal wire, $\Delta V = 0$).

Summing around the loop:

$$+12 - 4.0\,I_1 - 3.0\,I_3 = 0,$$

$$4.0\,I_1 + 3.0\,I_3 = 12.$$

Right loop (traverse clockwise starting from junction A):

- Go down through the right branch, in the *opposite* direction to I_2: the resistor R_2 is traversed opposite to I_2, contributing $+I_2R_2 = +6.0\,I_2$. The battery $\mathcal{E}_2$ is traversed from $+$ to $-$, contributing $-\mathcal{E}_2 = -6\,\text{V}$.
- Go across the bottom wire from right to middle (ideal wire, $\Delta V = 0$).
- Go up through the middle branch, in the *opposite* direction to I_3: resistor R_3 is traversed opposite to I_3, contributing $+I_3R_3 = +3.0\,I_3$.
- Go across the top wire back to A (ideal wire, $\Delta V = 0$).

Summing around the loop:

$$+6.0\,I_2 - 6 + 3.0\,I_3 = 0,$$

$$6.0\,I_2 + 3.0\,I_3 = 6.$$

Part (c). Solving for the currents. We have three equations:

$$I_3 = I_1 + I_2, \qquad \text{(junction)} \qquad (2.4)$$

$$4.0\,I_1 + 3.0\,I_3 = 12, \qquad \text{(left loop)} \qquad (2.5)$$

$$6.0\,I_2 + 3.0\,I_3 = 6. \qquad \text{(right loop)} \qquad (2.6)$$

Substitute equation (1) into equation (2):

$$4.0\,I_1 + 3.0\,(I_1 + I_2) = 12,$$

$$4.0\,I_1 + 3.0\,I_1 + 3.0\,I_2 = 12,$$

$$7.0\,I_1 + 3.0\,I_2 = 12.$$

Substitute equation (1) into equation (3):

$$6.0\,I_2 + 3.0\,(I_1 + I_2) = 6,$$

$$6.0\,I_2 + 3.0\,I_1 + 3.0\,I_2 = 6,$$

$$3.0\,I_1 + 9.0\,I_2 = 6.$$

Dividing by 3.0:

$$I_1 + 3.0\,I_2 = 2.0,$$

so

$$I_1 = 2.0 - 3.0\,I_2.$$

Substitute equation (3) into equation (2):

$$7.0\,(2.0 - 3.0\,I_2) + 3.0\,I_2 = 12,$$

$$14.0 - 21.0\,I_2 + 3.0\,I_2 = 12,$$

$$14.0 - 18.0\,I_2 = 12,$$

$$-18.0\,I_2 = -2.0,$$

$$I_2 = \frac{2.0}{18.0} = \frac{1}{9}\,\text{A}.$$

From equation (3):

$$I_1 = 2.0 - 3.0\left(\frac{1}{9}\right) = 2.0 - \frac{1}{3} = \frac{6}{3} - \frac{1}{3} = \frac{5}{3}\,\text{A}.$$

From the junction equation (1):

$$I_3 = I_1 + I_2 = \frac{5}{3} + \frac{1}{9} = \frac{15}{9} + \frac{1}{9} = \frac{16}{9}\,\text{A}.$$

Verification. Check against the left-loop equation:

$$4.0\left(\frac{5}{3}\right) + 3.0\left(\frac{16}{9}\right) = \frac{20}{3} + \frac{16}{3} = \frac{36}{3} = 12.$$

Check against the right-loop equation:

$$6.0\left(\frac{1}{9}\right) + 3.0\left(\frac{16}{9}\right) = \frac{6}{9} + \frac{48}{9} = \frac{54}{9} = 6.$$

Both are satisfied.

Part (d). Directions. All three currents are positive, confirming they flow in the assumed directions:

- $I_1 = 5/3\,\text{A}$ upward in the left branch,
- $I_2 = 1/9\,\text{A}$ upward in the right branch,
- $I_3 = 16/9\,\text{A}$ downward in the middle branch.

Therefore, the branch currents are:

$$I_1 = \frac{5}{3}\,\text{A}, \qquad I_2 = \frac{1}{9}\,\text{A}, \qquad I_3 = \frac{16}{9}\,\text{A},$$

all flowing in the assumed directions.

2.4.6 RC Transients and the Time Constant

This subsection introduces RC circuits – circuits containing resistors and capacitors – derives the first-order differential equation governing charge evolution during charging and discharging, solves it by separation of variables, and defines the time constant $\tau = RC$ as the characteristic timescale of the transient response.

Definition 2.4.11: RC circuit and transient response

A series *RC circuit* consists of a resistor R, a capacitor C, and (optionally) a battery of emf $\mathcal{E}$, all connected in a single closed loop. When the circuit is first connected (or when the battery is disconnected), the capacitor neither holds its initial charge nor its final charge instantaneously; instead, its charge evolves over time. This time-dependent behavior is called a *transient response*.

Note:-

At the instant a capacitor begins charging, it behaves like a short circuit: its voltage is zero and all of the battery voltage appears across the resistor. As the capacitor charges, its voltage increases and the current decreases. In the limit $t \to \infty$, the capacitor is fully charged to voltage $\mathcal{E}$ and the current drops to zero, so the capacitor acts like an open circuit.

Theorem 2.4.8 Charging a capacitor

A capacitor of capacitance C that is initially uncharged is connected in series at $t = 0$ with a resistor of resistance R and a battery of emf $\mathcal{E}$, forming a single-loop circuit. The charge on the capacitor at time t is

$$q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right),$$

and the current flowing through the resistor is

$$I(t) = \frac{\mathcal{E}}{R}\, e^{-t/RC}.$$

Here $q(0) = 0$ and $I(0) = \mathcal{E}/R$. As $t \to \infty$, $q \to C\mathcal{E}$ and $I \to 0$.

Derivation of charging equations: Apply Kirchhoff's voltage law around the loop. The potential drops across the resistor and capacitor sum to the battery emf:

$$\mathcal{E} - IR - \frac{q}{C} = 0,$$

where $I = \dfrac{dq}{dt}$ is the current (the rate at which charge accumulates on the capacitor). Substituting gives the first-order linear ODE

$$R\,\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}.$$

Divide by R:

$$\frac{dq}{dt} + \frac{1}{RC}\,q = \frac{\mathcal{E}}{R}.$$

This is a first-order linear ODE. Use the integrating factor $\mu(t) = e^{t/RC}$:

$$\frac{d}{dt}\left(q\,e^{t/RC}\right) = \frac{\mathcal{E}}{R}\,e^{t/RC}.$$

Integrate both sides from 0 to t, with $q(0) = 0$:

$$q(t)\,e^{t/RC} - q(0) = \frac{\mathcal{E}}{R}\int_0^t e^{t'/RC}\,dt' = \mathcal{E}C\left(e^{t/RC} - 1\right).$$

Solving for $q(t)$:

$$q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right).$$

The current is obtained from $I = dq/dt$:

$$I(t) = \frac{d}{dt}\left[C\mathcal{E}\left(1 - e^{-t/RC}\right)\right] = \frac{\mathcal{E}}{R}\, e^{-t/RC}.$$

The initial current is $I(0) = \mathcal{E}/R$, and as $t \to \infty$ the exponential vanishes, so $q \to C\mathcal{E}$ and $I \to 0$. ☺

Theorem 2.4.9 Discharging a capacitor

A capacitor of capacitance C that is initially charged to charge q_0 is connected at $t = 0$ with a resistor of resistance R, forming a single-loop circuit with no battery. The charge on the capacitor at time t is

$$q(t) = q_0\, e^{-t/RC},$$

and the rate of change of charge is

$$\frac{dq}{dt} = -\frac{q_0}{RC}\, e^{-t/RC}.$$

The negative sign indicates that charge is *decreasing*: the capacitor is discharging. The magnitude of the current through the resistor is

$$|I(t)| = \left|\frac{dq}{dt}\right| = \frac{q_0}{RC}\, e^{-t/RC}.$$

As $t \to \infty$, both $q \to 0$ and $I \to 0$.

Derivation of discharging equations: With no battery in the loop, Kirchhoff's voltage law gives

$$-\frac{q}{C} - IR = 0,$$

where I is the current through the resistor and q/C is the voltage across the capacitor. During discharge, charge flows off the capacitor, so $I = -dq/dt$ (current is positive while charge is decreasing). Substituting:

$$-\frac{q}{C} - R\left(-\frac{dq}{dt}\right) = 0,$$

which simplifies to

$$\frac{dq}{dt} = -\frac{q}{RC}.$$

Separate variables:

$$\frac{dq}{q} = -\frac{dt}{RC}.$$

Integrate from 0 to t, with $q(0) = q_0$:

$$\int_{q_0}^{q(t)} \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt', \qquad \ln\left(\frac{q(t)}{q_0}\right) = -\frac{t}{RC}.$$

Exponentiate:

$$q(t) = q_0\, e^{-t/RC}.$$

Differentiating gives the rate of change of charge:

$$\frac{dq}{dt} = -\frac{q_0}{RC}\, e^{-t/RC}.$$

The current flowing through the resistor (in the direction that discharges the capacitor) is $I = -dq/dt$, so

$$I(t) = \frac{q_0}{RC}\, e^{-t/RC}.$$

The voltage across the capacitor is $V_C = q/C = (q_0/C)e^{-t/RC}$, and the voltage across the resistor is $V_R = IR = (q_0/C)e^{-t/RC}$, so $V_R = V_C$ at every instant, consistent with the loop equation. ☺

Note:-

During charging, the current is positive and flows onto the positively charged plate of the capacitor. During discharging, the current flows off the capacitor, and the instantaneous current through the resistor has the same magnitude as the rate at which charge leaves the capacitor: $|dq/dt| = I$.

Definition 2.4.12: Time constant of an RC circuit

The *time constant* of an RC circuit is

$$\tau = RC,$$

where R is the resistance and C is the capacitance. The SI unit of τ is the second (s). The time constant sets the characteristic timescale of the transient response: at $t = \tau$, the capacitor charge during charging reaches $1 - e^{-1} \approx 63.2\%$ of its final value $C\mathcal{E}$, and during discharging the charge falls to $e^{-1} \approx 36.8\%$ of its initial value q_0. After 5τ, the transient is essentially over: $1 - e^{-5} \approx 0.993$ of the final charge has been reached during charging, and $e^{-5} \approx 0.0067$ of the initial charge remains during discharging.

Note:-

The time constant is the product of two quantities with SI units Ω (ohms) and F (farads). Since $\Omega = \text{V/A}$ and $\text{F} = \text{C/V}$, the product is $(\text{V/A})(\text{C/V}) = \text{C/A} = \text{C/(C/s)} = \text{s}$, confirming that τ has units of time. A larger resistance slows the charge flow, giving a longer time constant. A larger capacitance stores more charge per volt, also requiring more time to charge or discharge, giving a longer time constant.

Proposition 2.4.5 Charging and discharging at one time constant

For a charging capacitor, at $t = \tau = RC$:

$$q(\tau) = C\mathcal{E}\left(1 - \frac{1}{e}\right) \approx 0.632\, C\mathcal{E}, \qquad I(\tau) = \frac{\mathcal{E}}{R} \cdot \frac{1}{e} \approx 0.368\, \frac{\mathcal{E}}{R}.$$

For a discharging capacitor, at $t = \tau = RC$:

$$q(\tau) = \frac{q_0}{e} \approx 0.368\, q_0, \qquad |I(\tau)| = \frac{q_0}{RC} \cdot \frac{1}{e} \approx 0.368\, \frac{q_0}{RC}.$$

After n time constants ($t = n\tau$):

$$q_{\text{charge}} = C\mathcal{E}\,(1 - e^{-n}) \qquad \text{and} \qquad q_{\text{discharge}} = q_0\, e^{-n}.$$

Thus, after $n = 5$, charging reaches $1 - e^{-5} \approx 99.3\%$ of full charge and discharging leaves only $e^{-5} \approx 0.67\%$ of the initial charge.

Note:-

The time constant is independent of the initial conditions and of the battery emf. It depends only on the circuit geometry (through R and C). This is a hallmark of first-order linear systems: the timescale of the exponential decay is set by the coefficients of the differential equation, not by the particular solution's initial values.

Theorem 2.4.10 Energy during charging

When an initially uncharged capacitor C is charged through a resistor R by a battery of emf $\mathcal{E}$, the total energy supplied by the battery is

$$U_{\text{battery}} = C\mathcal{E}^2.$$

The energy finally stored in the capacitor is

$$U_C = \frac{1}{2}\, C\mathcal{E}^2.$$

The remaining half,

$$U_R = \frac{1}{2} C\mathcal{E}^2,$$

is dissipated as Joule heat in the resistor during the charging process. The fraction dissipated in the resistor is exactly 50%, independent of the value of R.

Energy during charging: The battery supplies energy at rate $dU_{\text{battery}}/dt = \mathcal{E}I(t)$, so the total energy delivered during charging is

$$U_{\text{battery}} = \int_0^\infty \mathcal{E}\, I(t)\, dt = \mathcal{E} \int_0^\infty \frac{\mathcal{E}}{R}\, e^{-t/RC}\, dt.$$

The integral is

$$\int_0^\infty e^{-t/RC}\, dt = RC,$$

so

$$U_{\text{battery}} = \frac{\mathcal{E}^2}{R} \cdot RC = C\mathcal{E}^2.$$

The energy stored in the capacitor at full charge ($q = C\mathcal{E}$) is

$$U_C = \frac{q^2}{2C} = \frac{(C\mathcal{E})^2}{2C} = \frac{1}{2} C\mathcal{E}^2.$$

By energy conservation, the energy dissipated in the resistor is

$$U_R = U_{\text{battery}} - U_C = C\mathcal{E}^2 - \frac{1}{2} C\mathcal{E}^2 = \frac{1}{2} C\mathcal{E}^2.$$

One can verify this directly:

$$U_R = \int_0^\infty I(t)^2 R\, dt = \int_0^\infty \frac{\mathcal{E}^2}{R}\, e^{-2t/RC}\, dt = \frac{\mathcal{E}^2}{R} \cdot \frac{RC}{2} = \frac{1}{2} C\mathcal{E}^2.$$

The result is independent of R, because a larger R gives less current but a proportionally longer charging time. 😊

Corollary 2.4.3 Energy during discharging

During complete discharge, the energy initially stored in the capacitor,

$$U_{\text{initial}} = \frac{q_0^2}{2C},$$

is entirely dissipated as Joule heat in the resistor:

$$U_R = \frac{q_0^2}{2C}.$$

This follows from

$$U_R = \int_0^\infty I(t)^2 R\, dt = \int_0^\infty \left(\frac{q_0}{RC}\, e^{-t/RC}\right)^2 R\, dt = \frac{q_0^2}{RC^2} \int_0^\infty e^{-2t/RC}\, dt = \frac{q_0^2}{RC^2} \cdot \frac{RC}{2} = \frac{q_0^2}{2C}.$$

Example 2.4.5 (Illustrative example)

If a capacitor of capacitance C is charged through a resistor R by a battery of emf $\mathcal{E}$, the time at which the current has dropped to half its initial value is found from $I(t) = (\mathcal{E}/R)e^{-t/RC} = (\mathcal{E}/2R)$, giving $e^{-t/RC} = 1/2$ and $t = RC\ln(2) \approx 0.693\,\tau$. At this instant, the charge on the capacitor is $q = C\mathcal{E}(1-1/2) = C\mathcal{E}/2$, exactly half of its final value. The energy stored in the capacitor is $\frac{1}{2}C(\mathcal{E}/2)^2 = \frac{1}{8}C\mathcal{E}^2 = 25\%$ of the energy that will ultimately be stored, while the battery has supplied $C\mathcal{E} \cdot (\mathcal{E}/2) = \frac{1}{2}C\mathcal{E}^2$, exactly half of the total energy it will supply.

Question 21: Worked example

A series RC circuit consists of a battery of emf

$$\mathcal{E} = 12.0\,\text{V},$$

a resistor of resistance

$$R = 2.00\,\text{k}\Omega = 2.00 \times 10^3\,\Omega,$$

and an initially uncharged capacitor of capacitance

$$C = 3.00\,\mu\text{F} = 3.00 \times 10^{-6}\,\text{F}.$$

At $t = 0$, a switch is closed connecting all three elements in series. Assume ideal wires and components. Find:

(a) the time constant τ of the circuit,

(b) the time $t_{1/2}$ at which the capacitor reaches 50.0% of its final (maximum) charge, and

(c) the current $I(t_{1/2})$ at that instant.

Solution: **Part (a).** The time constant is

$$\tau = RC = \left(2.00 \times 10^3\,\Omega\right)\left(3.00 \times 10^{-6}\,\text{F}\right) = 6.00 \times 10^{-3}\,\text{s}.$$

In more convenient units,

$$\tau = 6.00\,\text{ms}.$$

Part (b). The charge on the capacitor during charging is

$$q(t) = C\mathcal{E}\left(1 - e^{-t/\tau}\right).$$

The final (maximum) charge is

$$q_{\max} = C\mathcal{E} = \left(3.00 \times 10^{-6}\,\text{F}\right)(12.0\,\text{V}) = 36.0 \times 10^{-6}\,\text{C} = 36.0\,\mu\text{C}.$$

We want the time $t_{1/2}$ at which $q(t_{1/2}) = q_{\max}/2$. Setting the charge equation equal to $q_{\max}/2$:

$$C\mathcal{E}\left(1 - e^{-t_{1/2}/\tau}\right) = \frac{C\mathcal{E}}{2}.$$

Cancel $C\mathcal{E}$ and solve:

$$1 - e^{-t_{1/2}/\tau} = \frac{1}{2}, \qquad e^{-t_{1/2}/\tau} = \frac{1}{2}.$$

Taking the natural logarithm:

$$-\frac{t_{1/2}}{\tau} = \ln\left(\frac{1}{2}\right) = -\ln(2), \qquad t_{1/2} = \tau\ln(2).$$

Substitute $\tau = 6.00\,\text{ms}$:

$$t_{1/2} = (6.00\,\text{ms})\ln(2) = (6.00 \times 10^{-3}\,\text{s})\,(0.6931) = 4.16 \times 10^{-3}\,\text{s}.$$

Thus,

$$t_{1/2} = 4.16\,\text{ms}.$$

Part (c). The current during charging is

$$I(t) = \frac{\mathcal{E}}{R}\,e^{-t/\tau}.$$

At $t = t_{1/2}$, we found $e^{-t_{1/2}/\tau} = 1/2$, so

$$I(t_{1/2}) = \frac{\mathcal{E}}{R} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{12.0\,\text{V}}{2.00 \times 10^3\,\Omega}.$$

Compute the initial current:

$$\frac{\mathcal{E}}{R} = \frac{12.0}{2.00 \times 10^3}\,\text{A} = 6.00 \times 10^{-3}\,\text{A} = 6.00\,\text{mA}.$$

Therefore,

$$I(t_{1/2}) = \frac{6.00\,\text{mA}}{2} = 3.00\,\text{mA}.$$

Check. At $t_{1/2} = \tau \ln(2)$, the charge is $q = q_{\max}(1 - e^{-\ln 2}) = q_{\max}(1 - 1/2) = q_{\max}/2 = 18.0\,\mu\text{C}$, and the current is $I = (\mathcal{E}/R)e^{-\ln 2} = (\mathcal{E}/R)(1/2) = 3.00\,\text{mA}$. Both are consistent with the expected behavior of a charging RC circuit.

Therefore,

$$\tau = 6.00\,\text{ms}, \qquad t_{1/2} = 4.16\,\text{ms}, \qquad I(t_{1/2}) = 3.00\,\text{mA}.$$

2.4.7 Internal Resistance and Measurement Devices

This subsection introduces the real-battery model (ideal emf in series with an internal resistance), derives the terminal-voltage relation under load, and discusses how practical measurement devices (ammeters and voltmeters) affect the circuits they measure.

Definition 2.4.13: Real battery (emf and internal resistance)

A *real battery* is modelled as an ideal electromotive-force source $\mathcal{E}$ in series with an *internal resistance* r. The emf $\mathcal{E}$ represents the work per unit charge the battery can deliver when no current flows. The internal resistance r accounts for the finite conductivity of the electrolyte, electrode materials, and other dissipative processes inside the battery. The SI unit of emf is the volt (V), which is dimensionally equivalent to J/C.

Theorem 2.4.11 Terminal voltage of a real battery

Let a real battery with emf $\mathcal{E}$ and internal resistance r deliver a current I to an external load. The *terminal voltage* V across the battery's terminals is

$$V = \mathcal{E} - Ir.$$

When the battery is delivering current (discharging), the terminal voltage is *less* than the emf by the voltage drop Ir across the internal resistance. When $I = 0$ (open circuit), the terminal voltage equals the emf: $V = \mathcal{E}$. If the current is driven backwards through the battery (charging), then I is negative and $V > \mathcal{E}$.

Derivation of the terminal-voltage relation: Consider a real battery connected to an external load resistor R. The equivalent circuit consists of the emf $\mathcal{E}$, the internal resistance r, and the load R all in series. By Kirchhoff's loop rule, traversing the loop in the direction of current I:

$$\mathcal{E} - Ir - IR = 0.$$

Solving for the current gives

$$I = \frac{\mathcal{E}}{r + R}.$$

The terminal voltage V is the potential drop across the load, which also equals the potential drop from the battery's positive to negative terminal:

$$V = IR = \mathcal{E} - Ir.$$

This proves the relation $V = \mathcal{E} - Ir$. ☺

Theorem 2.4.12 Power delivered by a real battery

The power P delivered to an external load R by a real battery of emf $\mathcal{E}$ and internal resistance r is

$$P = I^2 R = \mathcal{E} I - I^2 r,$$

where $I = \mathcal{E}/(r + R)$. The first term $\mathcal{E}I$ is the total rate at which the battery converts chemical energy to electrical energy; the second term $I^2 r$ is the rate of internal dissipation as heat within the battery. The difference is the power delivered to the external circuit.

Theorem 2.4.13 Power transfer theorem (maximum power transfer)

For a real battery with fixed $\mathcal{E}$ and r connected to a variable load R, the power delivered to the load is

$$P(R) = \frac{\mathcal{E}^2 R}{(r+R)^2}.$$

This power is maximized when $R = r$, giving $P_{\max} = \mathcal{E}^2/(4r)$.

Maximum power transfer: From the preceding theorem, $P(R) = I^2 R$ with $I = \mathcal{E}/(r+R)$, so

$$P(R) = \frac{\mathcal{E}^2 R}{(r+R)^2}.$$

To find the maximum, differentiate with respect to R and set the derivative to zero:

$$\frac{dP}{dR} = \mathcal{E}^2 \frac{(r+R)^2 - R \cdot 2(r+R)}{(r+R)^4} = \mathcal{E}^2 \frac{(r+R) - 2R}{(r+R)^3} = \mathcal{E}^2 \frac{r-R}{(r+R)^3}.$$

Setting $dP/dR = 0$ gives $R = r$. For $R < r$ the derivative is positive (power increases); for $R > r$ it is negative (power decreases). Hence $R = r$ is a maximum. Substituting $R = r$ into the power expression gives $P_{\max} = \mathcal{E}^2/(4r)$. 😊

Corollary 2.4.4 Short circuit and open circuit limits

When the load is zero ($R = 0$, *short circuit*), the current is $I_{\text{sc}} = \mathcal{E}/r$ and the terminal voltage is $V = 0$. All power is dissipated internally: $P_{\text{load}} = 0$, and the battery heats up. When the load is infinite ($R \to \infty$, *open circuit*), the current is $I = 0$ and $V = \mathcal{E}$. No power is delivered.

Definition 2.4.14: Ammeter

An *ammeter* measures the current through a branch of a circuit. It is inserted in *series* with the branch. An ideal ammeter has zero resistance so it does not affect the circuit. A real ammeter has a small but finite resistance R_A (the *ammeter resistance*).

Definition 2.4.15: Voltmeter

A *voltmeter* measures the potential difference between two points in a circuit. It is connected in *parallel* between those points. An ideal voltmeter has infinite resistance so no current flows through it. A real voltmeter has a large but finite resistance R_V (the *voltmeter resistance*).

Theorem 2.4.14 Loading effects of measurement devices

When an ammeter of resistance R_A is inserted in series with a circuit of total resistance R_{eq} (not including R_A), the current measured is

$$I_{\text{measured}} = \frac{I_{\text{true}} R_{\text{eq}}}{R_{\text{eq}} + R_A},$$

where I_{true} is the current that would flow without the ammeter. The reading is smaller than the true value by a factor of $R_{\text{eq}}/(R_{\text{eq}} + R_A)$.

When a voltmeter of resistance R_V is connected across a component of resistance R that has voltage V across it (without the voltmeter), the measured voltage is

$$V_{\text{measured}} = V \frac{R R_V/(R+R_V)}{R R_V/(R+R_V) + R_{\text{series}}} = V \frac{R_V}{R_V + R_{\text{series}}(1 + R_V/R)}^{-1},$$

where R_{series} is the resistance in series with the component. In the common case where the component of interest is connected to a source with small internal resistance $r \ll R_V$, the correction is small: $V_{measured} \approx V(1 - r/R_V)$. The reading is smaller than the true voltage.

Theorem 2.4.15 Efficiency of power transfer

The *efficiency* η of a real battery delivering power to a load R is the ratio of power delivered to the load to the total power generated by the emf:

$$\eta = \frac{P_{load}}{P_{total}} = \frac{I^2 R}{\mathcal{E} I} = \frac{IR}{\mathcal{E}} = \frac{R}{R + r}.$$

Efficiency increases as R becomes large compared to r. When $R = r$, the efficiency is 50%: half the power is dissipated in the internal resistance.

Note:-

An ideal ammeter ($R_A = 0$) does not change the current in the branch it measures. A real ammeter always slightly *reduces* the current. An ideal voltmeter ($R_V = \infty$) draws no current and does not disturb the circuit. A real voltmeter always slightly *lowers* the voltage it is measuring because it provides an additional parallel current path. In well-designed circuits, $R_A \ll R_{branch}$ and $R_V \gg R_{parallel}$, so these perturbations are negligible.

Example 2.4.6 (Illustrative example)

A 9 V battery with internal resistance $r = 1\,\Omega$ is connected to a load $R = 8\,\Omega$. The current is $I = 9\,\text{V}/(1\,\Omega + 8\,\Omega) = 1.0\,\text{A}$, and the terminal voltage is $V = 9\,\text{V} - (1.0\,\text{A})(1\,\Omega) = 8\,\text{V}$.

Question 22: Worked example

A battery has an emf

$$\mathcal{E} = 12.0\,\text{V}$$

and an internal resistance

$$r = 2.0\,\Omega.$$

The battery is connected to an external load resistor $R = 10.0\,\Omega$, as shown in the circuit diagram below.

(Diagram description: A single loop consisting of an ideal emf source $\mathcal{E}$, an internal resistor r in series, and an external load resistor R in series, forming a closed circuit.)

Find:

(a) the current I in the circuit,

(b) the terminal voltage V across the battery,

(c) the power P_{load} delivered to the load resistor,

(d) the power P_{int} dissipated in the internal resistance,

(e) the total power P_{total} generated by the emf,

(f) the efficiency η of the battery, and

(g) the value of R that maximizes the power delivered to the load.

Solution: **Part (a).** The circuit consists of $\mathcal{E}$, r, and R in series. By Kirchhoff's loop rule,

$$\mathcal{E} - Ir - IR = 0,$$

so

$$I = \frac{\mathcal{E}}{r + R} = \frac{12.0\,\text{V}}{2.0\,\Omega + 10.0\,\Omega} = \frac{12.0\,\text{V}}{12.0\,\Omega} = 1.0\,\text{A}.$$

Part (b). The terminal voltage is

$$V = \mathcal{E} - Ir = 12.0\,\text{V} - (1.0\,\text{A})(2.0\,\Omega) = 12.0\,\text{V} - 2.0\,\text{V} = 10.0\,\text{V}.$$

Equivalently, $V = IR = (1.0\,\text{A})(10.0\,\Omega) = 10.0\,\text{V}$, which confirms the result.

Part (c). The power delivered to the load resistor is

$$P_{\text{load}} = I^2R = (1.0\,\text{A})^2(10.0\,\Omega) = 10.0\,\text{W}.$$

Alternatively, $P_{\text{load}} = VI = (10.0\,\text{V})(1.0\,\text{A}) = 10.0\,\text{W}$, giving the same result.

Part (d). The power dissipated in the internal resistance is

$$P_{\text{int}} = I^2r = (1.0\,\text{A})^2(2.0\,\Omega) = 2.0\,\text{W}.$$

Part (e). The total power generated by the emf is

$$P_{\text{total}} = \mathcal{E}I = (12.0\,\text{V})(1.0\,\text{A}) = 12.0\,\text{W}.$$

Check: $P_{\text{total}} = P_{\text{load}} + P_{\text{int}} = 10.0\,\text{W} + 2.0\,\text{W} = 12.0\,\text{W}$, so energy is conserved.

Part (f). The efficiency of the battery is

$$\eta = \frac{P_{\text{load}}}{P_{\text{total}}} = \frac{10.0\,\text{W}}{12.0\,\text{W}} = 0.833 = 83.3\%.$$

Equivalently, $\eta = R/(R + r) = 10.0\,\Omega/(10.0\,\Omega + 2.0\,\Omega) = 10/12 = 0.833 = 83.3\%$.

Part (g). By the maximum power transfer theorem, the power delivered to the load is maximized when the load resistance equals the internal resistance:

$$R = r = 2.0\,\Omega.$$

At this value, the maximum power delivered to the load is

$$P_{\max} = \frac{\mathcal{E}^2}{4r} = \frac{(12.0\,\text{V})^2}{4(2.0\,\Omega)} = \frac{144\,\text{V}^2}{8.0\,\Omega} = 18.0\,\text{W}.$$

Therefore,

$$I = 1.0\,\text{A}, \qquad V = 10.0\,\text{V}, \qquad P_{\text{load}} = 10.0\,\text{W},$$

$$P_{\text{int}} = 2.0\,\text{W}, \qquad P_{\text{total}} = 12.0\,\text{W}, \qquad \eta = 83.3\%, \qquad R_{\text{max power}} = 2.0\,\Omega.$$

2.5 Magnetism: Forces, Fields, and Sources

This unit develops the magnetic interaction, beginning with the force a magnetic field exerts on a moving charge — the vector cross-product law $\vec{F} = q\vec{v} \times \vec{B}$ — and the circular or helical motion that follows. In AP Physics C: Electricity and Magnetism, magnetic force is introduced as a velocity-dependent, perpendicular force that changes the direction of motion but does no work.

From particle motion the unit extends to macroscopic currents: the force and torque on current-carrying conductors and loops placed in a magnetic field. That current viewpoint then flips to its source side — the Biot–Savart law and Ampère's law — which let you calculate the magnetic field produced by steady currents using symmetry. The unit closes with solenoids, parallel currents, and magnetic dipoles as canonical configurations.

2.5.1 Magnetic Force on a Moving Charge

A charge that moves through a magnetic field experiences a force perpendicular to both its velocity and the field. This is the magnetic part of the Lorentz force. Unlike the electric force, the magnetic force acts only on moving charges and is always perpendicular to the direction of motion.

Definition 2.5.1: Magnetic force on a point charge

A particle of charge q moving with velocity $\vec{v}$ through a magnetic field $\vec{B}$ experiences a magnetic force

$$\vec{F}_B = q\,\vec{v} \times \vec{B}.$$

The magnitude of this force is

$$F_B = |q|\,v\,B\,\sin\theta,$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{B}$ measured in the plane they span. The direction of $\vec{F}_B$ is perpendicular to that plane and is determined by the right-hand rule, reversed for negative charge.

Note:-

The magnetic force vanishes when the charge is at rest ($\vec{v} = \vec{0}$), when $\vec{v}$ is parallel or antiparallel to $\vec{B}$ ($\theta = 0°$ or $180°$), or when $B = 0$. The force is maximal when $\vec{v} \perp \vec{B}$ ($\theta = 90°$).

Theorem 2.5.1 Lorentz magnetic force law

Let q be the charge of a particle, $\vec{v}$ its velocity vector, and $\vec{B}$ the magnetic field at the particle's position. Then the magnetic force on the particle is

$$\vec{F}_B = q\,\vec{v} \times \vec{B}.$$

- **Magnitude:** $F_B = |q|\,v\,B\,\sin\theta$, where θ is the angle between $\vec{v}$ and $\vec{B}$.
- **Direction:** Point the fingers of your right hand along $\vec{v}$, then curl them toward $\vec{B}$. Your thumb points in the direction of $\vec{F}_B$ if $q > 0$. If $q < 0$, the force is opposite to your thumb.
- **SI unit of B:** The tesla, $\mathrm{T} = \dfrac{\mathrm{N}}{\mathrm{C \cdot m/s}} = \dfrac{\mathrm{N}}{\mathrm{A \cdot m}} = \dfrac{\mathrm{kg}}{\mathrm{C \cdot s}}$.

Lorentz magnetic force law from cross-product geometry: The vector cross product $\vec{v} \times \vec{B}$ is defined to have magnitude $vB\sin\theta$ and direction given by the right-hand rule. Multiplying by q scales the magnitude by $|q|$ and reverses direction if $q < 0$. Thus

$$\vec{F}_B = q\,(\vec{v} \times \vec{B})$$

has magnitude $|q|vB\sin\theta$ and the correct directional behaviour. This is the experimentally determined magnetic force law for a point charge. ☺

Corollary 2.5.1 Charge at rest or parallel to field

When $\vec{v} = \vec{0}$ or when $\vec{v} \parallel \vec{B}$, we have $\sin\theta = 0$ and therefore $F_B = 0$. The magnetic field exerts no force on a stationary charge or on a charge moving exactly along the field lines.

Proposition 2.5.1 Magnetic force vs. electric force

For the same charge q placed in both an electric field $\vec{E}$ and a magnetic field $\vec{B}$, the total Lorentz force is

$$\vec{F} = q\,\vec{E} + q\,\vec{v} \times \vec{B}.$$

Key differences:

- $\vec{F}_E$ points parallel (or antiparallel) to $\vec{E}$, regardless of motion.
- $\vec{F}_B$ is always perpendicular to $\vec{v}$, so it does zero work on the charge.
- $\vec{F}_B$ vanishes when $\vec{v} = \vec{0}$; $\vec{F}_E$ does not.

Theorem 2.5.2 Magnetic force does no work

Since $\vec{F}_B \perp \vec{v}$ at every instant, the instantaneous power delivered by the magnetic force is

$$P = \vec{F}_B \cdot \vec{v} = q\,(\vec{v} \times \vec{B}) \cdot \vec{v} = 0.$$

The magnetic force can change the direction of a particle's velocity but never its kinetic energy. This is the mathematical expression of the scalar triple-product identity $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

Example 2.5.1 (Illustrative example)

When a charged particle enters a uniform magnetic field perpendicularly, it follows a circular path. The magnetic force provides the centripetal force:

$$|q|\,v\,B = \frac{m\,v^2}{R} \quad \Rightarrow \quad R = \frac{m\,v}{|q|\,B},$$

where R is the radius of the circular path. The period of revolution is

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{|q|\,B},$$

which is independent of the particle's speed. This circular-motion analysis is developed fully in section E12.2.

Question 23: Worked example

A proton (charge $q = +1.60 \times 10^{-19}$ C, mass $m = 1.67 \times 10^{-27}$ kg) moves with speed $v = 3.0 \times 10^6$ m/s in the $+\hat{\imath}$ direction. It enters a region with a uniform magnetic field $\vec{B} = (0.50\,\text{T})\,\hat{\jmath}$. The proton's velocity is perpendicular to the field.
Find:

(a) the magnitude of the magnetic force on the proton,

(b) the direction of the magnetic force as a unit vector, and

(c) the initial acceleration vector of the proton.

Solution: **(a) Force magnitude.** The magnetic force is $\vec{F}_B = q\,\vec{v} \times \vec{B}$. With $\vec{v} = v\,\hat{\imath}$ and $\vec{B} = B\,\hat{\jmath}$, the angle between them is $\theta = 90^\circ$ and $\sin\theta = 1$. The magnitude is

$$F_B = |q|\,v\,B\,\sin 90^\circ = |q|\,v\,B.$$

Substitute the given values:

$$F_B = (1.60 \times 10^{-19}\,\text{C})(3.0 \times 10^6\,\text{m/s})(0.50\,\text{T}).$$

Compute step by step:

$$(1.60 \times 10^{-19})(3.0 \times 10^6) = 4.8 \times 10^{-13},$$

$$(4.8 \times 10^{-13})(0.50) = 2.4 \times 10^{-13}\,\text{N}.$$

Thus

$$F_B = 2.4 \times 10^{-13}\,\text{N}.$$

(b) Force direction. Use the cross product directly:

$$\vec{F}_B = q\,(\vec{v} \times \vec{B}) = q\,(v\,\hat{\imath}) \times (B\,\hat{\jmath}) = q\,v\,B\,(\hat{\imath} \times \hat{\jmath}).$$

Since $\hat{\imath} \times \hat{\jmath} = \hat{k}$,

$$\vec{F}_B = q\,v\,B\,\hat{k}.$$

Because $q > 0$ and $v, B > 0$, the force points in the $+\hat{k}$ direction. In unit-vector form:

$$\vec{F}_B = (2.4 \times 10^{-13}\,\mathrm{N})\,\hat{k}.$$

The right-hand rule confirms this: fingers along $+\hat{\imath}$, curl toward $+\hat{\jmath}$, thumb points along $+\hat{k}$.

(c) Acceleration vector. By Newton's second law,

$$\vec{a} = \frac{\vec{F}_B}{m}.$$

Substitute:

$$\vec{a} = \frac{(2.4 \times 10^{-13}\,\mathrm{N})\,\hat{k}}{1.67 \times 10^{-27}\,\mathrm{kg}}.$$

Compute the magnitude:

$$\frac{2.4 \times 10^{-13}}{1.67 \times 10^{-27}} = \frac{2.4}{1.67} \times 10^{14} \approx 1.44 \times 10^{14}\,\mathrm{m/s^2}.$$

Thus

$$\vec{a} = (1.44 \times 10^{14}\,\mathrm{m/s^2})\,\hat{k}.$$

Final answers:

(a) $F_B = 2.4 \times 10^{-13}\,\mathrm{N}$

(b) $\vec{F}_B = (2.4 \times 10^{-13}\,\mathrm{N})\,\hat{k}$

(c) $\vec{a} = (1.44 \times 10^{14}\,\mathrm{m/s^2})\,\hat{k}$

2.5.2 Circular and Helical Motion in a Uniform Magnetic Field

When a charged particle moves through a uniform magnetic field, the magnetic force acts as a centripetal force, bending the particle's path. The resulting motion depends on the angle between the velocity $\vec{v}$ and the field $\vec{B}$: perpendicular entry produces circular motion, while oblique entry produces helical motion.

Definition 2.5.2: Circular motion of a charge in a uniform B-field

Let a particle of mass m and charge q move with speed v through a uniform magnetic field $\vec{B}$, with $\vec{v} \perp \vec{B}$. The magnetic force has constant magnitude $|q|vB$ and is always perpendicular to $\vec{v}$, so it provides the centripetal force for uniform circular motion:

$$|q|\, v\, B = \frac{m\, v^2}{R}.$$

Solving for the radius,

$$R = \frac{m\, v}{|q|\, B} = \frac{m\, v_\perp}{|q|\, B},$$

where $v_\perp = v$ is the perpendicular speed. The period of revolution is

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{|q|\, B}.$$

The frequency of revolution (cyclotron frequency) is

$$f = \frac{1}{T} = \frac{|q|\, B}{2\pi m},$$

and the angular frequency is

$$\omega = 2\pi f = \frac{|q|\, B}{m}.$$

Note that T, f, and ω are independent of the particle's speed.

Note:-

Because the magnetic force does no work (it is always perpendicular to $\vec{v}$), the particle's speed and kinetic energy remain constant throughout the circular motion. The magnetic field only changes the direction of the velocity, not its magnitude. This is why the radius depends on v but the period does not.

Theorem 2.5.3 Cyclotron motion

Let m be the mass and q the charge of a particle entering a uniform magnetic field $\vec{B}$ with velocity component $v_\perp$ perpendicular to $\vec{B}$. The particle undergoes uniform circular motion in the plane perpendicular to $\vec{B}$ with:

- **Radius:** $R = \dfrac{m\, v_\perp}{|q|\, B}$
- **Period:** $T = \dfrac{2\pi m}{|q|\, B}$
- **Cyclotron frequency:** $f = \dfrac{|q|\, B}{2\pi m}$
- **Angular frequency:** $\omega = \dfrac{|q|\, B}{m}$

The sense of rotation is counter-clockwise for $q > 0$ and clockwise for $q < 0$ when viewing along the direction of $\vec{B}$.

Cyclotron motion from Newton's second law: The magnetic force on the particle is $\vec{F}_B = q\, \vec{v} \times \vec{B}$. When $\vec{v} \perp \vec{B}$, the force magnitude is $F_B = |q|vB$ and its direction is always perpendicular to $\vec{v}$ (toward the center of curvature). For uniform circular motion, Newton's second law requires

$$F_{\text{net}} = \frac{m\, v^2}{R}.$$

Equating the magnetic force to the required centripetal force:

$$|q|\, v\, B = \frac{m\, v^2}{R}.$$

Solving for R:

$$R = \frac{m\, v^2}{|q|\, v\, B} = \frac{m\, v}{|q|\, B}.$$

The period is the circumference divided by the speed:

$$T = \frac{2\pi R}{v} = \frac{2\pi}{v} \cdot \frac{m\, v}{|q|\, B} = \frac{2\pi m}{|q|\, B}.$$

Thus T is independent of v and R. The angular frequency is $\omega = 2\pi/T = |q|B/m$, known as the cyclotron angular frequency.

Definition 2.5.3: Helical motion of a charge in a uniform B-field

Let a particle of mass m and charge q move with speed v through a uniform magnetic field $\vec{B}$, with the velocity making an angle α with $\vec{B}$ (where $0° < \alpha < 90°$). Decompose the velocity into components parallel and perpendicular to $\vec{B}$:

$$v_{\parallel} = v \cos\alpha, \qquad v_{\perp} = v \sin\alpha.$$

The perpendicular component produces circular motion with radius

$$R = \frac{m\, v_{\perp}}{|q|\, B} = \frac{m\, v\, \sin\alpha}{|q|\, B}.$$

The parallel component is unaffected by the magnetic force and produces uniform linear motion along $\vec{B}$. The combination is helical motion.
The *pitch* p of the helix is the distance advanced along $\vec{B}$ during one full revolution:

$$p = v_{\parallel}\, T = (v \cos\alpha) \cdot \frac{2\pi m}{|q|\, B} = \frac{2\pi m\, v\, \cos\alpha}{|q|\, B}.$$

The sense of the circular rotation follows the same rule as cyclotron motion: counter-clockwise for $q > 0$ and clockwise for $q < 0$, when viewing along $\vec{B}$.

Note:-

If $\alpha = 0°$, then $v_{\perp} = 0$ and the particle travels in a straight line along $\vec{B}$ (no magnetic force). If $\alpha = 90°$, then $v_{\parallel} = 0$ and the particle undergoes pure circular motion (no drift along $\vec{B}$). Helical motion interpolates between these two extremes. The pitch increases as $\alpha \to 0°$ and approaches zero as $\alpha \to 90°$.

Proposition 2.5.2 Cyclotron and helical motion parameters

For a particle of mass m and charge q in a uniform magnetic field $\vec{B}$, with velocity $\vec{v}$ at angle α to $\vec{B}$:

$$R = \frac{m\, v\, \sin\alpha}{|q|\, B} \quad \text{(helix radius)} \tag{2.7}$$

$$T = \frac{2\pi m}{|q|\, B} \quad \text{(period of revolution, independent of } v \text{ and } \alpha) \tag{2.8}$$

$$f = \frac{|q|\, B}{2\pi m} \quad \text{(cyclotron frequency)} \tag{2.9}$$

$$p = \frac{2\pi m\, v\, \cos\alpha}{|q|\, B} \quad \text{(helix pitch)} \tag{2.10}$$

Proposition 2.5.3 Magnetic force does no work

The magnetic force $\vec{F}_B = q\,\vec{v} \times \vec{B}$ is always perpendicular to $\vec{v}$, so

$$P = \vec{F}_B \cdot \vec{v} = 0 \qquad \text{and} \qquad \Delta K = 0.$$

The kinetic energy and speed of the particle remain constant. This holds for both pure circular motion and helical motion.

Example 2.5.2 (Illustrative example)

An electron and a proton, each with the same speed v, enter perpendicular to the same uniform magnetic field. The proton's radius is larger by the mass ratio $m_p/m_e \approx 1836$, but both complete one revolution in the same time $T = 2\pi m/|q|B$, because the proton's period is 1836 times longer due to its mass but it travels a proportionally longer path (radius 1836 times larger), so the time cancels out.

Question 24: Worked example

An electron (mass $m_e = 9.11 \times 10^{-31}$ kg, charge $q = -1.60 \times 10^{-19}$ C) enters a region with a uniform magnetic field $\vec{B} = (0.040\,\text{T})\,\hat{\jmath}$. At the moment it enters, its velocity is

$$\vec{v} = (4.0 \times 10^5\ \text{m/s})\,\hat{\imath} + (2.0 \times 10^5\ \text{m/s})\,\hat{k}.$$

Find:

(a) the radius of the helical path,

(b) the period of revolution,

(c) the pitch of the helix, and

(d) the sense of rotation (clockwise or counter-clockwise when viewing along $+\vec{B}$).

Solution: **(a) Radius of the helix.** Decompose the velocity into components parallel and perpendicular to $\vec{B}$ (which points along $\hat{\jmath}$):

$$v_\parallel = v_k = 2.0 \times 10^5\ \text{m/s}, \qquad v_\perp = \sqrt{v_i^2 + v_j^2} = \sqrt{(4.0 \times 10^5)^2 + 0^2} = 4.0 \times 10^5\ \text{m/s}.$$

Note that $v_j = 0$, so the entire x-component is perpendicular to $\vec{B}$.

The radius of the helical path is

$$R = \frac{m\,v_\perp}{|q|\,B}.$$

Substitute the values:

$$R = \frac{(9.11 \times 10^{-31}\ \text{kg})(4.0 \times 10^5\ \text{m/s})}{(1.60 \times 10^{-19}\ \text{C})(0.040\ \text{T})}.$$

Compute the numerator:

$$(9.11 \times 10^{-31})(4.0 \times 10^5) = 3.644 \times 10^{-25}\ \text{kg·m/s}.$$

Compute the denominator:

$$(1.60 \times 10^{-19})(0.040) = 6.4 \times 10^{-21}\ \text{C·T}.$$

Thus

$$R = \frac{3.644 \times 10^{-25}}{6.4 \times 10^{-21}}\ \text{m} = 5.69 \times 10^{-5}\ \text{m}.$$

So

$$R = 5.69 \times 10^{-5}\ \text{m} = 56.9\ \mu\text{m}.$$

(b) Period of revolution. The period depends only on the particle's properties and the field strength:

$$T = \frac{2\pi m}{|q|\,B}.$$

Substitute:

$$T = \frac{2\pi(9.11 \times 10^{-31}\,\text{kg})}{(1.60 \times 10^{-19}\,\text{C})(0.040\,\text{T})}.$$

The denominator is 6.4×10^{-21} C·T as computed above. The numerator is

$$2\pi(9.11 \times 10^{-31}) = 5.724 \times 10^{-30}\,\text{kg}.$$

Thus

$$T = \frac{5.724 \times 10^{-30}}{6.4 \times 10^{-21}}\,\text{s} = 8.94 \times 10^{-10}\,\text{s}.$$

So

$$T = 8.94 \times 10^{-10}\,\text{s} = 0.894\,\text{ns}.$$

(c) Pitch of the helix. The pitch is the distance advanced along $\vec{B}$ in one period:

$$p = v_{\parallel}\,T.$$

Substitute:

$$p = (2.0 \times 10^{5}\,\text{m/s})(8.94 \times 10^{-10}\,\text{s}).$$

Compute:

$$p = 1.788 \times 10^{-4}\,\text{m}.$$

So

$$p = 1.79 \times 10^{-4}\,\text{m} = 179\,\mu\text{m}.$$

(d) Sense of rotation. The magnetic field points in the $+\hat{\jmath}$ direction. To determine the sense of rotation, note the initial force on the electron at the entry point. The velocity has a $+x$ component, so at $t = 0$ the perpendicular velocity is $\vec{v}_{\perp} = (4.0 \times 10^{5}\,\text{m/s})\,\hat{\imath}$. The magnetic force is

$$\vec{F} = q\,(\vec{v} \times \vec{B}) = (-1.60 \times 10^{-19})\,[(4.0 \times 10^{5}\,\hat{\imath}) \times (0.040\,\hat{\jmath})].$$

Since $\hat{\imath} \times \hat{\jmath} = \hat{k}$,

$$\vec{F} = (-1.60 \times 10^{-19})(4.0 \times 10^{5})(0.040)\,\hat{k} = -(2.56 \times 10^{-15}\,\text{N})\,\hat{k}.$$

The initial force points in the $-z$ direction, meaning the electron curves toward $-z$ from its initial $+x$ direction. Viewing along $+\hat{\jmath}$ (the direction of $\vec{B}$), the $+x$ axis is to the right and the $+z$ axis points toward you. Starting at $+x$ and curving toward $-z$, the electron moves clockwise. This is consistent with the general rule: for negative charge, the rotation is clockwise when viewing along $\vec{B}$.

Final answers:

(a) $R = 5.69 \times 10^{-5}\,\text{m} = 56.9\,\mu\text{m}$

(b) $T = 8.94 \times 10^{-10}\,\text{s} = 0.894\,\text{ns}$

(c) $p = 1.79 \times 10^{-4}\,\text{m} = 179\,\mu\text{m}$

(d) Clockwise (viewing along $+\hat{\jmath}$)

2.5.3 Force on Current-Carrying Conductors and Loops

A current-carrying wire in a magnetic field experiences a force because the moving charge carriers inside the wire each feel a magnetic force. When the current flows through a closed loop, the forces on individual segments can produce a net torque, causing the loop to rotate. This is the operating principle of electric motors.

Definition 2.5.4: Force on a current-carrying wire

A wire carrying current I and placed in a magnetic field $\vec{B}$ experiences a magnetic force. For a straight wire segment of length ℓ carrying current I, with the vector $\vec{\ell}$ pointing in the direction of the current, the force is

$$\vec{F} = I\,\vec{\ell} \times \vec{B}.$$

The magnitude of this force is

$$F = I\,\ell\,B\,\sin\theta,$$

where θ is the angle between the current direction (the direction of $\vec{\ell}$) and the magnetic field $\vec{B}$. The direction is given by the right-hand rule for cross products, reversed for negative current carriers.

Note:-

The vector $\vec{\ell}$ has magnitude equal to the length of the wire segment and points along the wire in the direction of conventional current. In a curved wire, the total force is obtained by integrating the differential-force expression over the entire path: $\vec{F} = \oint I\,d\vec{\ell} \times \vec{B}$.

Theorem 2.5.4 Magnetic force on a current-carrying conductor

Let a wire carry steady current I through a magnetic field $\vec{B}$.

- **Straight wire:** For a straight wire segment of length ℓ, with $\vec{\ell}$ pointing along the wire in the current direction,

  $$\vec{F} = I\,\vec{\ell} \times \vec{B}.$$

  The magnitude is $F = I\,\ell\,B\,\sin\theta$, where θ is the angle between $\vec{\ell}$ and $\vec{B}$.

- **Differential element:** For an arbitrary wire path, the force on a differential element $d\vec{\ell}$ is

  $$d\vec{F} = I\,d\vec{\ell} \times \vec{B},$$

  and the total force is

  $$\vec{F} = \int_{\text{wire}} I\,d\vec{\ell} \times \vec{B}.$$

- **Right-hand rule:** Point your right hand's fingers along $\vec{\ell}$ (current direction), then curl them toward $\vec{B}$. Your thumb gives the direction of $\vec{F}$.

Force on a current-carrying wire from the force on moving charges: The force on a single charge q moving with drift velocity $\vec{v}_d$ is $\vec{F}_q = q\,\vec{v}_d \times \vec{B}$. In a wire segment of length ℓ and cross-sectional area A, the number of charge carriers is $N = n\,A\,\ell$, where n is the carrier number density. The total force is

$$\vec{F} = N\,q\,\vec{v}_d \times \vec{B} = n\,A\,\ell\,q\,\vec{v}_d \times \vec{B}.$$

The current is $I = n\,q\,v_d\,A$, and the direction of $\vec{v}_d$ for positive carriers is the current direction. Letting $\vec{\ell}$ point in that direction with magnitude ℓ, we have $n\,q\,\vec{v}_d = (I/A)\,\hat{\ell}$, and so

$$\vec{F} = I\,\vec{\ell} \times \vec{B},$$

where we used $\vec{\ell} = \ell\,\hat{\ell}$. This derivation confirms that the macroscopic force on a current-carrying wire follows directly from the Lorentz force on individual charge carriers. ☺

Corollary 2.5.2 Straight wire parallel or perpendicular to field

When the wire is parallel or antiparallel to $\vec{B}$ ($\theta = 0°$ or $180°$), then $\sin\theta = 0$ and $F = 0$. When the wire is perpendicular to $\vec{B}$ ($\theta = 90°$), the force is maximal: $F = I\,\ell\,B$.

Corollary 2.5.3 Closed loop in a uniform field

When a closed current loop sits entirely in a uniform magnetic field, the net force is zero:

$$\vec{F}_{\text{net}} = I \oint d\vec{\ell} \times \vec{B} = I \left(\oint d\vec{\ell} \right) \times \vec{B} = \vec{0}.$$

The integral $\oint d\vec{\ell}$ around any closed loop is the zero vector. Thus, no net force acts on a closed loop in a uniform field, though individual segments still feel forces.

Definition 2.5.5: Magnetic dipole moment of a current loop

A planar loop carrying current I with enclosed area A has a *magnetic dipole moment*

$$\vec{\mu} = N\,I\,A\,\hat{n},$$

where N is the number of turns in the loop, A is the area enclosed by one turn, and $\hat{n}$ is a unit vector perpendicular to the plane of the loop. The direction of $\hat{n}$ is given by the right-hand rule: curl the fingers of your right hand around the loop in the direction of the current, and your thumb points along $\hat{n}$.

Proposition 2.5.4 Torque and potential energy of a current loop in a uniform field

Let a planar current loop with magnetic dipole moment $\vec{\mu} = NIA\,\hat{n}$ be placed in a uniform magnetic field $\vec{B}$. Then:

1. The torque on the loop is
$$\vec{\tau} = \vec{\mu} \times \vec{B}.$$
The magnitude is
$$\tau = \mu\,B\,\sin\phi = N\,I\,A\,B\,\sin\phi,$$
where ϕ is the angle between $\hat{n}$ and $\vec{B}$.
2. The potential energy of the loop is
$$U = -\vec{\mu} \cdot \vec{B} = -\mu\,B\,\cos\phi = -N\,I\,A\,B\,\cos\phi.$$

Equilibrium occurs when $\phi = 0°$ (stable, $\hat{n} \parallel \vec{B}$, torque zero, energy minimum) or $\phi = 180°$ (unstable, $\hat{n}$ antiparallel to $\vec{B}$, torque zero, energy maximum).

Torque on a current loop in a uniform field: Consider a rectangular loop of width a (in the x-direction) and height b (in the y-direction), carrying current I in a uniform field $\vec{B} = B\,\hat{k}$ (out of the plane). Let the normal $\hat{n}$ to the loop make angle ϕ with $\hat{k}$. The loop rotates about an axis through its center, perpendicular to $\vec{B}$.

The forces on the top and bottom segments (length a) are equal and opposite and collinear, so they cancel. The forces on the two side segments (length b) are

$$\vec{F}_1 = I\,\vec{b}_1 \times \vec{B} \quad \text{and} \quad \vec{F}_2 = I\,\vec{b}_2 \times \vec{B},$$

with $\vec{b}_1 = -\vec{b}_2$. These forces have magnitude $F = I\,b\,B$ and act at perpendicular distances $(a/2)\sin\phi$ from the axis. Each produces a torque of magnitude

$$\tau_{\text{side}} = F \cdot \frac{a}{2}\,\sin\phi = I\,b\,B \cdot \frac{a}{2}\,\sin\phi.$$

Both sides contribute in the same rotational sense, so

$$\tau = 2 \cdot I\,b\,B \cdot \frac{a}{2}\,\sin\phi = I\,a\,b\,B\,\sin\phi.$$

Since $A = a\,b$ and $\vec{\mu}$ has magnitude $\mu = IA$ (for $N = 1$),

$$\tau = \mu\,B\,\sin\phi.$$

The vector form is $\vec{\tau} = \vec{\mu} \times \vec{B}$. For N turns, multiply by N.

For the potential energy, the torque tends to align $\hat{n}$ with $\vec{B}$. The work done by an external agent rotating the loop from angle ϕ_0 to ϕ equals the change in potential energy:

$$\Delta U = -\int_{\phi_0}^{\phi} \tau_{\text{ext}}\,d\phi' = -\int_{\phi_0}^{\phi} \mu B\,\sin\phi'\,d\phi' = -\mu B\left(\cos\phi_0 - \cos\phi\right).$$

Choosing the reference $U(\phi_0 = 90°) = 0$, we find

$$U = -\mu B\,\cos\phi = -\vec{\mu} \cdot \vec{B}.$$

Note:-

The torque tries to align $\vec{\mu}$ with $\vec{B}$. The stable equilibrium orientation has $\hat{n} \parallel \vec{B}$, i.e., the plane of the loop is perpendicular to the field. The unstable equilibrium has $\hat{n}$ antiparallel to $\vec{B}$. A DC motor exploits this by periodically reversing the current so the loop continues rotating.

😄

Note:-

The SI unit of magnetic dipole moment $\vec{\mu}$ is ampere-square meter, $\text{A} \cdot \text{m}^2$. This is equivalent to joules per tesla, J/T.

Example 2.5.3 (Illustrative example)

A coil with $N = 50$ turns, each of area $A = 8.0\,\text{cm}^2 = 8.0 \times 10^{-4}\,\text{m}^2$, carries current $I = 2.0\,\text{A}$ in a field $B = 0.40\,\text{T}$. The maximum torque occurs at $\phi = 90°$:

$$\tau_{\max} = N\,I\,A\,B = (50)(2.0\,\text{A})(8.0 \times 10^{-4}\,\text{m}^2)(0.40\,\text{T}) = 3.2 \times 10^{-2}\,\text{N} \cdot \text{m}.$$

Question 25: Worked example

A rectangular wire loop has width $a = 0.10\,\text{m}$ (horizontal side) and height $b = 0.050\,\text{m}$ (vertical side). The loop has $N = 100$ turns and carries current $I = 2.0\,\text{A}$ in the direction shown: clockwise when viewed from the front (from the $+x$-axis toward the yz-plane). The loop sits in a uniform magnetic field $\vec{B} = (0.60\,\text{T})\,\hat{\imath}$ pointing to the right. The normal vector $\hat{n}$ points along the $-\hat{k}$ direction (into the page) by the right-hand rule for clockwise current.
The loop lies in the xy-plane, centered at the origin, with its sides parallel to the x- and y-axes.
Find:

(a) the net magnetic force on the loop,

(b) the net magnetic torque on the loop (magnitude and direction), and

(c) the magnetic potential energy of the loop in this orientation, and determine whether the loop will rotate clockwise or counterclockwise when released.

Given quantities:

- Width (horizontal): $a = 0.10\,\text{m}$

- Height (vertical): $b = 0.050\,\text{m}$
- Number of turns: $N = 100$
- Current: $I = 2.0\,\text{A}$ (clockwise in the xy-plane)
- Magnetic field: $\vec{B} = (0.60\,\text{T})\,\hat{\imath}$
- Normal vector: $\hat{n} = -\hat{k}$

Solution: **(a) Net force.** The magnetic field is uniform, and the loop is a closed current loop. From the corollary, the net force on a closed loop in a uniform magnetic field is zero. We can verify this segment by segment.

The loop has four segments:

- *Bottom segment* (length a, current to the right): $\vec{\ell}_1 = a\,\hat{\imath}$, so $\vec{F}_1 = I\,(a\,\hat{\imath}) \times (B\,\hat{\imath}) = \vec{0}$ (parallel to $\vec{B}$).
- *Top segment* (length a, current to the left): $\vec{\ell}_3 = -a\,\hat{\imath}$, so $\vec{F}_3 = I\,(-a\,\hat{\imath}) \times (B\,\hat{\imath}) = \vec{0}$.
- *Right segment* (length b, current downward): $\vec{\ell}_2 = -b\,\hat{\jmath}$, so

$$\vec{F}_2 = I\,(-b\,\hat{\jmath}) \times (B\,\hat{\imath}) = I\,b\,B\,(-\hat{\jmath} \times \hat{\imath}) = I\,b\,B\,\hat{k}.$$

- *Left segment* (length b, current upward): $\vec{\ell}_4 = b\,\hat{\jmath}$, so

$$\vec{F}_4 = I\,(b\,\hat{\jmath}) \times (B\,\hat{\imath}) = I\,b\,B\,(\hat{\jmath} \times \hat{\imath}) = -I\,b\,B\,\hat{k}.$$

Summing:

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0} + I\,b\,B\,\hat{k} + \vec{0} - I\,b\,B\,\hat{k} = \vec{0}.$$

For N turns, each segment's force is multiplied by N, but they still cancel:

$$\boxed{\vec{F}_{\text{net}} = \vec{0}}.$$

(b) Net torque. The magnetic dipole moment has magnitude

$$\mu = N\,I\,A = N\,I\,(a\,b).$$

Substitute the values:

$$\mu = (100)(2.0\,\text{A})(0.10\,\text{m})(0.050\,\text{m}) = (100)(2.0)(0.0050)\,\text{A}\cdot\text{m}^2 = 1.0\,\text{A}\cdot\text{m}^2.$$

The dipole moment vector is $\vec{\mu} = \mu\,\hat{n} = -(1.0\,\text{A}\cdot\text{m}^2)\,\hat{k}$.

The torque is

$$\vec{\tau} = \vec{\mu} \times \vec{B} = [-(1.0)\,\hat{k}] \times [(0.60)\,\hat{\imath}] = -(1.0)(0.60)\,(\hat{k} \times \hat{\imath}).$$

Since $\hat{k} \times \hat{\imath} = \hat{\jmath}$,

$$\vec{\tau} = -(0.60)\,\hat{\jmath}\,\text{N}\cdot\text{m}.$$

The magnitude is

$$\tau = 0.60\,\text{N}\cdot\text{m}.$$

To interpret the direction, $-\hat{\jmath}$ points downward. Using the right-hand rule for torque, the loop tends to rotate such that $\vec{\mu}$ aligns with $\vec{B}$ — that is, $\hat{n}$ rotates from $-\hat{k}$ toward $+\hat{\imath}$. This corresponds to a rotation about the y-axis.

More physically: the right side of the loop (at $+x/2$) feels force $\vec{F}_2 = N\,I\,b\,B\,\hat{k} = 100(2.0)(0.050)(0.60)\,\hat{k} = 6.0\,\text{N}$ upward, while the left side (at $-x/2$) feels $6.0\,\text{N}$ downward. This pair of forces creates a torque that rotates the right side up and the left side down — a rotation about the y-axis.

$$\boxed{\tau = 0.60\,\text{N}\cdot\text{m}, \quad \text{rotation about the } -\hat{\jmath} \text{ axis (right side up, left side down)}}.$$

(c) Potential energy and rotational tendency. The potential energy is

$$U = -\vec{\mu} \cdot \vec{B} = -[-(1.0)\,\hat{k}] \cdot [(0.60)\,\hat{\imath}].$$

Since $\hat{k} \perp \hat{\imath}$, their dot product is 0, so

$$\boxed{U = 0}.$$

This corresponds to $\phi = 90^\circ$ between $\vec{\mu}$ (pointing in $-\hat{k}$) and $\vec{B}$ (pointing in $+\hat{\imath}$), since $\cos 90^\circ = 0$.

The loop will rotate toward the stable equilibrium orientation where $\vec{\mu} \parallel \vec{B}$. Currently $\vec{\mu}$ points into the page ($-\hat{k}$) while $\vec{B}$ points right ($+\hat{\imath}$). The stable orientation has $\hat{n}$ aligned with $+\hat{\imath}$, meaning the loop's plane is perpendicular to the field (normal pointing along $\vec{B}$). The torque computed in part (b) drives this rotation: the right side is pushed upward and the left side downward, rotating the loop about the y-axis. Viewed from the $+y$ direction (looking down from above), this rotation appears counterclockwise.

$$\boxed{\text{Rotates about the } y\text{-axis toward stable equilibrium (right side up, left side down).}}$$

Summary of results:

(a) $\vec{F}_{\text{net}} = \vec{0}$

(b) $\tau = 0.60\,\text{N} \cdot \text{m}$, rotation about the $-\hat{\jmath}$ axis

(c) $U = 0$, loop rotates toward stable equilibrium where $\hat{n} \parallel \vec{B}$

2.5.4 The Biot–Savart Law

Currents produce magnetic fields. The Biot–Savart law gives the magnetic field at a point in space due to a steady current distribution. It is the magnetic analogue of Coulomb's law in electrostatics: just as Coulomb's law tells you the electric field of a charge distribution by integrating over point charges, the Biot–Savart law tells you the magnetic field of a current distribution by integrating over current elements.

Definition 2.5.6: Biot–Savart law (differential form)

Let a steady current I flow through a thin wire. Consider a differential element of the wire of length $d\ell$ carrying current I, represented by the vector $d\vec{\ell}$ pointing in the direction of the current. Let P be an observation point, and let $\vec{r}$ be the displacement vector from the current element to P, with magnitude $r = |\vec{r}|$ and unit vector $\hat{r} = \vec{r}/r$. The differential magnetic field at P due to this current element is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{\ell} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{I\,d\vec{\ell} \times \vec{r}}{r^3}.$$

Here $\mu_0 = 4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A}$ is the permeability of free space (the magnetic constant).

Note:-

The Biot–Savart law is fundamentally a *superposition principle*: the total field is the vector sum (integral) of all the differential contributions from every current element. Because each contribution involves a cross product, $d\vec{B}$ is always perpendicular to both the current element $d\vec{\ell}$ and the displacement $\vec{r}$. The $1/r^2$ dependence mirrors Coulomb's law, making the Biot–Savart law a Green's-function solution to the static Maxwell equations for $\vec{B}$.

Theorem 2.5.5 Biot–Savart law

For a steady current I flowing along a wire path C, the total magnetic field at a point P is

$$\vec{B} = \int_C \frac{\mu_0}{4\pi} \frac{I\,d\vec{\ell} \times \hat{r}}{r^2}.$$

- **Constants:** $\mu_0 = 4\pi \times 10^{-7}\,\text{T} \cdot \text{m/A}$ (permeability of free space). In SI units, $d\vec{B}$ has units of tesla

(T).

- **Geometry:** $d\vec{\ell}$ points along the wire in the direction of conventional current. The vector $\vec{r}$ points *from* the current element *to* the observation point. The unit vector $\hat{r} = \vec{r}/r$ is the normalized version of this displacement.
- **Direction:** The direction of $d\vec{B}$ is given by the right-hand rule for the cross product $d\vec{\ell} \times \hat{r}$. $d\vec{B}$ is perpendicular to the plane containing $d\vec{\ell}$ and $\vec{r}$.
- **Magnitude:** The magnitude of the differential field is

$$dB = \frac{\mu_0}{4\pi} \frac{I\, d\ell \, \sin\theta}{r^2},$$

where θ is the angle between $d\vec{\ell}$ and $\vec{r}$.
- **Symmetry considerations:** In many symmetric geometries (long straight wires, circular loops, solenoids), symmetry allows you to argue that certain components of $\vec{B}$ cancel upon integration, dramatically simplifying the calculation.

Biot–Savart law from the macroscopic field of a long wire and superposition: The field of a long straight wire carrying current I is known experimentally (from Ampère's law or direct measurement) to be

$$B = \frac{\mu_0 I}{2\pi s},$$

where s is the perpendicular distance from the wire. The Biot–Savart law is the differential statement that, when integrated for an infinite straight wire, reproduces this result.

Consider an infinite straight wire along the z-axis carrying current I in the $+\hat{k}$ direction. The observation point is in the xy-plane at distance s from the wire. A current element at position z has

$$d\vec{\ell} = dz\, \hat{k}, \qquad \vec{r} = s\,\hat{s} - z\,\hat{k}, \qquad r = \sqrt{s^2 + z^2}.$$

The cross product is

$$d\vec{\ell} \times \vec{r} = dz\, \hat{k} \times (s\,\hat{s} - z\,\hat{k}) = s\, dz\, (\hat{k} \times \hat{s}) = s\, dz\, \hat{\phi},$$

which points in the azimuthal direction. The magnitude of the field contribution is

$$dB = \frac{\mu_0}{4\pi} \frac{I\, s\, dz}{(s^2 + z^2)^{3/2}}.$$

Integrating from $z = -\infty$ to $z = +\infty$ using $z = s\tan\theta$, $dz = s\,\sec^2\theta\, d\theta$:

$$B = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{s\, dz}{(s^2 + z^2)^{3/2}} = \frac{\mu_0 I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{s^2\,\sec^2\theta}{s^3\,\sec^3\theta}\, d\theta = \frac{\mu_0 I}{4\pi s} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s} \Big[\sin\theta\Big]_{-\pi/2}^{\pi/2} = \frac{\mu_0 I}{2\pi s}.$$

This matches the known result for the field of an infinite wire. By linearity, the Biot–Savart law applied to any current distribution gives the correct total field via superposition. 😃 😃

Proposition 2.5.5 Field of a long straight wire

An infinitely long straight wire carrying steady current I produces a magnetic field at perpendicular distance s given by

$$B = \frac{\mu_0 I}{2\pi s}.$$

The field lines are concentric circles around the wire, with direction given by the right-hand rule: point your right thumb along the current, and your fingers curl in the direction of $\vec{B}$. In cylindrical coordinates

(s, ϕ, z) with the wire along the z-axis,

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\phi}.$$

Proposition 2.5.6 Field at the center of a circular current loop

A circular loop of radius R carrying steady current I produces a magnetic field at its center given by

$$B = \frac{\mu_0 I}{2R}.$$

The direction is perpendicular to the plane of the loop, given by the right-hand rule: curl your fingers in the direction of the current, and your thumb points along $\vec{B}$. For N tightly wound turns, multiply by N:

$$B = \frac{\mu_0 N I}{2R}.$$

Proposition 2.5.7 Field on the axis of a circular current loop

A circular loop of radius R carrying steady current I produces a magnetic field on its symmetry axis at distance z from the center given by

$$B(z) = \frac{\mu_0 I\, R^2}{2\,(R^2 + z^2)^{3/2}}.$$

The field points along the axis in the direction given by the right-hand rule. At the center ($z = 0$), this reduces to $B = \mu_0 I/(2R)$. Far from the loop ($z \gg R$), the field falls off as

$$B \approx \frac{\mu_0 I\, R^2}{2\, z^3} = \frac{\mu_0}{4\pi}\,\frac{2\pi R^2 I}{z^3} = \frac{\mu_0}{4\pi}\,\frac{2\mu}{z^3},$$

which is the dipole-field form, where $\mu = \pi R^2 I$ is the magnetic dipole moment of the loop.

Corollary 2.5.4 Far-field (dipole) limit of a current loop

Far from any compact current loop, the magnetic field has the universal dipole form

$$\vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}}{r^3}\right],$$

where $\vec{\mu} = I\,A\,\hat{n}$ is the magnetic dipole moment (A is the loop area, $\hat{n}$ its normal). This is the analogue of the electric dipole field in electrostatics.

Corollary 2.5.5 Straight wire of finite length

For a finite straight wire segment of length L carrying current I, the field at a point perpendicular to the midpoint of the wire at distance s is

$$B = \frac{\mu_0 I}{2\pi s}\,\frac{L/2}{\sqrt{s^2 + (L/2)^2}}.$$

In the limit $L \to \infty$, the fraction approaches 1, recovering the infinite-wire result $B = \mu_0 I/(2\pi s)$.

Example 2.5.4 (Illustrative example)

A square loop of side a carries current I. Each of the four sides contributes equally. From the finite-wire formula with $L = a$ and the perpendicular distance from the midpoint to the center being $s = a/2$:

$$B_{\text{one side}} = \frac{\mu_0 I}{2\pi(a/2)} \cdot \frac{a/2}{\sqrt{(a/2)^2 + (a/2)^2}} = \frac{\mu_0 I}{\pi a} \cdot \frac{a/2}{a/\sqrt{2}} = \frac{\mu_0 I}{\pi a} \cdot \frac{\sqrt{2}}{2}.$$

Four sides contribute in the same direction (perpendicular to the square's plane), so

$$B_{\text{center}} = 4 \cdot \frac{\mu_0 I\sqrt{2}}{2\pi a} = \frac{2\sqrt{2}\,\mu_0 I}{\pi a}.$$

☺

Note:-

Key symmetry principles for Biot–Savart calculations:

- **Straight wire segments aimed directly at (or away from) the observation point** contribute *zero* field: when $d\vec{\ell} \parallel \vec{r}$, the cross product $d\vec{\ell} \times \hat{r} = \vec{0}$. This is a very useful shortcut.
- **Circular arcs** centered on the observation point contribute field proportional to the arc angle. For an arc of angle θ (in radians) and radius R, $B = (\mu_0 I/4\pi R) \cdot \theta$.
- **Perpendicular geometry** maximises the contribution: when $d\vec{\ell} \perp \vec{r}$ at every point (as on a circular arc centered at P), the magnitude is $dB = (\mu_0/4\pi)\, I\, d\ell/R^2$ with no angle factor.

Question 26: Worked example

A wire bent into the shape shown carries a steady current I in the direction indicated. The wire consists of three segments:

(i) A straight horizontal segment running from $x = -\infty$ to $x = -R$ along the line $y = 0$, approaching the origin.

(ii) A circular arc of radius R centered at the origin, extending from the point $(R, 0)$ counterclockwise through the upper half-plane to the point $(-R, 0)$ (a semicircle).

(iii) A straight horizontal segment running from $x = -R$ along the line $y = 0$ to $x = +\infty$, extending to the right.

Find the magnetic field $\vec{B}$ at the origin O (the center of the circular arc).
Assume the wire lies entirely in the xy-plane and the current I flows to the right on the incoming straight segment, then counterclockwise along the arc, then to the right on the outgoing straight segment. Use $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$.

Given quantities:

- Current: I
- Arc radius: R
- Arc: semicircle in the upper half-plane ($y \geqslant 0$), counterclockwise
- Observation point: origin O

Solution: **Strategy.** By the superposition principle, $\vec{B}_{\text{total}} = \vec{B}_{\text{straight, incoming}} + \vec{B}_{\text{arc}} + \vec{B}_{\text{straight, outgoing}}$. We evaluate each contribution separately.

(i) Incoming straight wire (segment from $x = -\infty$ to $x = -R$, along $y = 0$).

The current flows along the x-axis (the line $y = 0$) toward the origin. The observation point (the origin) lies *on the line* of the wire. For every current element $d\vec{\ell}$ on this segment, the displacement vector $\vec{r}$ from the element to the origin points along the $+x$ direction, which is *parallel* to $d\vec{\ell}$ (current flows in the $+x$ direction). Therefore

$$d\vec{\ell} \times \hat{r} = 0$$

for all elements of this segment, and

$$\vec{B}_{\text{incoming}} = \vec{0}.$$

The same reasoning applies to the outgoing wire.

(ii) Outgoing straight wire (segment from $x = -R$ to $x = +\infty$, along $y = 0$).

Again, the observation point lies on the line of the wire. The displacement $\vec{r}$ from every current element to the origin is collinear with $d\vec{\ell}$, so $d\vec{\ell} \times \hat{r} = \vec{0}$. Thus

$$\vec{B}_{\text{outgoing}} = \vec{0}.$$

(iii) Semicircular arc (radius R, upper half-plane, counterclockwise).

For the circular arc, every current element is at distance R from the origin, and every $d\vec{\ell}$ is tangent to the circle. The displacement vector from each element to the origin points radially inward (toward the center). The angle between $d\vec{\ell}$ (tangential) and $\vec{r}$ (radial) is $90°$, so $\sin 90° = 1$ everywhere.

The magnitude of each differential contribution is

$$dB = \frac{\mu_0}{4\pi} \frac{I\, d\ell}{R^2}.$$

The direction: by the right-hand rule, $d\vec{\ell} \times \hat{r}$ for counterclockwise current on the upper semicircle points in the $+\hat{k}$ direction (out of the page) everywhere along the arc.

The total magnitude is

$$B_{\text{arc}} = \int_{\text{arc}} \frac{\mu_0 I}{4\pi R^2}\, d\ell = \frac{\mu_0 I}{4\pi R^2} \int_{\text{arc}} d\ell = \frac{\mu_0 I}{4\pi R^2} \cdot (\pi R) = \frac{\mu_0 I}{4R}.$$

Here we used that the arc length is πR (a semicircle).

In vector form, with $\hat{k}$ pointing out of the xy-plane:

$$\vec{B}_{\text{arc}} = \frac{\mu_0 I}{4R}\,\hat{k}.$$

(iv) Total field.

Summing the three contributions:

$$\vec{B}_{\text{total}} = \vec{0} + \frac{\mu_0 I}{4R}\,\hat{k} + \vec{0} = \frac{\mu_0 I}{4R}\,\hat{k}.$$

Final answer:

$$\boxed{\vec{B} = \frac{\mu_0 I}{4R}\,\hat{k}}$$

The field points out of the page (perpendicular to the wire plane, in the $+\hat{k}$ direction by the right-hand rule for the counterclockwise arc current).

Check. If the arc were a full loop, we would recover $B = \mu_0 I/(2R)$ (the result from the centre-of-loop formula). Since we have a semicircle (half a loop), the field should be half of that: $B = \mu_0 I/(4R)$. This matches our result, confirming consistency.

2.5.5 Ampère's Law and Symmetry Reduction

This subsection states Ampère's law and shows how symmetry can reduce a difficult line integral to simple algebra when the current distribution is highly symmetric. It is the magnetic analogue of Gauss's law.

Definition 2.5.7: Amperian loop and enclosed current

Let C be any closed curve in space (called an *Amperian loop*). The *enclosed current* I_{enc} is the algebraic sum of all steady currents passing through any open surface S bounded by C. The sign of each current is determined by the right-hand rule: curl the fingers of your right hand along the direction of integration around C; if your thumb points in the direction of the current, that current counts as positive. Currents in the opposite direction count as negative.

Note:-

Just as with Gauss's law, Ampère's law is always true, but it is not always useful for finding $\vec{B}$. In a general asymmetric current distribution, knowing only the total enclosed current does not tell you the field at each point on the loop. The main strategy is therefore: first identify strong symmetry, then choose an Amperian loop matched to that symmetry so that $B = |\vec{B}|$ is constant on the field-contributing parts of the loop and the angle between $\vec{B}$ and $d\vec{\ell}$ is everywhere 0°, 180°, or 90°.

Theorem 2.5.6 Ampère's law and when symmetry makes it useful

Let C be any closed curve and I_{enc} the net steady current passing through any surface bounded by C. Then Ampère's law states

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}.$$

This law is always true. It becomes a practical method for solving for the magnetic field when the current distribution has enough symmetry that one can choose an Amperian loop for which the magnitude $B = |\vec{B}|$ is constant on each field-contributing part of the loop and the angle between $\vec{B}$ and $d\vec{\ell}$ is everywhere 0°, 180°, or 90°. Then the line integral reduces to algebraic terms such as $B\ell$, $-B\ell$, or 0. Common useful cases are cylindrical symmetry (long straight wires), planar symmetry (infinite current sheets), and solenoidal symmetry (ideal solenoids). The direction of $\vec{B}$ follows the right-hand rule relative to the enclosed current: if the thumb of your right hand points in the direction of the current, your fingers curl in the direction of the magnetic field circulation.

Note:-

Ampère's law is the magnetic analogue of Gauss's law. Gauss's law relates the electric field flux through a closed surface to the enclosed charge, $\oint \vec{E} \cdot d\vec{A} = q_{\text{enc}}/\varepsilon_0$. Ampère's law relates the magnetic field circulation around a closed loop to the enclosed current, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$. Both are universally valid but are practically useful for finding fields only when the source distribution has high symmetry. The matching of symmetry to geometry is parallel: spherical symmetry → spherical Gaussian surface, cylindrical symmetry → circular Amperian loop, planar symmetry → rectangular Amperian loop.

How symmetry reduces the line integral: Let a long straight wire carry current I along the $+z$ axis. By cylindrical symmetry, the magnetic field circulates around the wire in concentric circles in planes perpendicular to the wire, and its magnitude $B(r)$ depends only on the radial distance r from the wire axis. Choose a circular Amperian loop of radius r centred on the wire. Along this loop, $\vec{B}$ is everywhere tangent to $d\vec{\ell}$, so $\vec{B} \cdot d\vec{\ell} = B(r)\, d\ell$, and $B(r)$ is constant everywhere on the loop. Therefore,

$$\oint_C \vec{B} \cdot d\vec{\ell} = B(r) \oint_C d\ell = B(r)(2\pi r).$$

If the enclosed current is I_{enc}, Ampère's law gives

$$B(r)(2\pi r) = \mu_0 I_{\text{enc}}.$$

The law itself is general, but the symmetry is what allowed $B(r)$ to be pulled outside the integral. ☺

Proposition 2.5.8 Magnetic field of a long straight wire

Let $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A}$ be the permeability of free space. Consider a long straight wire of radius R carrying total steady current I with uniform current density J across its cross section. The magnetic field magnitude is

$$B(r) = \begin{cases} \dfrac{\mu_0 I\, r}{2\pi R^2} & \text{for } r < R \text{ (inside the wire)}, \\ \dfrac{\mu_0 I}{2\pi r} & \text{for } r > R \text{ (outside the wire)}. \end{cases}$$

Inside the wire the field grows linearly with r; outside it falls as $1/r$. In both regions, $\vec{B}$ is tangent to circles centred on the wire axis, in the direction given by the right-hand rule.

Corollary 2.5.6 Continuity of B at the wire surface

At the surface of the wire ($r = R$), both the inside and outside formulas give the same result:

$$B(R) = \frac{\mu_0 I\, R}{2\pi R^2} = \frac{\mu_0 I}{2\pi R}.$$

The magnetic field is continuous at the boundary even though the functional form changes. The maximum field occurs at the surface.

Proposition 2.5.9 Magnetic field of an infinite current sheet

An infinite planar sheet carrying uniform surface current density K (current per unit width, in units of A/m). Choose a rectangular Amperian loop of length L straddling the sheet, with two long sides parallel to the field and two short sides perpendicular. By symmetry, the field has constant magnitude on each side of the sheet and is parallel to the sheet but perpendicular to the current direction. Ampère's law gives

$$2BL = \mu_0(KL) \quad \Rightarrow \quad B = \frac{\mu_0 K}{2}.$$

The field reverses direction on opposite sides of the sheet. The field outside is independent of distance from the sheet.

Example 2.5.5 (Illustrative example)

A square loop of side 0.1 m carries a current $I = 2\,\mathrm{A}$ in a uniform magnetic field $B = 0.5\,\mathrm{T}$ perpendicular to the plane of the loop. The magnetic flux through the loop is

$$\Phi_B = BA = (0.5\,\mathrm{T})(0.1\,\mathrm{m})^2 = 5.0 \times 10^{-3}\,\mathrm{T\cdot m^2}.$$

Question 27: Worked example

A long cylindrical wire of radius $R = 2.0 \times 10^{-3}\,\mathrm{m} = 2.0\,\mathrm{mm}$ carries a steady current $I = 10\,\mathrm{A}$ uniformly distributed across its cross section. The current flows in the $+\hat{k}$ direction.
Find the magnetic field magnitude and direction at:

(a) $r_1 = 1.0 \times 10^{-3}\,\mathrm{m} = 1.0\,\mathrm{mm}$ (inside the wire), and

(b) $r_2 = 5.0 \times 10^{-3}\,\mathrm{m} = 5.0\,\mathrm{mm}$ (outside the wire).

Solution: Let the wire lie along the z axis with current flowing in the $+\hat{k}$ direction. By cylindrical symmetry, the magnetic field circulates around the wire in concentric circles in planes perpendicular to the wire. The field magnitude depends only on the radial distance r from the wire axis.

Choose a circular Amperian loop of radius r centred on the wire axis. Along this loop, $\vec{B}$ is everywhere tangent to $d\vec{\ell}$, so $\vec{B}\cdot d\vec{\ell} = B(r)\,d\ell$. The line integral becomes

$$\oint_C \vec{B}\cdot d\vec{\ell} = B(r)\oint_C d\ell = B(r)(2\pi r).$$

The enclosed current depends on whether the loop is inside or outside the wire. The uniform current density is

$$J = \frac{I}{\pi R^2}.$$

(a) Inside the wire ($r_1 < R$). The enclosed current is the fraction of the total current passing through the area inside the Amperian loop:

$$I_{\text{enc}} = J(\pi r_1^2) = \frac{I}{\pi R^2}(\pi r_1^2) = I\,\frac{r_1^2}{R^2}.$$

Substitute the values:

$$I_{\text{enc}} = (10\,\text{A})\,\frac{(1.0\times 10^{-3}\,\text{m})^2}{(2.0\times 10^{-3}\,\text{m})^2} = (10\,\text{A})\,\frac{1.0\times 10^{-6}}{4.0\times 10^{-6}}.$$

Thus

$$I_{\text{enc}} = (10\,\text{A})\times\frac{1}{4} = 2.5\,\text{A}.$$

Apply Ampère's law:

$$B(r_1)(2\pi r_1) = \mu_0 I_{\text{enc}}.$$

Solve for $B(r_1)$:

$$B(r_1) = \frac{\mu_0 I_{\text{enc}}}{2\pi r_1} = \frac{(4\pi\times 10^{-7}\,\text{T}\cdot\text{m/A})(2.5\,\text{A})}{2\pi(1.0\times 10^{-3}\,\text{m})}.$$

Compute step by step:

$$(4\pi\times 10^{-7})(2.5) = 10\pi\times 10^{-7},$$
$$2\pi(1.0\times 10^{-3}) = 2\pi\times 10^{-3},$$
$$B(r_1) = \frac{10\pi\times 10^{-7}}{2\pi\times 10^{-3}}\,\text{T} = 5.0\times 10^{-4}\,\text{T}.$$

The direction follows the right-hand rule: thumb along $+\hat{k}$ (current direction), fingers curl counterclockwise when viewed from above.

(b) Outside the wire ($r_2 > R$). The Amperian loop encloses the entire wire, so

$$I_{\text{enc}} = I = 10\,\text{A}.$$

Apply Ampère's law:

$$B(r_2)(2\pi r_2) = \mu_0 I.$$

Solve for $B(r_2)$:

$$B(r_2) = \frac{\mu_0 I}{2\pi r_2} = \frac{(4\pi\times 10^{-7}\,\text{T}\cdot\text{m/A})(10\,\text{A})}{2\pi(5.0\times 10^{-3}\,\text{m})}.$$

Compute step by step:

$$(4\pi\times 10^{-7})(10) = 40\pi\times 10^{-7},$$
$$2\pi(5.0\times 10^{-3}) = 10\pi\times 10^{-3},$$
$$B(r_2) = \frac{40\pi\times 10^{-7}}{10\pi\times 10^{-3}}\,\text{T} = 4.0\times 10^{-4}\,\text{T}.$$

The direction is again given by the right-hand rule: counterclockwise when viewed from above.

Final answers:

(a) $B(r_1) = 5.0\times 10^{-4}\,\text{T} = 0.50\,\text{mT}$, directed counterclockwise around the wire.

(b) $B(r_2) = 4.0\times 10^{-4}\,\text{T} = 0.40\,\text{mT}$, directed counterclockwise around the wire.

2.5.6 Solenoids, Parallel Currents, and Magnetic Dipoles

This subsection covers three closely related topics: the magnetic field inside a solenoid and a toroid (derived from Ampère's law), the magnetic force per unit length between two parallel current-carrying wires, and the magnetic dipole moment of a current loop. Together these describe how steady currents in geometrically ordered configurations produce well-defined magnetic fields and forces.

Definition 2.5.8: Ideal solenoid

An *ideal solenoid* is a long, tightly wound helical coil of wire. When the winding is close and the length L is much greater than the radius R, the magnetic field inside is uniform, axial, and of magnitude

$$B = \mu_0 \, n \, I,$$

where I is the current, $n = N/L$ is the number of turns per unit length (N is the total number of turns, L is the length of the solenoid), and $\mu_0 = 4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A}$ is the permeability of free space. Outside the ideal solenoid the field is approximately zero. The field direction follows the right-hand rule: curl the fingers of your right hand in the direction of the current around the coil, and your thumb points along $\vec{B}$ inside the solenoid.

Note:-

The field inside a solenoid depends only on the turn density n and the current I; it is independent of the radius of the coil. This is analogous to the electric field inside a uniformly charged infinite sheet, which depends only on the surface charge density and not on the lateral dimensions.

Theorem 2.5.7 Magnetic field of a long solenoid

For an ideal solenoid of length $L \gg R$ with N turns carrying current I, the magnetic field inside is uniform and directed along the axis. Its magnitude is

$$B = \mu_0 \, n \, I \qquad \text{where} \qquad n = \frac{N}{L}.$$

Outside the solenoid $B \approx 0$. The field lines are straight, parallel lines inside and loop back around outside (forming closed loops).

Definition 2.5.9: Toroid

A *toroid* is a solenoid bent into a circle (a doughnut shape). It consists of N turns wound around a circular core of mean radius r. By applying Ampère's law to a circular Amperian loop of radius r inside the toroid, the magnetic field inside the torus is

$$B = \frac{\mu_0 \, N \, I}{2\pi \, r}.$$

Outside the toroid (r is not within the winding region), $B \approx 0$. The field lines are circles concentric with the toroid axis.

Note:-

The toroid field formula $B = \mu_0 NI/(2\pi r)$ shows a $1/r$ dependence, unlike the uniform solenoid. For a toroid with a large mean radius and small cross-section (so r varies little across the windings), the field is nearly uniform and approximately $B \approx \mu_0 nI$ where $n = N/(2\pi r_{\text{avg}})$.

Proposition 2.5.10 Solenoid and toroid magnetic fields

The magnetic field produced by steady currents in these common device geometries is:

- **Long solenoid (inside):** $B = \mu_0 \, n \, I$, where $n = N/L$ is the turn density. The field is uniform and axial.

- **Toroid (inside the windings):** $B = \dfrac{\mu_0 N I}{2\pi r}$, where r is the radial distance from the toroid centre. The field circulates along circular field lines and decreases as $1/r$.
- **Outside both devices:** $B \approx 0$ (ideal case).

Solenoid field from Ampère's law (outline): Consider an ideal solenoid with n turns per unit length. Choose a rectangular Amperian loop with one long side (length ℓ) inside the solenoid parallel to the axis, the opposite side outside, and the two short sides perpendicular to the axis. The line integral of $\vec{B} \cdot d\vec{\ell}$ around the loop receives contributions only from the inside segment, because $\vec{B} \approx 0$ outside and $\vec{B} \perp d\vec{\ell}$ on the perpendicular segments. Thus

$$\oint \vec{B} \cdot d\vec{\ell} = B\,\ell.$$

The enclosed current is $I_{\text{enc}} = n\,\ell\,I$ (each of the $n\ell$ turns inside the loop carries current I). By Ampère's law, $B\,\ell = \mu_0\,n\,\ell\,I$, giving $B = \mu_0 n I$. 😊

Toroid field from Ampère's law: Choose a circular Amperian loop of radius r inside the toroid, concentric with the toroid axis. By symmetry, $\vec{B}$ is tangent to the circle and has constant magnitude B at fixed r. The line integral is

$$\oint \vec{B} \cdot d\vec{\ell} = B\,(2\pi r).$$

The loop encloses N turns each carrying current I, so $I_{\text{enc}} = NI$. Ampère's law gives $B(2\pi r) = \mu_0 N I$, yielding $B = \mu_0 N I/(2\pi r)$. 😊

Note:-

Both derivations rely on Ampère's law $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}$ and the presence of sufficient symmetry to pull B out of the integral. The solenoid requires the infinite-length approximation; the toroid requires circular symmetry. These are the two magnetostatic configurations in the AP Physics C curriculum where Ampère's law gives a clean result.

Definition 2.5.10: Magnetic force between two parallel current-carrying wires

Two long, straight, parallel wires separated by distance d, carrying steady currents I_1 and I_2, exert magnetic forces on each other. Wire 1 produces a magnetic field $B_1 = \dfrac{\mu_0 I_1}{2\pi d}$ at the location of wire 2. The force per unit length on wire 2 due to wire 1 is

$$\frac{F_{12}}{L} = \frac{\mu_0\,I_1\,I_2}{2\pi\,d}.$$

The force is *attractive* when the currents flow in the *same* direction and *repulsive* when they flow in *opposite* directions. By Newton's third law, the force per unit length on wire 1 due to wire 2 has the same magnitude and opposite direction.

Note:-

The rule for attraction and repulsion is the *opposite* of what happens with electric charges. Parallel currents (same direction) *attract*, while like electric charges *repel*. A useful mnemonic: "like currents attract, unlike repel" – but remember this refers to current *directions*, not charge types.

Theorem 2.5.8 Force per unit length between two parallel wires

Two long straight parallel wires separated by distance d carry currents I_1 and I_2. The magnitude of the magnetic force per unit length is

$$\frac{F}{L} = \frac{\mu_0\,I_1\,I_2}{2\pi\,d}.$$

- Currents in the **same direction** $\rightarrow$ **attractive** force.

- Currents in **opposite directions** $\rightarrow$ **repulsive** force.

The direction of the force on either wire is perpendicular to the wire and toward (or away from) the other wire, along the line connecting the wires.

Force between parallel wires from the Lorentz force: Wire 1 (carrying I_1) produces a magnetic field at the position of wire 2. By the right-hand rule for a long straight wire, B_1 circles wire 1 and has magnitude $B_1 = \mu_0 I_1/(2\pi d)$. The field at wire 2 is perpendicular to wire 2. The Lorentz force on a length L of wire 2 is $\vec{F}_{12} = I_2\, \vec{L} \times \vec{B}_1$. Since $\vec{L} \perp \vec{B}_1$, the magnitude is

$$F_{12} = I_2\, L\, B_1 = I_2\, L\, \frac{\mu_0 I_1}{2\pi d}.$$

Dividing by L gives $F_{12}/L = \mu_0 I_1 I_2/(2\pi d)$. The direction follows from the cross product: if both currents point upward, $\vec{B}_1$ at wire 2 points into the page, and $\vec{L}_2 \times \vec{B}_1$ points toward wire 1 (attractive). 😄

Theorem 2.5.9 Magnetic dipole moment of a current loop

A planar current loop carrying current I and enclosing area A has a *magnetic dipole moment*

$$\vec{\mu} = I\, A\, \hat{n},$$

where $\hat{n}$ is the unit normal to the plane of the loop, determined by the right-hand rule: curl the fingers of your right hand in the direction of the current, and your thumb points along $\hat{n}$. The SI unit of μ is the ampere-square metre (A·m^2), equivalent to joule per tesla (J/T). For a coil with N turns, $\vec{\mu} = NIA\, \hat{n}$.

Note:-

The magnetic dipole moment characterises the torque a current loop experiences in a uniform external field: $\vec{\tau} = \vec{\mu} \times \vec{B}$. It is also the quantity that determines the far-field of the loop – at distances much greater than the loop size, a current loop produces the same magnetic field as a magnetic dipole. This is the quantum-mechanical basis of atomic magnetism (electron orbital and spin angular momenta give rise to magnetic dipole moments).

Proposition 2.5.11 Magnetic dipole properties

For a planar current loop (or coil of N turns) with area A and current I:

- **Dipole moment:** $\mu = NIA$. The direction is given by the right-hand rule.
- **Torque in a uniform field:** $\vec{\tau} = \vec{\mu} \times \vec{B}$, with magnitude $\tau = \mu B \sin\theta$, where θ is the angle between $\vec{\mu}$ and $\vec{B}$.
- **Potential energy:** $U = -\vec{\mu} \cdot \vec{B} = -\mu B \cos\theta$. The minimum energy (stable equilibrium) occurs when $\vec{\mu}$ aligns with $\vec{B}$ ($\theta = 0$).

Corollary 2.5.7 Rectangular and circular loops

For a rectangular loop of sides a and b, $A = a\, b$. For a circular loop of radius R, $A = \pi R^2$. In both cases $\mu = IA$ for a single turn, and the dipole moment points along the axis of symmetry.

Example 2.5.6 (Illustrative example)

Two parallel wires are separated by $d = 8.0\,\text{cm}$ and carry currents $I_1 = 3.0\,\text{A}$ and $I_2 = 5.0\,\text{A}$ in the same direction. The force per unit length is

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{(4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(3.0\,\text{A})(5.0\,\text{A})}{2\pi(0.080\,\text{m})} = \frac{6 \times 10^{-6}}{0.080}\,\text{N/m} = 7.5 \times 10^{-5}\,\text{N/m},$$

attractive. A circular loop of radius 5.0 cm carrying 2.0 A has dipole moment $\mu = I(\pi R^2) = (2.0)(\pi)(0.050)^2 = 1.57 \times 10^{-2}\,\text{A·m}^2$.

Question 28: Worked example

A solenoid is 40.0 cm long, has $N = 600$ turns uniformly distributed along its length, and a circular cross-section of diameter 3.0 cm. A steady current $I = 4.0\,\text{A}$ flows through the wire.
Find:

(a) the turn density n of the solenoid,

(b) the magnitude of the magnetic field inside the solenoid,

(c) the direction of the magnetic field if the current, viewed from the left end, flows counterclockwise,

(d) the magnetic dipole moment of the solenoid, and

(e) the torque on the solenoid if it is placed in a uniform external magnetic field $B_{\text{ext}} = 0.15\,\text{T}$ with its axis at 30° to the field.

Solution: **Part (a).** The turn density is the number of turns divided by the length:

$$n = \frac{N}{L} = \frac{600}{0.400\,\text{m}} = 1500\,\text{turns/m}.$$

Part (b). The magnetic field inside the solenoid is

$$B = \mu_0\, n\, I.$$

Substitute the values:

$$B = \left(4\pi \times 10^{-7}\,\text{T·m/A}\right)\left(1500\,\text{m}^{-1}\right)\left(4.0\,\text{A}\right).$$

Compute step by step:

$$1500 \times 4.0 = 6000,$$

$$B = 4\pi \times 10^{-7} \times 6000 = 4\pi \times 6.0 \times 10^{-4} = 24\pi \times 10^{-4}\,\text{T}.$$

Using $\pi \approx 3.1416$:

$$B = 24 \times 3.1416 \times 10^{-4}\,\text{T} = 7.54 \times 10^{-3}\,\text{T} = 7.54\,\text{mT}.$$

Part (c). The current, viewed from the left end, flows counterclockwise. By the right-hand rule: curl the fingers of your right hand counterclockwise (as seen from the left), and your thumb points to the right. Thus the magnetic field inside the solenoid points to the **right** (along the solenoid axis, away from the viewer of the left end).

Part (d). The magnetic dipole moment of a solenoid is $\mu = NIA$, where A is the cross-sectional area of one turn. The radius of the solenoid is

$$R = \frac{3.0\,\text{cm}}{2} = 1.5\,\text{cm} = 0.015\,\text{m}.$$

The area is

$$A = \pi R^2 = \pi (0.015\,\text{m})^2 = \pi \times 2.25 \times 10^{-4}\,\text{m}^2 = 7.07 \times 10^{-4}\,\text{m}^2.$$

The dipole moment magnitude is

$$\mu = NIA = (600)(4.0\,\text{A})(7.07 \times 10^{-4}\,\text{m}^2).$$

Compute:

$$600 \times 4.0 = 2400,$$

$$\mu = 2400 \times 7.07 \times 10^{-4}\,\text{A·m}^2 = 1.70\,\text{A·m}^2.$$

The direction of $\vec{\mu}$ is along the axis to the right (same as $\vec{B}$ inside).

Part (e). The torque on a magnetic dipole in a uniform field is $\vec{\tau} = \vec{\mu} \times \vec{B}_{\text{ext}}$. Its magnitude is

$$\tau = \mu\, B_{\text{ext}}\, \sin\theta,$$

where $\theta = 30°$. Substituting:

$$\tau = (1.70\,\text{A}\cdot\text{m}^2)(0.15\,\text{T})\,\sin 30°.$$

Since $\sin 30° = 0.5$:

$$\tau = 1.70 \times 0.15 \times 0.5 = 0.1275\,\text{N}\cdot\text{m} \approx 0.13\,\text{N}\cdot\text{m}.$$

Final answers:

(a) $n = 1500\,\text{turns/m}$

(b) $B = 7.54\,\text{mT}$ (or $7.5 \times 10^{-3}\,\text{T}$)

(c) To the right (along the solenoid axis)

(d) $\mu = 1.70\,\text{A}\cdot\text{m}^2$

(e) $\tau = 0.13\,\text{N}\cdot\text{m}$

2.6 Electromagnetic Induction

This unit introduces the phenomena where changing magnetic conditions produce electric effects. In AP Physics C: Electricity and Magnetism, the core ideas begin with magnetic flux — a scalar measure of how much magnetic field penetrates a surface — and then culminate in Faraday's law, which quantifies how a time-varying magnetic flux induces an electromotive force. These concepts are the gateway from electrostatics to the full unification of electricity and magnetism.

The flow starts with magnetic flux and Faraday's law, then addresses the direction of induced currents through Lenz's law. From there, the unit covers motional emf (emf generated by conductors moving through a field), inductance and the magnetic energy stored in fields, and finally the transient behavior of LR circuits and the harmonic oscillations of LC circuits.

2.6.1 Magnetic Flux

Magnetic flux is a scalar quantity that measures the total magnetic field passing through a given surface. It is the magnetic analogue of electric flux and provides the natural framework for understanding Faraday's law of induction, which is the subject of the next section.

Definition 2.6.1: Magnetic flux

Let $\vec{B}$ denote the magnetic-field vector and let S be an oriented surface. The *magnetic flux* through S is the surface integral

$$\Phi_B = \iint_S \vec{B} \cdot d\vec{A},$$

where $d\vec{A}$ is the vector area element: its magnitude is the area of the infinitesimal surface patch, and its direction is normal to the surface. For an open surface the orientation (and hence the direction of $d\vec{A}$) must be specified; for a closed surface the convention is that $d\vec{A}$ points outward.

The magnetic flux Φ_B is a scalar quantity. Its sign depends on the relative orientation of the surface normal and the field: positive when $\vec{B}$ has a component along the surface normal, negative when it opposes the normal, and zero when $\vec{B}$ is tangent to the surface.

Note:-

Think of magnetic flux as the "number of magnetic field lines" passing through a surface. A larger surface catches more lines; tilting the surface reduces the effective area and hence the flux; reversing the surface normal flips the sign of the flux. This picture is qualitative, but it is extremely useful for building intuition about how

Φ_B changes when the surface moves or rotates.

Theorem 2.6.1 Magnetic flux through a flat surface in a uniform field

When the magnetic field $\vec{B}$ is uniform (constant in magnitude and direction) and the surface is flat with area A and unit normal $\hat{n}$, the surface integral simplifies to a single dot product:

$$\Phi_B = \vec{B} \cdot \vec{A} = B\,A\,\cos\theta,$$

where $\vec{A} = A\,\hat{n}$ is the area vector, $B = |\vec{B}|$, and θ is the angle between $\vec{B}$ and the area normal $\hat{n}$.

Note:-

The angle θ is always measured between $\vec{B}$ and the *surface normal*, *not* between $\vec{B}$ and the surface itself. When $\vec{B}$ is perpendicular to the surface, $\theta = 0°$ and $\cos\theta = 1$, giving maximum flux $\Phi_B = BA$. When $\vec{B}$ is parallel to the surface, $\theta = 90°$ and $\cos\theta = 0$, giving zero flux. These are the two extremes that frequently appear on exams.

Proposition 2.6.1 Magnetic flux: key properties

- **SI unit:** The weber, Wb, where $1\,\text{Wb} = 1\,\text{T}\cdot\text{m}^2$.
- **Scalar:** Φ_B is a scalar. It can be positive, negative, or zero, depending on the choice of surface orientation.
- **Orientation dependence:** $\Phi_B = BA\cos\theta$, where θ is the angle between $\vec{B}$ and the surface normal. Flux is maximal when $\vec{B} \perp$ surface ($\theta = 0°$) and zero when $\vec{B} \parallel$ surface ($\theta = 90°$).
- **Comparison with electric flux:** The electric flux through a surface is $\Phi_E = \iint_S \vec{E} \cdot d\vec{A}$. The mathematical structure is identical; the only difference is that Φ_E can be non-zero for closed surfaces enclosing net charge ($\Phi_E = Q_{\text{enc}}/\varepsilon_0$, Gauss's law), while Φ_B is always zero through a closed surface.
- **Time dependence:** If any of B, A, or θ changes with time, the flux changes: $\Delta\Phi_B = \Phi_{B,f} - \Phi_{B,i}$. A changing magnetic flux is the fundamental ingredient behind electromagnetic induction (Faraday's law).

Corollary 2.6.1 Gauss's law for magnetism (qualitative)

The net magnetic flux through any *closed* surface is zero:

$$\oiint_{\text{closed}} \vec{B} \cdot d\vec{A} = 0.$$

This is Gauss's law for magnetism. It reflects the experimental fact that magnetic monopoles have never been observed: magnetic field lines always form closed loops, so every field line that enters a closed surface must also exit it. This stands in contrast to electric flux through a closed surface, which is proportional to the enclosed charge.

Example 2.6.1 (Illustrative example)

A square loop of side 0.20 m sits in a uniform magnetic field $B = 0.30\,\text{T}$. The field makes an angle $\theta = 35°$ with the loop's normal. The area is $A = (0.20\,\text{m})^2 = 0.040\,\text{m}^2$, so the flux is

$$\Phi_B = B\,A\,\cos\theta = (0.30\,\text{T})(0.040\,\text{m}^2)\cos 35° = 0.012 \times 0.8192 = 9.83 \times 10^{-3}\,\text{Wb}.$$

Question 29: Worked example

A rectangular loop of wire has width $w = 12.0\,\text{cm} = 0.120\,\text{m}$ and height $h = 8.00\,\text{cm} = 0.0800\,\text{m}$. The loop is situated in a uniform magnetic field of magnitude $B = 0.450\,\text{T}$. The field is constant in space. The loop has area $A = w\,h$.
Find the magnetic flux Φ_B through the loop for each of the following orientations:

(a) The loop is in the xy-plane and the magnetic field points in the $+\hat{k}$ direction (perpendicular to the loop, with the area normal also taken as $+\hat{k}$).

(b) The loop is in the xz-plane and the magnetic field points in the $+\hat{k}$ direction (the field is parallel to the loop).

(c) The loop is in the xy-plane and the magnetic field lies in the xz-plane, making an angle $\theta = 30.0°$ below the x-axis. Take the area normal as $+\hat{k}$.

(d) The loop is in the yz-plane and the magnetic field points in the $+\hat{k}$ direction (the field is perpendicular to the loop, with the area normal taken as $+\hat{k}$).

(e) The loop is in the xy-plane. The magnetic field is $\vec{B} = (0.450\,\text{T})\,\hat{\imath} + (0.300\,\text{T})\,\hat{\jmath}$. Take the area normal as $+\hat{k}$.

Solution: First compute the loop's area:

$$A = w\,h = (0.120\,\text{m})(0.0800\,\text{m}) = 9.60 \times 10^{-3}\,\text{m}^2.$$

The area vector is $\vec{A} = A\,\hat{n}$, where $\hat{n}$ is the unit normal determined by the loop's orientation. Since $\vec{B}$ is uniform, we use $\Phi_B = \vec{B} \cdot \vec{A} = B\,A\,\cos\theta$, where θ is the angle between $\vec{B}$ and $\hat{n}$.

(a) The loop is in the xy-plane with normal $\hat{n} = +\hat{k}$, and $\vec{B} = B\,\hat{k}$. The angle between $\vec{B}$ and $\hat{n}$ is $\theta = 0°$:

$$\Phi_B = B\,A\,\cos 0° = (0.450\,\text{T})(9.60 \times 10^{-3}\,\text{m}^2)(1).$$

Compute:

$$\Phi_B = 0.450 \times 9.60 \times 10^{-3}\,\text{Wb} = 4.32 \times 10^{-3}\,\text{Wb}.$$

(b) The loop is in the xz-plane with normal $\hat{n} = +\hat{\jmath}$ (by the right-hand rule, or equivalently, the area vector points along y). The field is $\vec{B} = B\,\hat{k}$. The angle between $\vec{B}$ and $\hat{n}$ is $\theta = 90°$:

$$\Phi_B = B\,A\,\cos 90° = 0.$$

No field lines pass through the loop; the field runs parallel to the plane of the loop.

(c) The loop is in the xy-plane with normal $\hat{n} = +\hat{k}$. The field is $\vec{B}$ lying in the xz-plane at $30.0°$ below the x-axis. We need the angle between $\vec{B}$ and $\hat{k}$. Since $\vec{B}$ is in the xz-plane and makes $30.0°$ with the x-axis, the angle with the z-axis ($\hat{k}$) is $\theta = 90° - 30.0° = 60.0°$:

$$\Phi_B = B\,A\,\cos 60.0° = (0.450\,\text{T})(9.60 \times 10^{-3}\,\text{m}^2)\,(0.500).$$

Compute:

$$\Phi_B = 0.450 \times 9.60 \times 10^{-3} \times 0.500\,\text{Wb} = 2.16 \times 10^{-3}\,\text{Wb}.$$

(d) The loop is in the yz-plane with normal $\hat{n} = +\hat{k}$, and the field is $\vec{B} = B\,\hat{k}$. Again $\theta = 0°$:

$$\Phi_B = B\,A\,\cos 0° = (0.450\,\text{T})(9.60 \times 10^{-3}\,\text{m}^2)(1).$$

$$\Phi_B = 4.32 \times 10^{-3}\,\text{Wb}.$$

The orientation of the loop in space does not matter; only the relative angle between $\vec{B}$ and the area normal matters. Since $\vec{B}$ is again perpendicular to the loop and aligned with the normal, the flux is the same as in part (a).

(e) The loop is in the xy-plane with normal $\hat{n} = +\hat{k}$. The field is $\vec{B} = (0.450\,\mathrm{T})\,\hat{\imath} + (0.300\,\mathrm{T})\,\hat{\jmath}$. The area vector is $\vec{A} = A\,\hat{k}$. The flux is the dot product:

$$\Phi_B = \vec{B}\cdot\vec{A} = \left[(0.450)\,\hat{\imath} + (0.300)\,\hat{\jmath}\right]\cdot\left[(9.60\times10^{-3})\,\hat{k}\right].$$

Since $\hat{\imath}\cdot\hat{k} = 0$ and $\hat{\jmath}\cdot\hat{k} = 0$:

$$\Phi_B = 0.$$

The field lies entirely in the xy-plane, parallel to the loop, so no flux passes through.

Final answers:

(a) $\Phi_B = 4.32\times10^{-3}\,\mathrm{Wb} = 4.32\,\mathrm{mWb}$

(b) $\Phi_B = 0$

(c) $\Phi_B = 2.16\times10^{-3}\,\mathrm{Wb} = 2.16\,\mathrm{mWb}$

(d) $\Phi_B = 4.32\times10^{-3}\,\mathrm{Wb} = 4.32\,\mathrm{mWb}$

(e) $\Phi_B = 0$

2.6.2 Faraday's Law of Induction

This subsection states Faraday's law of induction, explains the meaning of the sign in the law via Lenz's law, and derives the motional-EMF formula $\mathcal{E} = B\ell v$ as a special case. It establishes the connection between Faraday's law and energy conservation.

Definition 2.6.2: Electromotive force (EMF)

The *electromotive force* (EMF) $\mathcal{E}$ around a closed loop C is the work done per unit charge by the non-electrostatic forces that drive charge carriers around the loop. In terms of the total force per unit charge $\vec{f}_{\text{non-e}}$ (which may include magnetic Lorentz forces or induced electric fields),

$$\mathcal{E} = \oint_C \vec{f}_{\text{non-e}}\cdot d\vec{\ell}.$$

In a resistive loop of total resistance R, the induced current obeys $\mathcal{E} = IR$.

Theorem 2.6.2 Faraday's law of induction

Let C be a closed conducting loop and S any surface bounded by C. The magnetic flux through S is

$$\Phi_B = \int_S \vec{B}\cdot d\vec{A},$$

where $d\vec{A}$ is the oriented area element (direction given by the right-hand rule from the chosen circulation direction around C). Faraday's law states that the induced electromotive force around the loop is

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

The minus sign encodes Lenz's law: the induced current produces a magnetic field that opposes the change in flux that created it.

Derivation from Faraday's law: The EMF is the work per unit charge by the induced non-conservative field: $\mathcal{E} = \oint_C \vec{E}\cdot d\vec{\ell}$. When the magnetic flux $\Phi_B = \int_S \vec{B}\cdot d\vec{A}$ through the loop changes in time, a non-conservative electric field is induced with non-zero circulation. Energy conservation requires this circulation to equal the rate of flux change (with the minus sign from Lenz's law):

$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_B}{dt},$$

which is Faraday's law of induction. ☺

Note:-

The surface S is not unique — any surface bounded by the same loop C gives the same flux because $\nabla \cdot \vec{B} = 0$. The orientation of $d\vec{A}$ is fixed by the right-hand rule applied to the chosen direction of integration around C: if your fingers curl along the integration direction, your thumb points along $d\vec{A}$. The sign of Φ_B then tells you whether the field generally threads the loop in the "positive" direction (along $d\vec{A}$) or the "negative" direction (opposite to $d\vec{A}$). A negative $d\Phi_B/dt$ means the flux in the positive direction is decreasing, so $\mathcal{E} > 0$ and the induced EMF drives current in the positive (integration) direction.

Note:-

Faraday's law can be understood through two complementary mechanisms that both change the flux:

- **Transformer EMF:** The loop is stationary but the magnetic field $\vec{B}(t)$ changes with time. An induced non-conservative electric field $\vec{E}$ is created by $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$, and this field drives the current.
- **Motional EMF:** The magnetic field is static but the loop moves, changes shape, or rotates. The magnetic component of the Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$ on the charge carriers drives the current.

Both mechanisms give the same $\mathcal{E} = -d\Phi_B/dt$; the distinction is frame-dependent.

Proposition 2.6.2 Differential form (Maxwell–Faraday equation)

A time-varying magnetic field produces a spatially non-conservative electric field. In differential form,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

By Stokes' theorem, for a stationary loop C bounding surface S, this is equivalent to the integral form:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{A} = -\frac{d\Phi_B}{dt}.$$

This equation describes the transformer-EMF mechanism and holds even in the absence of any physical conductor.

Proposition 2.6.3 EMF for a coil of N turns

If a tightly wound coil has N identical turns and the same flux Φ_B passes through each turn, the total EMF is

$$\mathcal{E} = -N\,\frac{d\Phi_B}{dt}.$$

The product $N\Phi_B$ is called the *flux linkage*. The factor N appears because the EMFs of the individual turns add in series.

Proposition 2.6.4 Motional EMF

When a straight conductor of length ℓ moves with velocity $\vec{v}$ perpendicular to a uniform magnetic field $\vec{B}$, the magnetic Lorentz force on the charge carriers produces an EMF

$$\mathcal{E} = B\,\ell\,v.$$

More generally, for a moving loop the motional EMF is $\mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{\ell}$, and the total EMF (including any transformer contribution) is $\mathcal{E} = -d\Phi_B/dt$.

Corollary 2.6.2 Energy conservation and the sign of Faraday's law

The minus sign in Faraday's law is required by conservation of energy. If the induced current reinforced the flux change instead of opposing it, a small perturbation would produce a positive feedback loop that creates energy from nothing. The minus sign ensures that mechanical work done to change the flux is converted to electrical energy (dissipated as Joule heat in the resistance of the loop).

Example 2.6.2 (Illustrative example)

A conducting rod of length $\ell = 0.50\,\mathrm{m}$ slides at constant speed $v = 4.0\,\mathrm{m/s}$ on frictionless rails in a uniform magnetic field $B = 0.30\,\mathrm{T}$ perpendicular to the plane of the circuit. The rod and rails form a closed loop of total resistance $R = 2.0\,\Omega$. The induced EMF is $\mathcal{E} = B\ell v = (0.30\,\mathrm{T})(0.50\,\mathrm{m})(4.0\,\mathrm{m/s}) = 0.60\,\mathrm{V}$, and the induced current is $I = \mathcal{E}/R = 0.30\,\mathrm{A}$. The magnetic force on the rod is $F = BI\ell = 0.036\,\mathrm{N}$, opposing the motion.

Question 30: Worked example

A rectangular conducting loop has horizontal width $L = 0.400\,\mathrm{m}$ and vertical height $w = 0.200\,\mathrm{m}$, and total resistance $R = 5.00\,\Omega$. The loop lies in the xy-plane and is partially inside a region of uniform magnetic field $\vec{B} = 0.750\,\mathrm{T}$ pointing in the $-\hat{k}$ direction (into the page). The field region is confined to the half-space $x < x_0$, and the loop is being pulled to the right with constant velocity $v = 3.00\,\mathrm{m/s}$, as shown. At the instant shown, the right edge of the loop is outside the field region and the left edge is still inside it.

(a) Find the magnitude and direction of the induced current in the loop.

(b) Find the magnitude and direction of the applied force required to maintain the constant velocity.

(c) Verify that the mechanical power input by the applied force equals the electrical power dissipated in the loop's resistance.

Solution: **Geometry and sign conventions.** The loop lies in the xy-plane. The magnetic field is $\vec{B} = -(0.750\,\mathrm{T})\,\hat{k}$. Choose the integration direction around the loop to be clockwise (as viewed from above the xy-plane). By the right-hand rule, this makes the area vector point in the $-\hat{k}$ direction (into the page), so $d\vec{A} = dA\,(-\hat{k})$. The area vector and the field are in the same direction, so the flux Φ_B is positive.

Let x be the horizontal extent of the portion of the loop still inside the field region. At the instant shown, the loop is partially out, so $0 < x < L$. The flux is

$$\Phi_B = \vec{B} \cdot \vec{A} = BA = B\,w\,x,$$

where $A = w\,x$ is the area of the loop inside the field. As the loop is pulled to the right, x decreases, so $dx/dt = -v$.

(a) Induced current. The rate of change of flux is

$$\frac{d\Phi_B}{dt} = B\,w\,\frac{dx}{dt} = B\,w\,(-v) = -B\,w\,v.$$

By Faraday's law, the EMF is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = B\,w\,v.$$

Substitute the values:

$$\mathcal{E} = (0.750\,\mathrm{T})(0.200\,\mathrm{m})(3.00\,\mathrm{m/s}).$$

Compute step by step:

$$(0.750)(0.200) = 0.150, \qquad (0.150)(3.00) = 0.450,$$

so

$$\mathcal{E} = 0.450\,\mathrm{V}.$$

The EMF is positive, meaning the induced current flows in the chosen (clockwise) direction. Using Ohm's law:

$$I = \frac{\mathcal{E}}{R} = \frac{0.450\,\mathrm{V}}{5.00\,\Omega} = 0.0900\,\mathrm{A}.$$

The current flows **clockwise** as viewed from above.

(b) Applied force. The portion of the loop inside the field that carries current and experiences a magnetic force is the left (leading) vertical segment, of length $w = 0.200\,\mathrm{m}$. The current in this segment flows downward (consistent with the clockwise circulation). The magnetic field points into the page $(-\hat{k})$. The magnetic force on this segment is

$$\vec{F}_{\rm mag} = I\vec{\ell} \times \vec{B} = Iw(-\hat{\jmath}) \times (-B\,\hat{k}) = IBw\,(\hat{\jmath} \times \hat{k}) = IBw\,\hat{\imath}.$$

The magnetic force points to the right (in the direction of motion). To maintain constant velocity, the net force must be zero, so the applied force must balance the magnetic force:

$$\vec{F}_{\rm app} + \vec{F}_{\rm mag} = 0 \quad \Rightarrow \quad \vec{F}_{\rm app} = -IBw\,\hat{\imath}.$$

The applied force points to the **left** (opposite to the direction of motion) with magnitude

$$F_{\rm app} = IBw = BIw.$$

Substitute the values:

$$F_{\rm app} = (0.750\,\mathrm{T})(0.0900\,\mathrm{A})(0.200\,\mathrm{m}).$$

Compute:

$$(0.750)(0.0900) = 0.0675, \qquad (0.0675)(0.200) = 0.0135,$$

so

$$F_{\rm app} = 0.0135\,\mathrm{N}.$$

(c) Energy conservation. The electrical power dissipated in the loop is

$$P_{\rm elec} = I^2R = I\mathcal{E} = I(Bwv).$$

Substitute:

$$P_{\rm elec} = (0.0900\,\mathrm{A})(0.450\,\mathrm{V}) = 0.0405\,\mathrm{W}.$$

The mechanical power delivered by the applied force is

$$P_{\rm mech} = \vec{F}_{\rm app} \cdot \vec{v}.$$

Since $\vec{F}_{\rm app}$ points left and $\vec{v}$ points right, $\vec{F}_{\rm app} \cdot \vec{v} = -F_{\rm app}v$.

$$P_{\rm mech} = -(0.0135\,\mathrm{N})(3.00\,\mathrm{m/s}) = -0.0405\,\mathrm{W}.$$

The mechanical power is negative because the applied force opposes the motion (you are holding back the loop). The magnetic force does positive work at rate $+0.0405\,\mathrm{W}$, which exactly equals the electrical power dissipated. Thus,

$$|P_{\rm mech}| = P_{\rm elec} = 0.0405\,\mathrm{W}.$$

This verifies energy conservation: mechanical energy lost by the system equals electrical energy dissipated as heat.

Final answers:

(a) $I = 0.0900\,\mathrm{A}$, clockwise.

(b) $F_{\rm app} = 0.0135\,\mathrm{N}$, to the left.

(c) $P_{\rm mech} = -0.0405\,\mathrm{W}$, $P_{\rm elec} = 0.0405\,\mathrm{W}$; magnitudes equal.

2.6.3 Lenz's Law and Induced Current Direction

This subsection explains how the sign in Faraday's law encodes the direction of induced current, states Lenz's law in its operational form, and connects it to energy conservation.

Definition 2.6.3: Lenz's law and induced current direction

Let a closed conducting loop bound an oriented surface S with unit normal $\hat{n}$ determined by the right-hand rule from the chosen circulation direction. The magnetic flux through the loop is

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}.$$

Faraday's law gives the induced electromotive force around the loop:

$$\mathcal{E} = -\frac{d\Phi_B}{dt}.$$

Lenz's law states that the induced current flows in the direction that produces a magnetic field $\vec{B}_{\text{ind}}$ opposing the change in flux Φ_B.
The operational procedure is:

(1) Determine the direction of the external magnetic field $\vec{B}_{\text{ext}}$ through the loop.

(2) Determine whether Φ_B is increasing or decreasing.

(3) The induced field $\vec{B}_{\text{ind}}$ points in the same direction as $\vec{B}_{\text{ext}}$ if the flux is decreasing, and in the opposite direction if the flux is increasing.

(4) Use the right-hand rule on $\vec{B}_{\text{ind}}$ to find the induced current direction: curl the fingers of your right hand in the current direction; your thumb points along $\vec{B}_{\text{ind}}$.

Note:-

The negative sign in Faraday's law *is* Lenz's law written as an equation. If the sign were positive, the induced current would reinforce the flux change, producing more flux in the same direction, which would drive yet more current — an energy-creating runaway. Lenz's law prevents this by ensuring the induced field opposes the change.

Theorem 2.6.3 Lenz's law (energy-conservation form)

The direction of induced current in any closed loop is always such that the magnetic force or torque on the loop opposes the motion or change that produced the induction. Equivalently, mechanical work must be done against the magnetic forces to sustain the change in flux; this work is converted to electrical energy (and ultimately to thermal energy in the resistance of the loop).

Lenz's law from energy conservation: Suppose a magnet is pushed toward a conducting loop. The induced current creates a magnetic field $\vec{B}_{\text{ind}}$ that opposes the approaching magnet. An external agent must do positive work against the magnetic repulsion to keep the magnet moving. This work supplies the electrical energy dissipated as Joule heating in the loop.

If Lenz's law were reversed — if the induced field *aided* the approaching magnet — the magnet would accelerate toward the loop without any external work, increasing both the kinetic energy of the magnet and the electrical energy dissipated in the loop, with no energy input. This violates conservation of energy. Therefore, the minus sign in Faraday's law is required by energy conservation.

Corollary 2.6.3 Flux-change sign convention

When the flux through a loop is increasing ($d\Phi_B/dt > 0$), the induced EMF is negative and the induced current flows in the direction that creates a field opposing the external field. When the flux is decreasing ($d\Phi_B/dt < 0$), the induced EMF is positive and the induced current flows in the direction that reinforces the external field.

Example 2.6.3 (Illustrative example)

A circular loop of radius R lies in a uniform magnetic field $\vec{B}$ pointing out of the page. The radius is shrunk at constant speed. Since the outward flux $\Phi_B = BR^2\pi$ is decreasing, the induced current must create an outward field to oppose the decrease. By the right-hand rule, this corresponds to a counterclockwise current.

Question 31: Worked example

A bar magnet with its north pole facing downward is released from rest above a horizontal copper ring. The magnet falls along the central axis of the ring.

(a) As the north pole *approaches* the ring from above, what is the direction of the induced current in the ring as viewed from above?

(b) Once the magnet has passed through and the south pole is *leaving* the ring from below, what is the direction of the induced current in the ring as viewed from above?

Solution: We view the ring from above (looking downward along the axis of the magnet). Let downward be the direction of the external field $\vec{B}_{\text{ext}}$ through the ring (the field lines emerge from the north pole and point downward through the ring while the north pole is above it, and continue downward through the ring while the south pole is below it).

Part (a): As the north pole approaches the ring from above, the downward magnetic field through the ring becomes stronger. The downward flux Φ_B is therefore *increasing*. By Lenz's law, the induced current must create an induced magnetic field $\vec{B}_{\text{ind}}$ that opposes this increase, so $\vec{B}_{\text{ind}}$ must point *upward* (out of the page). Using the right-hand rule, with the thumb pointing upward, the fingers curl *counterclockwise*. The induced current is **counterclockwise** as viewed from above.

Part (b): As the south pole leaves the ring from below, the downward magnetic field through the ring becomes weaker. The downward flux Φ_B is therefore *decreasing*. By Lenz's law, the induced current must create an induced magnetic field $\vec{B}_{\text{ind}}$ that opposes this decrease, so $\vec{B}_{\text{ind}}$ must point *downward* (into the page) to supplement the collapsing field. Using the right-hand rule, with the thumb pointing downward, the fingers curl *clockwise*. The induced current is **clockwise** as viewed from above.

Therefore,

$$\text{(a) counterclockwise,} \qquad \text{(b) clockwise.}$$

2.6.4 Motional Electromotive Force

When a conductor moves through a magnetic field, the magnetic force on the charge carriers inside the conductor separates charge and produces an electromotive force. This effect, called *motional emf*, is one of the two fundamental mechanisms of electromagnetic induction (the other being a time-varying magnetic field).

Definition 2.6.4: Motional emf

When a conductor moves with velocity $\vec{v}$ through a magnetic field $\vec{B}$, each charge carrier of charge q experiences the magnetic force

$$\vec{F}_B = q\,\vec{v} \times \vec{B}.$$

This force acts as a non-electrostatic force per unit charge, $\vec{v}\times\vec{B}$, which drives charges along the conductor. The *motional emf* along a path from point a to point b within the conductor is the line integral

$$\mathcal{E} = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{\ell},$$

where $d\vec{\ell}$ is an infinitesimal displacement vector along the conductor from a to b. The motional emf represents the work done per unit charge by the magnetic force as charges move along the conductor.

Note:-

Even though the magnetic force itself does no net work on a moving charge (since $\vec{F}_B \perp \vec{v}$), the motional emf arises because the conductor's motion provides the energy transfer mechanism. The external agent pushing the conductor does mechanical work; the magnetic field acts as the intermediary that converts this work into electrical energy.

Note:-

Inside the moving conductor (the "source"), the emf drives current from lower potential to higher potential, just as a battery drives current from its negative to its positive terminal. The moving conductor thus acts as a battery with emf $\mathcal{E}$ and zero internal resistance (if the conductor is ideal).

Theorem 2.6.4 Motional emf of a straight conductor in a uniform field

Let a straight conductor of length ℓ move with constant velocity $\vec{v}$ through a uniform magnetic field $\vec{B}$.

- **General case:** The motional emf between the two ends of the conductor is

$$\mathcal{E} = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{\ell},$$

  where the integral is taken along the conductor from end a to end b.

- **Mutually perpendicular case:** When $\vec{v}$, $\vec{B}$, and the conductor are mutually perpendicular, the motional emf simplifies to

$$\mathcal{E} = B\,\ell\,v.$$

- **Direction of emf:** The end of the conductor toward which positive charges are pushed by $\vec{v} \times \vec{B}$ is at higher potential. Use the right-hand rule for the cross product $\vec{v} \times \vec{B}$: point your fingers along $\vec{v}$, curl toward $\vec{B}$; your thumb points in the direction of $\vec{v} \times \vec{B}$, which is the direction positive charges move inside the conductor.

Motional emf from the Lorentz force: Consider a straight conducting bar of length ℓ moving with velocity $\vec{v}$ to the right through a uniform magnetic field $\vec{B}$ pointing out of the page. The bar is oriented vertically, perpendicular to both $\vec{v}$ and $\vec{B}$.

Each conduction electron of charge $-e$ experiences a magnetic force

$$\vec{F}_e = -e\,(\vec{v} \times \vec{B}).$$

With $\vec{v}$ to the right $(+\hat{\imath})$ and $\vec{B}$ out of the page $(+\hat{k})$,

$$\vec{v} \times \vec{B} = vB\,(\hat{\imath} \times \hat{k}) = -vB\,\hat{\jmath},$$

so the force on electrons is $-e(-vB\,\hat{\jmath}) = evB\,\hat{\jmath}$, pointing *upward* along the bar. Electrons accumulate at the top, leaving the bottom end positively charged.

Charge separation continues until the resulting electrostatic field $\vec{E}_{\text{ind}}$ balances the magnetic force:

$$-e\,\vec{E}_{\text{ind}} + (-e)(\vec{v}\times\vec{B}) = \vec{0} \quad \Rightarrow \quad \vec{E}_{\text{ind}} = -(\vec{v}\times\vec{B}).$$

The motional emf is the line integral of the non-electrostatic force per unit charge along the bar. Equivalently, it equals the potential difference between the ends:

$$\mathcal{E} = \int_{\text{bottom}}^{\text{top}} \vec{E}_{\text{ind}}\cdot d\vec{\ell} = E_{\text{ind}}\,\ell.$$

Since $\vec{E}_{\text{ind}} = vB\,\hat{\jmath}$ and the bar extends along $\hat{\jmath}$, we obtain

$$\mathcal{E} = vB\,\ell.$$

More generally, without assuming perpendicularity,

$$\mathcal{E} = \int_a^b (\vec{v}\times\vec{B})\cdot d\vec{\ell},$$

which reduces to $B\ell v$ when $\vec{v}$, $\vec{B}$, and $d\vec{\ell}$ are mutually perpendicular.

Proposition 2.6.5 Motional emf in a circuit with resistance

A conducting bar of length ℓ slides on frictionless conducting rails in a uniform magnetic field B, with the rails connected by a resistor R. The bar moves with instantaneous speed v perpendicular to B. Then:

1. The motional emf is
$$\mathcal{E} = B\,\ell\,v.$$

2. The induced current in the circuit (magnitude) is
$$I = \frac{\mathcal{E}}{R} = \frac{B\,\ell\,v}{R}.$$

3. The magnetic force on the bar (magnitude) opposes the motion:
$$F_{\text{mag}} = I\,\ell\,B = \frac{B^2\,\ell^2\,v}{R}.$$

4. The external force needed to maintain constant velocity v equals the magnetic force in magnitude:
$$F_{\text{ext}} = \frac{B^2\,\ell^2\,v}{R}.$$

5. Power dissipated in the resistor:
$$P_{\text{elec}} = I^2 R = \frac{B^2\,\ell^2\,v^2}{R}.$$

6. Mechanical power delivered by the external agent:
$$P_{\text{mech}} = F_{\text{ext}}\,v = \frac{B^2\,\ell^2\,v^2}{R}.$$

Thus $P_{\text{mech}} = P_{\text{elec}}$, consistent with conservation of energy.

Corollary 2.6.4 Motional emf and Faraday's law agree

For a conducting bar sliding on rails, Faraday's law gives the induced emf as $\mathcal{E} = -\frac{d\Phi_B}{dt}$. If the bar moves a distance dx in time dt, the area changes by $dA = \ell\, dx$ and the flux by $d\Phi_B = B\,\ell\, dx$, so

$$\left|\frac{d\Phi_B}{dt}\right| = B\,\ell\,\frac{dx}{dt} = B\,\ell\, v.$$

This is exactly the motional-emf result $\mathcal{E} = B\ell v$. Faraday's law provides the same emf through the "flux-change" perspective, while the motional-emf derivation provides it through the "force-on-charges" perspective.

Example 2.6.4 (Illustrative example)

A metal rod of length ℓ rotates with angular speed ω about one end in a uniform magnetic field B perpendicular to the plane of rotation. Different points on the rod have different speeds $v(r) = r\,\omega$, so the emf must be computed by integration:

$$\mathcal{E} = \int_0^{\ell} B\,(r\omega)\,dr = \frac{1}{2}\,B\,\omega\,\ell^2.$$

Question 32: Worked example

A conducting bar of length $\ell = 0.50\,\text{m}$ slides on two horizontal, frictionless, conducting rails connected by a resistor $R = 4.0\,\Omega$ at the left end, as shown in the figure. A uniform magnetic field $\vec{B} = (0.30\,\text{T})\,\hat{k}$ points straight upward (out of the horizontal plane of the rails). At a particular instant, the bar is moving to the right with velocity $v = 2.0\,\text{m/s}$.

(a) Find the motional emf induced in the circuit at this instant.

(b) Find the induced current: magnitude and direction (clockwise or counterclockwise as viewed from above).

(c) Find the magnitude and direction of the external force that must be applied to the bar to maintain this constant velocity.

(d) Find the power dissipated in the resistor. Verify that the mechanical power delivered by the external force equals the electrical power dissipated.

Given quantities:

- Bar length: $\ell = 0.50\,\text{m}$
- Resistance: $R = 4.0\,\Omega$
- Magnetic field: $B = 0.30\,\text{T}$ (out of page)
- Bar velocity: $v = 2.0\,\text{m/s}$ (to the right)
- $\vec{v}$, $\vec{B}$, and the bar are mutually perpendicular.

Solution: **(a) Motional emf.** Since $\vec{v}$, $\vec{B}$, and the bar are mutually perpendicular, the motional emf is

$$\mathcal{E} = B\,\ell\,v.$$

Substitute the given values:

$$\mathcal{E} = (0.30\,\text{T})(0.50\,\text{m})(2.0\,\text{m/s}) = (0.30)(1.0)\,\text{V} = 0.30\,\text{V}.$$

(b) Induced current. The magnitude of the induced current is given by Ohm's law:

$$I = \frac{\mathcal{E}}{R} = \frac{0.30\,\text{V}}{4.0\,\Omega} = 0.075\,\text{A} = 75\,\text{mA}.$$

To find the direction, apply Lenz's law. The bar moves to the right, so the area of the loop increases, and the upward magnetic flux Φ_B through the loop is *increasing*. Lenz's law requires the induced current to create a magnetic field $\vec{B}_{\text{ind}}$ that opposes this increase, so $\vec{B}_{\text{ind}}$ must point *into* the page (downward). By the right-hand rule, a current that produces an into-the-page field flows *clockwise* as viewed from above.

Alternatively, use the force-on-charges argument: inside the moving bar, positive charge carriers experience $\vec{v}\times\vec{B}$, which points upward along the bar (since $\hat{\imath}\times\hat{k} = -\hat{\jmath}$ and positive charges move in the $+\hat{\jmath}$ direction if the bar is oriented in that direction). The upper end is thus at higher potential. Current flows from high to low potential through the external circuit (the resistor), i.e., clockwise.

$$I = 0.075\,\text{A}, \quad \text{clockwise (as viewed from above).}$$

(c) External force. The induced current in the bar flows *downward* (from the top rail to the bottom rail), i.e., in the $-\hat{\jmath}$ direction. Therefore, $\vec{\ell} = -\ell\,\hat{\jmath}$, and the magnetic force on the bar is

$$\vec{F}_{\text{mag}} = I\,\vec{\ell}\times\vec{B} = I\,(-\ell\,\hat{\jmath})\times(B\,\hat{k}) = -I\,\ell\,B\,(\hat{\jmath}\times\hat{k}) = -I\,\ell\,B\,\hat{\imath}.$$

The magnetic force on the bar points to the *left*, opposing the motion. Its magnitude is

$$F_{\text{mag}} = I\,\ell\,B = (0.075\,\text{A})(0.50\,\text{m})(0.30\,\text{T}) = (0.075)(0.15)\,\text{N} = 0.01125\,\text{N}.$$

To maintain constant velocity (zero net force), the external force must exactly balance the magnetic force:

$$\vec{F}_{\text{ext}} = -\vec{F}_{\text{mag}} = +(0.01125\,\text{N})\,\hat{\imath},$$

i.e., $F_{\text{ext}} = 0.01125\,\text{N}$ to the *right*.

Rounding to two significant figures (matching the given data):

$$F_{\text{ext}} = 0.011\,\text{N to the right.}$$

(d) Power. The electrical power dissipated in the resistor is

$$P_{\text{elec}} = I^2R = (0.075\,\text{A})^2(4.0\,\Omega) = (0.005625)(4.0)\,\text{W} = 0.0225\,\text{W}.$$

Equivalently,

$$P_{\text{elec}} = \frac{\mathcal{E}^2}{R} = \frac{(0.30\,\text{V})^2}{4.0\,\Omega} = \frac{0.090}{4.0}\,\text{W} = 0.0225\,\text{W}.$$

The mechanical power delivered by the external force is

$$P_{\text{mech}} = F_{\text{ext}}\,v = (0.01125\,\text{N})(2.0\,\text{m/s}) = 0.0225\,\text{W}.$$

Since $P_{\text{mech}} = P_{\text{elec}} = 0.0225\,\text{W}$, mechanical energy is fully converted to electrical energy (Joule heating), as expected from energy conservation.

$$P_{\text{elec}} = 0.0225\,\text{W} = 22.5\,\text{mW}, \qquad P_{\text{mech}} = 0.0225\,\text{W}.$$

Summary of results:

(a) $\mathcal{E} = 0.30\,\text{V}$

(b) $I = 0.075\,\text{A} = 75\,\text{mA}$, clockwise

(c) $F_{\text{ext}} = 0.011\,\text{N}$ to the right

(d) $P_{\text{elec}} = 0.0225\,\text{W}$, verified $P_{\text{mech}} = P_{\text{elec}}$

2.6.5 Inductance and Magnetic Energy Storage

This subsection covers self-inductance, mutual inductance, and the energy stored in magnetic fields. When the current through a conductor changes, the magnetic flux it produces also changes, inducing an electromotive force (EMF) that opposes that change — a phenomenon governed by Faraday's law and Lenz's law. The energy required to establish the current is stored in the magnetic field and can be recovered when the current decreases.

Definition 2.6.5: Self-inductance

The *self-inductance* L of a circuit (or coil) quantifies the magnetic flux linkage per unit current. If a current I through a coil of N turns produces a magnetic flux Φ_B through each turn, the total flux linkage is $N\Phi_B$. The self-inductance is defined by

$$L = \frac{N\,\Phi_B}{I}.$$

Equivalently, from Faraday's law of induction, a changing current induces an EMF

$$\mathcal{E} = -L\,\frac{dI}{dt},$$

where the negative sign reflects Lenz's law: the induced EMF opposes the change in current. The SI unit of inductance is the henry (H), where $1\ \text{H} = 1\ \text{V}\cdot\text{s/A} = 1\ \text{kg}\cdot\text{m}^2/(\text{s}^2\cdot\text{A}^2)$.

Note:-

Inductance is purely a geometric property. For a fixed geometry (and no ferromagnetic material near the coil), L is constant and independent of the current. The larger the coil, the more turns, and the greater the flux linkage per unit current, the larger the inductance. A coil with $L = 1$ H and $dI/dt = 1$ A/s develops a 1 V back EMF.

Theorem 2.6.5 Solenoid self-inductance

For an ideal long solenoid of length $\ell \gg R$, total turns N, cross-sectional area $A = \pi R^2$, and vacuum permeability $\mu_0 = 4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A}$, the magnetic field inside is $B = \mu_0\, n\, I$ where $n = N/\ell$. The flux through each turn is $\Phi_B = B\,A = \mu_0 N I A/\ell$. Therefore

$$L = \frac{N\,\Phi_B}{I} = \frac{\mu_0\,N^2\,A}{\ell}.$$

Note:-

The solenoid inductance scales as N^2 because each additional turn both increases the flux linkage (N factor) and increases the magnetic field produced per unit current (N factor through $n = N/\ell$). The inductance is proportional to the cross-sectional area A and inversely proportional to the length ℓ.

Definition 2.6.6: Mutual inductance

Consider two coils placed near each other. If a current I_1 in coil 1 produces a magnetic flux Φ_{21} through each turn of coil 2 (which has N_2 turns), the *mutual inductance* M_{21} is defined as

$$M_{21} = \frac{N_2\,\Phi_{21}}{I_1}.$$

By Faraday's law, a changing current in coil 1 induces an EMF in coil 2:

$$\mathcal{E}_2 = -M_{21}\,\frac{dI_1}{dt}.$$

Mutual inductance is symmetric: $M_{21} = M_{12} = M$. The SI unit is the henry (H).

Note:-

Mutual inductance is the operating principle of transformers. The amount of flux from coil 1 that threads coil 2 depends on their relative orientation, separation, and the presence of magnetic materials. When the coils are perfectly coupled (all flux from one threads the other), M reaches its maximum value. In practice, the coupling is characterized by the coefficient $k = M/\sqrt{L_1 L_2}$, where $0 \leqslant k \leqslant 1$.

Theorem 2.6.6 Magnetic energy and energy density

When a current I flows through an inductor of inductance L, the magnetic field stores energy. The total energy stored is

$$U = \frac{1}{2} L I^2.$$

Inside an ideal solenoid, the magnetic field is uniform with magnitude $B = \mu_0 n I$. The energy can be expressed in terms of B by noting that the energy is distributed throughout the volume $V = A\ell$. The *magnetic energy density* (energy per unit volume) in vacuum is

$$u_B = \frac{U}{V} = \frac{B^2}{2\mu_0}.$$

Magnetic energy from power: The power delivered to an inductor by an external source to drive current I against the back EMF $\mathcal{E} = -L\,dI/dt$ is

$$P = -\mathcal{E}\, I = L\, I \frac{dI}{dt}.$$

The rate of energy storage is $dU/dt = P$, so

$$\frac{dU}{dt} = L\, I \frac{dI}{dt}.$$

Integrating with respect to time as the current rises from 0 to I:

$$U = \int_0^I L\, i\, di = \frac{1}{2} L I^2.$$

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Energy density of the B field (solenoid derivation): Consider an ideal solenoid with N turns, length ℓ, cross-sectional area A, carrying current I. Its inductance is $L = \mu_0 N^2 A/\ell$, and the stored energy is

$$U = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{\ell} I^2.$$

The magnetic field inside is $B = \mu_0 N I/\ell$, so $I = B\,\ell/(\mu_0 N)$. Substituting:

$$U = \frac{1}{2} \frac{\mu_0 N^2 A}{\ell} \left(\frac{B\,\ell}{\mu_0 N}\right)^2 = \frac{1}{2} \frac{\mu_0 N^2 A}{\ell} \frac{B^2 \ell^2}{\mu_0^2 N^2} = \frac{B^2}{2\mu_0} A\,\ell.$$

The volume is $V = A\,\ell$, so the energy density is

$$u_B = \frac{U}{V} = \frac{B^2}{2\mu_0}.$$

This result is general: the energy density $u_B = B^2/(2\mu_0)$ holds at any point in space in vacuum wherever a magnetic field B exists. 😊

Note:-

The magnetic energy density $u_B = B^2/(2\mu_0)$ is the magnetic analogue of the electric energy density $u_E = \frac{1}{2}\varepsilon_0 E^2$. In both cases, energy is stored in the field itself, distributed throughout space. This field-energy viewpoint is essential in electrodynamics: changing fields carry energy via the Poynting vector $\vec{S} = \vec{E} \times \vec{B}/\mu_0$.

Proposition 2.6.6 Key inductance and magnetic-energy formulas

- **Self-inductance definition:** $L = N\,\Phi_B/I = -\mathcal{E}/(dI/dt)$.
- **Solenoid inductance:** $L = \mu_0 N^2 A/\ell$.
- **Mutual inductance:** $M = N_2\,\Phi_{21}/I_1$.
- **Magnetic energy:** $U = \frac{1}{2}LI^2$.
- **Magnetic energy density (vacuum):** $u_B = B^2/(2\mu_0)$.

Corollary 2.6.5 Units check

Energy density u_B has units of joules per cubic metre ($\mathrm{J/m^3}$). Since 1 J = 1 N·m and 1 N = 1 T·A·m, we have B^2/μ_0 in units of $(\mathrm{T})^2/(\mathrm{T\cdot m/A}) = \mathrm{T\cdot A/m} = (\mathrm{N/(A\cdot m)})\cdot(\mathrm{A/m}) = \mathrm{N/m^2} = \mathrm{J/m^3}$, as expected.

Example 2.6.5 (Illustrative example)

A toroidal solenoid has mean circumference 0.40 m, cross-sectional area $2.0\times10^{-3}\,\mathrm{m^2}$, and $N = 500$ turns. Its inductance is $L = \mu_0 N^2 A/\ell = (4\pi\times10^{-7})(500)^2(2.0\times10^{-3})/(0.40) = 1.57\times10^{-3}\,\mathrm{H} = 1.57\,\mathrm{mH}$. At $I = 3.0\,\mathrm{A}$, the stored energy is $U = \frac{1}{2}(1.57\times10^{-3})(3.0)^2 = 7.07\times10^{-3}\,\mathrm{J} = 7.1\,\mathrm{mJ}$.

Question 33: Worked example

An ideal solenoid is 50.0 cm long and has $N = 1200$ turns uniformly distributed along its length. Its circular cross-section has radius $R = 2.0\,\mathrm{cm}$. A steady current $I = 5.0\,\mathrm{A}$ flows through the wire. Take $\mu_0 = 4\pi\times10^{-7}\,\mathrm{T\cdot m/A}$.
Find:

(a) the self-inductance L of the solenoid,

(b) the magnetic energy U stored in the solenoid,

(c) the magnitude of the magnetic field B inside the solenoid, and

(d) the magnetic energy density u_B inside the solenoid. Verify that $U = u_B\,V$, where V is the interior volume of the solenoid.

Solution: **Part (a).** The cross-sectional area of the solenoid is

$$A = \pi R^2 = \pi\,(0.020\,\mathrm{m})^2 = \pi\times4.0\times10^{-4}\,\mathrm{m^2} = 1.26\times10^{-3}\,\mathrm{m^2}.$$

The length is $\ell = 0.500\,\mathrm{m}$. Using the solenoid inductance formula:

$$L = \frac{\mu_0\,N^2\,A}{\ell}.$$

Substitute the values:

$$L = \frac{(4\pi\times10^{-7}\,\mathrm{T\cdot m/A})(1200)^2(1.26\times10^{-3}\,\mathrm{m^2})}{0.500\,\mathrm{m}}.$$

Compute step by step:

$$(1200)^2 = 1.44\times10^6,$$

$$\mu_0\,(1200)^2 = (4\pi\times10^{-7})(1.44\times10^6) = 4\pi\times0.144 = 1.810\times10^{-6}\,\mathrm{T\cdot m^2/A}.$$

Then

$$L = \frac{(1.810\times10^{-6})(1.26\times10^{-3})}{0.500}\,\mathrm{H} = \frac{2.28\times10^{-9}}{0.500}\,\mathrm{H} = 4.56\times10^{-3}\,\mathrm{H}.$$

More precisely, carrying π through:

$$L = \frac{4\pi \times 10^{-7} \times 1.44 \times 10^{6} \times \pi \times 4.0 \times 10^{-4}}{0.500} = \frac{4\pi^2 \times 5.76 \times 10^{-5}}{0.500} = 8\pi^2 \times 5.76 \times 10^{-5}.$$

$$\pi^2 \approx 9.87, \qquad L = 8 \times 9.87 \times 5.76 \times 10^{-5}\,\text{H} = 4.55 \times 10^{-3}\,\text{H}.$$

So

$$L = 4.55 \times 10^{-3}\,\text{H} = 4.55\,\text{mH}.$$

Part (b). The stored magnetic energy is

$$U = \frac{1}{2}\,L\,I^2.$$

Substitute:

$$U = \frac{1}{2}\,(4.55 \times 10^{-3}\,\text{H})\,(5.0\,\text{A})^2.$$

Since $(5.0)^2 = 25.0$:

$$U = \frac{1}{2}\,(4.55 \times 10^{-3})(25.0)\,\text{J} = \frac{1}{2}\,(0.114)\,\text{J} = 5.68 \times 10^{-2}\,\text{J}.$$

So

$$U = 5.68 \times 10^{-2}\,\text{J} = 56.8\,\text{mJ}.$$

Part (c). The magnetic field inside the solenoid is

$$B = \mu_0\,n\,I,$$

where $n = N/\ell$ is the turn density:

$$n = \frac{1200}{0.500\,\text{m}} = 2400\,\text{turns/m}.$$

Substitute:

$$B = (4\pi \times 10^{-7}\,\text{T}\cdot\text{m/A})(2400\,\text{m}^{-1})(5.0\,\text{A}).$$

Compute:

$$2400 \times 5.0 = 1.20 \times 10^{4},$$

$$B = 4\pi \times 10^{-7} \times 1.20 \times 10^{4} = 4.80\pi \times 10^{-3}\,\text{T}.$$

Using $\pi \approx 3.1416$:

$$B = 4.80 \times 3.1416 \times 10^{-3}\,\text{T} = 1.51 \times 10^{-2}\,\text{T}.$$

So

$$B = 1.51 \times 10^{-2}\,\text{T} = 15.1\,\text{mT}.$$

Part (d). The magnetic energy density is

$$u_B = \frac{B^2}{2\,\mu_0}.$$

Substitute $B = 1.51 \times 10^{-2}$ T:

$$B^2 = (1.51 \times 10^{-2})^2 = 2.28 \times 10^{-4}\,\text{T}^2.$$

Then

$$u_B = \frac{2.28 \times 10^{-4}}{2(4\pi \times 10^{-7})}\,\text{J/m}^3 = \frac{2.28 \times 10^{-4}}{2.51 \times 10^{-6}}\,\text{J/m}^3 = 9.09 \times 10^{1}\,\text{J/m}^3.$$

More precisely:

$$u_B = \frac{2.283 \times 10^{-4}}{2.513 \times 10^{-6}}\,\text{J/m}^3 = 90.9\,\text{J/m}^3.$$

So

$$u_B = 90.9\,\text{J/m}^3.$$

Now verify $U = u_B V$. The interior volume of the solenoid is

$$V = A\,\ell = (\pi \times 4.0 \times 10^{-4}\,\text{m}^2)(0.500\,\text{m}) = 6.28 \times 10^{-4}\,\text{m}^3.$$

Then

$$u_B V = (90.9\,\text{J/m}^3)(6.28 \times 10^{-4}\,\text{m}^3) = 5.71 \times 10^{-2}\,\text{J}.$$

Using more precise intermediate values: $B = 1.508 \times 10^{-2}$ T, $B^2 = 2.274 \times 10^{-4}$, $u_B = 90.6\,\text{J/m}^3$, $V = 6.283 \times 10^{-4}\,\text{m}^3$, giving $u_B V = 5.69 \times 10^{-2}$ J, which matches $U = 5.68 \times 10^{-2}$ J within rounding. The relation $U = u_B V$ holds, confirming that the energy is uniformly distributed throughout the solenoid volume at density $u_B = B^2/(2\mu_0)$.

Final answers:

(a) $L = 4.55 \times 10^{-3}\,\text{H} = 4.55\,\text{mH}$

(b) $U = 5.68 \times 10^{-2}\,\text{J} = 56.8\,\text{mJ}$

(c) $B = 1.51 \times 10^{-2}\,\text{T} = 15.1\,\text{mT}$

(d) $u_B = 90.9\,\text{J/m}^3$, and $U = u_B V$ verified

2.6.6 LR Circuits and Transients

This subsection introduces the series LR circuit, derives the time-dependent current during both the growth and decay phases, and discusses the energy stored in the inductor's magnetic field. The inductor opposes changes in current through a self-induced back EMF, producing an exponential transient with characteristic time constant $\tau = L/R$.

Definition 2.6.7: LR circuit and self-induced EMF

Consider a series circuit consisting of a battery with constant EMF $\mathcal{E}$, a resistor of resistance R, an inductor of inductance L, and a switch, all connected in a single closed loop. When current I flows through the inductor, any change in current induces a back EMF across the inductor given by

$$\mathcal{E}_L = -L\,\frac{dI}{dt}.$$

By Kirchhoff's loop rule, the sum of potential differences around the loop is zero. Traversing the loop in the direction of the current:

$$\mathcal{E} - IR - L\,\frac{dI}{dt} = 0.$$

This first-order differential equation governs the time evolution of the current $I(t)$. The inductor's back EMF opposes any change in current, analogous to how inertia opposes changes in velocity in mechanics.

Note:-

The inductor acts as a "magnetic inertia" element: just as a mass cannot change its velocity instantaneously, current through an inductor cannot change instantaneously. At the instant the switch is closed, the current is zero and the full battery EMF appears across the inductor. After a long time, the current reaches a steady value and the inductor behaves as a short circuit (zero voltage drop).

Proposition 2.6.7 Time constant of an LR circuit

The *time constant* τ characterizes how quickly the current changes in an LR circuit:

$$\tau = \frac{L}{R}.$$

The SI unit of inductance is the henry (H) and the SI unit of resistance is the ohm (Ω). Since $1\,\text{H} = 1\,\text{V·s/A}$ and $1\,\Omega = 1\,\text{V/A}$, the ratio L/R has units of seconds, confirming that τ is a characteristic time. At $t = \tau$, the current during growth has reached $(1 - e^{-1}) \approx 63.2\%$ of its maximum value.

Theorem 2.6.7 Current growth in a series LR circuit

Consider a series LR circuit with battery EMF $\mathcal{E}$, resistance R, and inductance L. The switch is closed at $t = 0$, with the initial current $I(0) = 0$. The current as a function of time is

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right),$$

where $\tau = L/R$. The maximum (steady-state) current is

$$I_{\max} = \frac{\mathcal{E}}{R}.$$

Current growth derivation: We solve the differential equation obtained from Kirchhoff's loop rule:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0.$$

Rearrange to isolate the time derivative:

$$L\frac{dI}{dt} = \mathcal{E} - IR.$$

Separate variables, bringing all I terms to one side:

$$\frac{dI}{\mathcal{E} - IR} = \frac{dt}{L}.$$

Integrate both sides. On the left, substitute $u = \mathcal{E} - IR$, so $du = -R\,dI$:

$$\int_0^{I(t)} \frac{dI}{\mathcal{E} - IR} = \int_0^t \frac{dt}{L}.$$

The left-hand integral is

$$-\frac{1}{R}\ln\left(\mathcal{E} - IR\right)\Big|_0^{I(t)} = -\frac{1}{R}\ln\left(\frac{\mathcal{E} - IR(t)}{\mathcal{E}}\right).$$

The right-hand integral is t/L. Thus,

$$-\frac{1}{R}\ln\left(\frac{\mathcal{E} - IR(t)}{\mathcal{E}}\right) = \frac{t}{L}.$$

Multiply by $-R$:

$$\ln\left(\frac{\mathcal{E} - IR(t)}{\mathcal{E}}\right) = -\frac{R}{L}t.$$

Exponentiate both sides:

$$\frac{\mathcal{E} - IR(t)}{\mathcal{E}} = e^{-Rt/L}.$$

Solve for $I(t)$:

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-Rt/L}\right).$$

Since $\tau = L/R$, this is equivalent to $I(t) = (\mathcal{E}/R)(1 - e^{-t/\tau})$. In the limit $t \to \infty$, the exponential vanishes and $I \to \mathcal{E}/R$.

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Theorem 2.6.8 Current decay in a series LR circuit

Consider a series LR circuit carrying an initial current I_0 (established in steady state with a battery that is then disconnected, leaving only R and L in a closed loop). With $I(0) = I_0$, the current as a function of time is

$$I(t) = I_0\, e^{-t/\tau},$$

where $\tau = L/R$.

Current decay derivation: With the battery removed, Kirchhoff's loop rule gives

$$IR + L\,\frac{dI}{dt} = 0.$$

Separate variables:

$$\frac{dI}{I} = -\frac{R}{L}\,dt.$$

Integrate both sides from $t = 0$ to t:

$$\int_{I_0}^{I(t)} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt.$$

The left-hand side gives $\ln(I/I_0)$ and the right-hand side gives $-(R/L)\,t$. Thus,

$$\ln\!\left(\frac{I(t)}{I_0}\right) = -\frac{R}{L}\,t,$$

and exponentiating,

$$I(t) = I_0\, e^{-Rt/L} = I_0\, e^{-t/\tau}.$$

In the limit $t \to \infty$, $I \to 0$.

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Corollary 2.6.6 Back EMF across the inductor (growth phase)

Differentiating the growth current gives the self-induced EMF across the inductor:

$$\mathcal{E}_L = -L\,\frac{dI}{dt} = -L\,\frac{\mathcal{E}}{R}\,\frac{R}{L}\,e^{-t/\tau} = -\mathcal{E}\,e^{-t/\tau}.$$

At $t = 0$, $\mathcal{E}_L = -\mathcal{E}$ (the full battery EMF opposes the change). As $t \to \infty$, $\mathcal{E}_L \to 0$ (the inductor becomes a short circuit). The magnitude of the back EMF decays exponentially with the same time constant τ.

Corollary 2.6.7 Voltage across the resistor (growth phase)

The voltage across the resistor is

$$V_R = IR = \mathcal{E}\left(1 - e^{-t/\tau}\right).$$

At $t = 0$, $V_R = 0$. As $t \to \infty$, $V_R \to \mathcal{E}$, so the full battery EMF appears across the resistor in steady state.

Proposition 2.6.8 Energy stored in an inductor

When a current I flows through an inductor of inductance L, energy is stored in the magnetic field:

$$U_B = \frac{1}{2}\,L\,I^2.$$

The rate at which energy is stored is

$$\frac{dU_B}{dt} = L\,I\,\frac{dI}{dt}.$$

The SI unit of energy is the joule (J). During current growth in an LR circuit, the battery supplies energy at rate $\mathcal{E}I$, part of which is dissipated as Joule heating I^2R in the resistor and part is stored in the inductor's magnetic field.

Note:-

At $t = \tau = L/R$, the current reaches $I(\tau) = (\mathcal{E}/R)(1 - e^{-1}) \approx 0.632\,(\mathcal{E}/R)$. The inductor EMF has dropped to $e^{-1} \approx 36.8\%$ of its initial value. After 5τ, the current is within 1% of its steady-state value and the circuit is effectively in steady state.

Example 2.6.6 (Illustrative example)

A series LR circuit with $\mathcal{E} = 12\,\text{V}$, $R = 6.0\,\Omega$, and $L = 3.0\,\text{H}$ has time constant $\tau = L/R = 0.50\,\text{s}$. The maximum current is $I_{\text{max}} = \mathcal{E}/R = 2.0\,\text{A}$.

(1) At $t = 0$, $I = 0$ and $\mathcal{E}_L = -12\,\text{V}$.

(2) At $t = \tau = 0.50\,\text{s}$, $I = 2.0(1 - e^{-1}) = 1.26\,\text{A}$ and $\mathcal{E}_L = -12\,e^{-1} = -4.41\,\text{V}$.

(3) At $t = 2\tau = 1.0\,\text{s}$, $I = 2.0(1 - e^{-2}) = 1.73\,\text{A}$ and $\mathcal{E}_L = -12\,e^{-2} = -1.62\,\text{V}$.

(4) At $t \to \infty$, $I = 2.0\,\text{A}$ and $\mathcal{E}_L = 0\,\text{V}$.

Question 34: Worked example

A series LR circuit consists of a battery with emf $\mathcal{E} = 24\,\text{V}$, a resistor with resistance $R = 4.0\,\Omega$, and an inductor with inductance $L = 1.0\,\text{H}$, all connected in series with a switch. The switch is closed at time $t = 0$.

(a) Find the time constant τ of the circuit.

(b) Find the current at $t = 0.25\,\text{s}$.

(c) Find the maximum (steady-state) current.

(d) Find the energy stored in the inductor at $t = 0.25\,\text{s}$.

Solution: **Part (a):** The time constant of a series LR circuit is

$$\tau = \frac{L}{R}.$$

Substituting the given values:

$$\tau = \frac{1.0\,\text{H}}{4.0\,\Omega} = 0.25\,\text{s}.$$

Part (b): During the growth phase, the current is

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right).$$

At $t = 0.25\,\text{s}$ and with $\tau = 0.25\,\text{s}$:

$$I(0.25) = \frac{24\,\text{V}}{4.0\,\Omega}\left(1 - e^{-0.25/0.25}\right) = 6.0\,\text{A}\left(1 - e^{-1}\right).$$

Using $e^{-1} \approx 0.3679$:

$$I(0.25) = 6.0\,\text{A} \times (1 - 0.3679) = 6.0\,\text{A} \times 0.6321 = 3.79\,\text{A}.$$

Part (c): The maximum (steady-state) current is obtained as $t \to \infty$, when the exponential term vanishes:

$$I_{\text{max}} = \frac{\mathcal{E}}{R} = \frac{24\,\text{V}}{4.0\,\Omega} = 6.0\,\text{A}.$$

Part (d): The energy stored in the inductor's magnetic field is

$$U_B = \frac{1}{2}\,L\,I^2.$$

Using the current from part (b):

$$U_B = \frac{1}{2}\,(1.0\,\text{H})\,(3.79\,\text{A})^2 = 0.5 \times 14.36\,\text{J} = 7.2\,\text{J}.$$

Therefore,

$$\tau = 0.25\,\text{s}, \qquad I(0.25\,\text{s}) = 3.79\,\text{A}, \qquad I_{\text{max}} = 6.0\,\text{A}, \qquad U_B(0.25\,\text{s}) = 7.2\,\text{J}.$$

2.6.7 LC Oscillations

This subsection introduces the LC circuit – a capacitor and inductor connected in a closed loop with no resistance – derives the second-order differential equation governing charge evolution, identifies the sinusoidal oscillation of charge and current, defines the natural angular frequency $\omega = 1/\sqrt{LC}$, and analyses the continuous energy exchange between the capacitor and inductor.

Definition 2.6.8: LC circuit

An *LC circuit* consists of an inductor of inductance L and a capacitor of capacitance C connected in a single closed loop. No resistor is present (ideal components). If the capacitor is initially charged and then connected to the inductor, the charge on the capacitor and the current through the inductor oscillate sinusoidally in time.

Note:-

Think of an LC circuit as the electrical analogue of a frictionless spring-mass system. The capacitor stores energy in its electric field (just as a spring stores potential energy), and the inductor stores energy in its magnetic field (just as a moving mass has kinetic energy). The charge oscillates between the two plates of the capacitor while the current alternates direction through the inductor, and the total energy remains constant.

Theorem 2.6.9 Oscillation of charge and current

In a series LC circuit, let L be the inductance and C the capacitance. If at $t = 0$ the capacitor carries charge $q(0) = Q_{\text{max}}$ and the current is $I(0) = 0$, then the charge on the capacitor at time t is

$$q(t) = Q_{\text{max}}\,\cos(\omega t),$$

and the current through the inductor is

$$I(t) = \frac{dq}{dt} = -\,\omega\,Q_{\text{max}}\,\sin(\omega t).$$

Here $\omega = \dfrac{1}{\sqrt{LC}}$ is the *angular frequency* of oscillation (in rad/s), Q_{max} is the maximum charge on the capacitor, and the phase constant is $\phi = 0$ for these initial conditions. The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}.$$

The maximum current (current amplitude) is

$$I_{\text{max}} = \omega\,Q_{\text{max}} = \frac{Q_{\text{max}}}{\sqrt{LC}}.$$

Derivation of LC oscillations: Apply Kirchhoff's voltage law around the loop. Going around, the voltage drop across the capacitor is q/C and the voltage drop across the inductor is $L\,(dI/dt)$. Since the sum of voltage drops around a closed loop is zero:

$$\frac{q}{C} + L\,\frac{dI}{dt} = 0.$$

The current is the rate of change of charge on the capacitor. When the capacitor discharges, charge leaves the plate, so $I = -\,dq/dt$. (This sign convention is consistent: as q decreases, $dq/dt < 0$ and $I > 0$, meaning positive current flows off the positively charged plate.) Substituting:

$$\frac{q}{C} + L\,\frac{d}{dt}\left(-\,\frac{dq}{dt}\right) = 0,$$

which simplifies to

$$\frac{d^2q}{dt^2} + \frac{1}{LC}\,q = 0.$$

This is the *simple harmonic oscillator equation* (second-order linear ODE with constant coefficients). Comparing with the mechanical oscillator equation $\ddot{x} + (\kappa/m)\,x = 0$, the angular frequency is

$$\omega = \frac{1}{\sqrt{LC}}.$$

The general solution is

$$q(t) = A\,\cos(\omega t) + B\,\sin(\omega t) = Q_{\max}\,\cos(\omega t + \phi),$$

where $Q_{\max} = \sqrt{A^2 + B^2}$ is the amplitude and $\phi = \tan^{-1}(-B/A)$ is the phase constant. With $q(0) = Q_{\max}$ and $I(0) = 0$:

$$q(0) = A = Q_{\max}, \qquad I(0) = -\omega B = 0 \;\Rightarrow\; B = 0.$$

Thus $\phi = 0$ and

$$q(t) = Q_{\max}\,\cos(\omega t).$$

The current is

$$I(t) = \frac{dq}{dt} = -\,\omega\,Q_{\max}\,\sin(\omega t).$$

The maximum current occurs when $|\sin(\omega t)| = 1$:

$$I_{\max} = \omega\,Q_{\max} = \frac{Q_{\max}}{\sqrt{LC}}.$$

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Note:-

The initial conditions determine the amplitude $Q_{\max}$ and phase ϕ. If the capacitor is initially charged to $q(0) = q_0$ with $I(0) = 0$, then $Q_{\max} = q_0$ and $\phi = 0$. If the current has an initial value $I(0) = I_0$ while $q(0) = 0$, then $Q_{\max} = I_0/\omega$ and $\phi = \pi/2$.

Theorem 2.6.10 Energy conservation in an LC circuit

The energy stored in the capacitor at time t is

$$U_C(t) = \frac{q(t)^2}{2C} = \frac{Q_{\max}^2}{2C}\,\cos^2(\omega t),$$

and the energy stored in the inductor is

$$U_L(t) = \frac{1}{2}\,L\,I(t)^2 = \frac{1}{2}\,L\,\omega^2\,Q_{\max}^2\,\sin^2(\omega t).$$

Using $\omega^2 = 1/(LC)$, the inductor energy becomes

$$U_L(t) = \frac{Q_{\max}^2}{2C}\,\sin^2(\omega t).$$

The total energy is therefore

$$U_{\text{total}} = U_C(t) + U_L(t) = \frac{Q_{\max}^2}{2C}\,\bigl(\cos^2(\omega t) + \sin^2(\omega t)\bigr) = \frac{Q_{\max}^2}{2C}.$$

The total energy is *constant* and does not depend on time.

Energy conservation: The energy stored in a capacitor with charge q is $U_C = q^2/(2C)$, and the energy stored in an inductor with current I is $U_L = \frac{1}{2}LI^2$. Substituting the oscillation formulas:

$$U_C(t) = \frac{Q_{\max}^2}{2C}\cos^2(\omega t), \qquad U_L(t) = \frac{1}{2}L\left(-\omega Q_{\max}\sin(\omega t)\right)^2 = \frac{1}{2}L\omega^2 Q_{\max}^2\sin^2(\omega t).$$

Since $\omega^2 = 1/(LC)$:

$$U_L(t) = \frac{1}{2}L\cdot\frac{1}{LC}\cdot Q_{\max}^2\sin^2(\omega t) = \frac{Q_{\max}^2}{2C}\sin^2(\omega t).$$

Therefore,

$$U_{\text{total}} = U_C + U_L = \frac{Q_{\max}^2}{2C}\left(\cos^2(\omega t) + \sin^2(\omega t)\right) = \frac{Q_{\max}^2}{2C}.$$

This is time-independent, confirming energy conservation. ☺

Proposition 2.6.9 Energy exchange in an LC circuit

The total energy is

$$U = \frac{Q_{\max}^2}{2C} = \frac{1}{2}LI_{\max}^2,$$

where we used $I_{\max} = \omega Q_{\max} = Q_{\max}/\sqrt{LC}$, so $\frac{1}{2}LI_{\max}^2 = \frac{1}{2}L(Q_{\max}^2/LC) = Q_{\max}^2/(2C)$.
The energy is purely capacitive at times t when $\cos(\omega t) = \pm 1$ (i.e., $t = 0, T/2, T, \ldots$):

$$U_C = \frac{Q_{\max}^2}{2C} = U_{\text{total}}, \qquad U_L = 0.$$

The energy is purely inductive at times t when $\sin(\omega t) = \pm 1$ (i.e., $t = T/4, 3T/4, \ldots$):

$$U_L = \frac{Q_{\max}^2}{2C} = U_{\text{total}}, \qquad U_C = 0.$$

At all other times, the energy is shared between the two elements.

Corollary 2.6.8 Analogy to spring-mass simple harmonic motion

The LC circuit is directly analogous to a frictionless spring-mass oscillator:

	Spring–mass system	LC circuit
Displacement	$x(t) = A\cos(\omega t + \phi)$	Charge $q(t) = Q_{\max}\cos(\omega t + \phi)$
Velocity	$v(t) = -\omega A\sin(\omega t + \phi)$	Current $I(t) = -\omega Q_{\max}\sin(\omega t + \phi)$
Mass	m	Inductance L
Spring constant	k	Inverse capacitance $1/C$
Angular frequency	$\omega = \sqrt{k/m}$	$\omega = 1/\sqrt{LC}$
Potential energy	$\frac{1}{2}kx^2$	$\frac{1}{2}q^2/C$
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}LI^2$
Total energy	$\frac{1}{2}kA^2$	$\frac{Q_{\max}^2}{2C} = \frac{1}{2}LI_{\max}^2$

This analogy is useful for building physical intuition about LC circuits.

Example 2.6.7 (Illustrative example)

An LC circuit has $L = 40\,\text{mH}$ and $C = 2.0\,\mu\text{F}$, initially charged to $Q_{\max} = 5.0\,\mu\text{C}$. The angular frequency is $\omega = 1/\sqrt{LC} = 1/\sqrt{(40\times10^{-3})(2.0\times10^{-6})} = 1/\sqrt{8.0\times10^{-8}} = 1/(8.94\times10^{-4}) = 112\,\text{rad/s}$. The total energy is $U = Q_{\max}^2/(2C) = (5.0\times10^{-6})^2/(2\cdot2.0\times10^{-6}) = 6.25\times10^{-6}\,\text{J} = 6.25\,\mu\text{J}$. The maximum current is $I_{\max} = \omega Q_{\max} = 112\times5.0\times10^{-6} = 5.6\times10^{-4}\,\text{A} = 0.56\,\text{mA}$.

Question 35: Worked example

An LC circuit consists of an ideal inductor with inductance

$$L = 25.0\,\text{mH} = 25.0 \times 10^{-3}\,\text{H},$$

and an ideal capacitor with capacitance

$$C = 5.00\,\mu\text{F} = 5.00 \times 10^{-6}\,\text{F}.$$

At $t = 0$, the capacitor carries its maximum charge

$$Q_{\max} = 10.0\,\mu\text{C} = 1.00 \times 10^{-5}\,\text{C},$$

and the current is $I(0) = 0$. The phase constant is therefore $\phi = 0$, and

$$q(t) = Q_{\max}\cos(\omega t), \qquad I(t) = -\,\omega\,Q_{\max}\sin(\omega t),$$

with $\omega = 1/\sqrt{LC}$.
Find:

(a) the oscillation frequency f of the circuit,

(b) the maximum current $I_{\max}$,

(c) the energy U_L stored in the inductor at time $t = T/4$ (one-quarter of the oscillation period), and

(d) the energy U_C stored in the capacitor at time $t_{1/2}$, the first instant at which the energy stored in the capacitor equals one-half of the total energy.

Solution: **Part (a).** The angular frequency of oscillation is

$$\omega = \frac{1}{\sqrt{LC}}.$$

Substitute $L = 25.0 \times 10^{-3}\,\text{H}$ and $C = 5.00 \times 10^{-6}\,\text{F}$:

$$LC = (25.0 \times 10^{-3})(5.00 \times 10^{-6}) = 1.25 \times 10^{-7}\,\text{H}\cdot\text{F}.$$

Thus

$$\omega = \frac{1}{\sqrt{1.25 \times 10^{-7}}} = \frac{1}{3.536 \times 10^{-4}} = 2828\,\text{rad/s}.$$

The oscillation frequency is

$$f = \frac{\omega}{2\pi} = \frac{2828}{2\pi}\,\text{Hz} = 450\,\text{Hz}.$$

Rounded to three significant figures:

$$f = 450\,\text{Hz}.$$

(Equivalently, 4.50×10^2 Hz.)

Part (b). The maximum current is

$$I_{\max} = \omega\,Q_{\max}.$$

Substitute $\omega = 2828\,\text{rad/s}$ and $Q_{\max} = 1.00 \times 10^{-5}\,\text{C}$:

$$I_{\max} = (2828)\,(1.00 \times 10^{-5})\,\text{A} = 2.83 \times 10^{-2}\,\text{A} = 28.3\,\text{mA}.$$

Part (c). At $t = T/4$, the angular argument is $\omega t = (2\pi/T)(T/4) = \pi/2$. Therefore:

$$\cos\left(\frac{\pi}{2}\right) = 0, \qquad \sin\left(\frac{\pi}{2}\right) = 1.$$

The charge and current at this instant are:

$$q\left(\frac{T}{4}\right) = Q_{\max}\cos\left(\frac{\pi}{2}\right) = 0, \qquad I\left(\frac{T}{4}\right) = -\,\omega\, Q_{\max}\sin\left(\frac{\pi}{2}\right) = -\,\omega\, Q_{\max} = -\,I_{\max}.$$

The energy stored in the capacitor is $U_C = q^2/(2C) = 0$, so *all* the energy is in the inductor:

$$U_L\left(\frac{T}{4}\right) = U_{\text{total}} = \frac{Q_{\max}^2}{2C}.$$

Substitute the values:

$$U_L\left(\frac{T}{4}\right) = \frac{(1.00\times10^{-5}\ \text{C})^2}{2\,(5.00\times10^{-6}\ \text{F})} = \frac{1.00\times10^{-10}}{1.00\times10^{-5}}\ \text{J} = 1.00\times10^{-5}\ \text{J}.$$

Part (d). We seek the first time $t_{1/2}$ such that the capacitor energy is half the total:

$$U_C = \frac{q^2}{2C} = \frac{1}{2}\,U_{\text{total}} = \frac{1}{2}\cdot\frac{Q_{\max}^2}{2C} = \frac{Q_{\max}^2}{4C}.$$

Multiply both sides by $2C$:

$$q^2 = \frac{Q_{\max}^2}{2}, \qquad q = \frac{Q_{\max}}{\sqrt{2}}.$$

Now use $q(t) = Q_{\max}\cos(\omega t)$:

$$Q_{\max}\cos(\omega t_{1/2}) = \frac{Q_{\max}}{\sqrt{2}}, \qquad \cos(\omega t_{1/2}) = \frac{1}{\sqrt{2}}.$$

The first solution (smallest positive angle) is $\omega t_{1/2} = \pi/4$. We will not need the numerical value of $t_{1/2}$ to find the energy. The capacitor energy at this instant is

$$U_C = \frac{1}{2}\,U_{\text{total}} = \frac{1}{2}\cdot\frac{Q_{\max}^2}{2C} = \frac{Q_{\max}^2}{4C}.$$

Substitute the values:

$$U_C = \frac{(1.00\times10^{-5}\ \text{C})^2}{4\,(5.00\times10^{-6}\ \text{F})} = \frac{1.00\times10^{-10}}{2.00\times10^{-5}}\ \text{J} = 5.00\times10^{-6}\ \text{J}.$$

Check. At $t = T/4$, the charge is zero and the energy is entirely in the inductor: $U_L = 1.00\times10^{-5}\ \text{J} = U_{\text{total}}$, which checks out. At $t = t_{1/2}$, the capacitor holds half the total energy, so $U_C = 5.00\times10^{-6}$ J, and the inductor holds the other half: $U_L = U_{\text{total}} - U_C = 5.00\times10^{-6}$ J, as expected.

Therefore,

$$f = 450\,\text{Hz}, \qquad I_{\max} = 28.3\,\text{mA}, \qquad U_L\left(\frac{T}{4}\right) = 1.00\times10^{-5}\ \text{J}, \qquad U_C(t_{1/2}) = 5.00\times10^{-6}\ \text{J}.$$

Part III

Advanced Topics

Chapter 3

Advanced Analytical Mechanics

The Hamilton-Jacobi (HJ) formulation is the final reformulation of classical mechanics, expressing the entire dynamics of a system as a single first-order partial differential equation for a scalar function S, called the **principal function**. Solving the HJ equation by separation of variables often yields complete solutions more directly than the Lagrange or Hamilton equations – especially for systems with symmetries and cyclic coordinates. The HJ framework also provides the classical foundation for the WKB approximation and connects to the Schrodinger equation in the $\hbar \to 0$ limit.

This chapter is organized in three parts. Section 3.1 develops the HJ equation from Hamiltonian mechanics and introduces separation of variables, action-angle variables, and electromagnetic minimal coupling. Section 3.2 applies the HJ formalism to classical mechanics problems: the free particle, projectile motion, the simple harmonic oscillator, the Kepler (two-body) problem, and the rigid rotator on a sphere. Section 3.3 treats problems from electromagnetism, including charged particles in uniform $\vec{E}$-fields, cyclotron motion, and $\vec{E} \times \vec{B}$ drift, showing that the HJ approach recovers all standard results with a unified method.

3.1 Hamilton-Jacobi Fundamentals

3.1.1 Derivation of the Hamilton-Jacobi Equation

This subsection derives the Hamilton–Jacobi partial differential equation from the Lagrangian formulation through successive Legendre transforms and a special canonical transformation, and states Jacobi's theorem that reduces the solution of the mechanics problem to finding a complete integral of the resulting PDE.

Definition 3.1.1: Hamilton's principal function and the Hamilton–Jacobi action

Let $q_1, \ldots, q_n$ be generalized coordinates and let $p_1, \ldots, p_n$ be the corresponding canonical momenta. Hamilton's principal function $\mathcal{S}(q_1, \ldots, q_n, t)$ is a generating function whose spatial partial derivatives equal the canonical momenta:

$$p_i = \frac{\partial \mathcal{S}}{\partial q_i}$$

for each $i = 1, \ldots, n$. When $\mathcal{S}$ satisfies the Hamilton–Jacobi equation, it encodes the complete solution to the equations of motion.

Note:-

The Hamilton–Jacobi approach replaces the $2n$ coupled first-order Hamiltonian equations of motion with a single first-order nonlinear PDE for $\mathcal{S}$. The trade-off is between solving a system of coupled ODEs and solving a nonlinear PDE. In practice, the PDE is often separable, reducing to a set of ordinary equations that integrate more easily.

Theorem 3.1.1 The Hamilton–Jacobi partial differential equation

Let $\mathcal{H}(q_1, \dots, q_n, p_1, \dots, p_n, t)$ be the Hamiltonian of a system with n degrees of freedom. Then Hamilton's principal function $\mathcal{S}(q_1, \dots, q_n, t)$ satisfies

$$\mathcal{H}\left(q_1, \dots, q_n, \frac{\partial \mathcal{S}}{\partial q_1}, \dots, \frac{\partial \mathcal{S}}{\partial q_n}, t\right) + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Derivation from Lagrangian through canonical transformation to HJ: . Begin with the Lagrangian $\mathcal{L}(q, \dot{q}, t)$ for a system with generalized coordinates $q_1, \dots, q_n$. Define the canonical momenta through the Legendre transform:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

for each i. The Hamiltonian is the Legendre transform of the Lagrangian:

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L},$$

expressed as a function of (q, p, t) after eliminating $\dot{q}$ in favor of p using the inverse of the Legendre map.

Hamilton's canonical equations follow directly from this construction:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}.$$

These $2n$ first-order coupled equations fully determine the dynamics of the system once initial conditions are specified.

Now seek a type-2 canonical transformation from (q, p) to new variables (Q, P) that simplifies the dynamics. The generating function $F_2(q, P, t)$ defines the transformation through the relations

$$p_i = \frac{\partial F_2}{\partial q_i}, \qquad Q_i = \frac{\partial F_2}{\partial P_i}.$$

The new Hamiltonian $\mathcal{K}(Q, P, t)$ is related to the original Hamiltonian $\mathcal{H}$ by the standard transformation rule

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}.$$

Choose the generating function so that the new Hamiltonian vanishes identically: $\mathcal{K} = 0$. With this choice, all new coordinates and momenta are constant in time by Hamilton's equations in the new variables:

$$\dot{Q}_i = \frac{\partial \mathcal{K}}{\partial P_i} = 0, \qquad \dot{P}_i = -\frac{\partial \mathcal{K}}{\partial Q_i} = 0.$$

This makes every Q_i and P_i a constant of motion, which trivializes the dynamics in the transformed variables. Setting $\mathcal{K} = 0$ in the transformation rule gives the key relation

$$\mathcal{H} = -\frac{\partial F_2}{\partial t}.$$

Rename the generating function F_2 as $\mathcal{S}(q_1, \dots, q_n, P_1, \dots, P_n, t)$ and use $p_i = \frac{\partial \mathcal{S}}{\partial q_i}$ to substitute the momenta inside the Hamiltonian. The previous equation reads

$$\mathcal{H}\left(q_1, \dots, q_n, \frac{\partial \mathcal{S}}{\partial q_1}, \dots, \frac{\partial \mathcal{S}}{\partial q_n}, t\right) = -\frac{\partial \mathcal{S}}{\partial t}.$$

Rearranging gives the Hamilton–Jacobi partial differential equation:

$$\mathcal{H}\left(q_1, \dots, q_n, \frac{\partial \mathcal{S}}{\partial q_1}, \dots, \frac{\partial \mathcal{S}}{\partial q_n}, t\right) + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Every solution $\mathcal{S}$ of this PDE generates a canonical transformation to action-angle-like variables in which the motion is completely trivial. 😊

Corollary 3.1.1 Time-independent HJ equation (Hamilton–Charpit–Jacobi)

When the Hamiltonian does not depend explicitly on time, $\frac{\partial \mathcal{H}}{\partial t} = 0$ and the Hamiltonian is a conserved quantity, $\mathcal{H} = E$. In this case the time dependence of $\mathcal{S}$ separates as $\mathcal{S} = W(q_1, \ldots, q_n) - Et$, and the Hamilton–Jacobi equation reduces to

$$\mathcal{H}\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = E.$$

This is the time-independent Hamilton–Jacobi equation, sometimes called the Hamilton–Charpit–Jacobi equation. Solving for W and appending $-Et$ gives the complete principal function.

Proposition 3.1.1 Jacobi's theorem on complete integrals

Let $\mathcal{S}(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)$ denote a complete integral of the Hamilton–Jacobi equation, meaning it contains n independent non-additive separation constants $\alpha_1, \ldots, \alpha_n$ plus one true additive constant. Jacobi's theorem provides the equations of motion:

1. For each $i = 1, \ldots, n$, the equation
$$\frac{\partial \mathcal{S}}{\partial \alpha_i} = \beta_i$$
determines a relation among the coordinates, time, and the two constants α_i and β_i. The constants $\beta_1, \ldots, \beta_n$ are fixed by the initial conditions.
2. The n equations from the previous point determine $q_1(t), \ldots, q_n(t)$, thereby solving the mechanics problem entirely. The canonical momenta follow from
$$p_i = \frac{\partial \mathcal{S}}{\partial q_i}.$$
3. In the language of canonical transformations, the separation constants α_i play the role of the new momenta P_i and the constants β_i play the role of the new coordinates Q_i. All are constant in time because the new Hamiltonian has been chosen to vanish.

Note:-

Connection to Hamilton's principle of least action

The Hamilton–Jacobi formalism is deeply connected to Hamilton's principle of least action. Hamilton's principal function $\mathcal{S}(q, t)$ evaluated along the actual physical path coincides with the action integral $\int_{t_0}^{t} \mathcal{L}\, dt'$ computed along that trajectory. The Hamilton–Jacobi equation itself can be viewed as the condition that the action integral be stationary under variations of the endpoint, generalizing the Euler–Lagrange equations to a differential equation for the action. This unifies the variational and canonical formulations of classical mechanics into a single framework based on the propagating wavefront of constant action.

Note:-

Connection to Maupertuis' principle

The time-independent Hamilton–Jacobi equation also connects to Maupertuis' principle of least action, which characterizes true trajectories as geodesics in configuration space with a metric scaled by kinetic energy. The reduced action $W(q)$ satisfies $dW = p_i\, dq_i$, so that integrating dW along a trajectory is equivalent to integrating the momentum one-form. When $H = E$ is held fixed, the paths that extremize $\int p_i\, dq_i$ are the same paths found by solving the time-independent equation for W.

Example 3.1.1 (Complete integral for a free particle)

For a free particle in one dimension, $\mathcal{H} = p^2/2m$. The Hamilton–Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

We try the additive separation ansatz $\mathcal{S}(x,t) = W(x) + T(t)$. Substituting into the PDE gives

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}T}{\mathrm{d}t} = 0.$$

Since the first term depends only on x and the second only on t, each must equal a constant. We choose the separation constant as $\alpha^2/(2m)$, where α will turn out to be the constant momentum:

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 = \frac{\alpha^2}{2m}, \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\alpha^2}{2m}.$$

The spatial equation gives $\frac{\mathrm{d}W}{\mathrm{d}x} = \pm\alpha$. Choosing the positive sign (the negative case is covered by allowing α to be either positive or negative) and integrating with respect to x gives $W(x) = \alpha x + c_W$. Similarly, integrating the temporal equation gives $T(t) = -\alpha^2 t/(2m) + c_T$. Absorbing the two additive constants into a single overall additive constant (which does not affect the physics) yields the complete integral

$$\mathcal{S}(x,t;\alpha) = \alpha x - \frac{\alpha^2}{2m}\,t.$$

We can verify this solution directly by substitution. Computing $\frac{\partial \mathcal{S}}{\partial x} = \alpha$ and $\frac{\partial \mathcal{S}}{\partial t} = -\alpha^2/(2m)$, the left-hand side of the PDE becomes $\frac{1}{2m}\alpha^2 - \alpha^2/(2m) = 0$, confirming the result. Here α is the constant momentum and serves as the single separation constant for this one-degree-of-freedom system. Because the Hamiltonian does not depend explicitly on time, energy conservation $\mathcal{H} = E = \alpha^2/(2m)$ determines the relationship between the separation constant and the total mechanical energy of the particle.

Question 1: Hamilton–Jacobi for a one-dimensional Hamiltonian

A one-dimensional system has Hamiltonian

$$\mathcal{H}(x,p) = \frac{p^2}{2m} + V(x).$$

(a) Write the Hamilton–Jacobi PDE explicitly for this system.

(b) For the free-particle case $V(x) = 0$, find the complete integral $\mathcal{S}(x,t;\alpha)$ by separation of variables, identifying α as the constant momentum.

(c) Show from Jacobi's theorem that $\frac{\partial \mathcal{S}}{\partial \alpha} = \beta$ yields the trajectory $x(t) = (\alpha/m)t + \beta$, and verify that this satisfies the free-particle equation of motion $m\ddot{x} = 0$.

Solution: **Part (a).** The Hamilton–Jacobi equation is obtained by replacing the canonical momentum p with the partial derivative $\frac{\partial \mathcal{S}}{\partial x}$. Substituting this into the given Hamiltonian gives

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + V(x) + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

This is the explicit Hamilton–Jacobi PDE for a one-dimensional particle in potential $V(x)$.

Part (b). For $V(x) = 0$, the Hamilton–Jacobi equation reduces to

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Use the separation ansatz $\mathcal{S}(x,t) = W(x) + T(t)$. Substituting and using ordinary derivatives gives

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}T}{\mathrm{d}t} = 0.$$

The first term depends only on x and the second only on t, so each must equal a constant. Choose the separation constant so that the spatial derivative equals the momentum α:

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 = \frac{\alpha^2}{2m}, \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\alpha^2}{2m}.$$

Solving the spatial part, take the square root of both sides:

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \pm\alpha.$$

Choosing the positive sign (the negative case is covered by allowing α to be negative) and integrating with respect to x gives $W(x) = \alpha x$. Integrating the temporal equation gives $T(t) = -\alpha^2 t/(2m)$. Therefore,

$$\mathcal{S}(x,t;\alpha) = \alpha x - \frac{\alpha^2}{2m}\,t.$$

This is the complete integral: it solves the HJ PDE and contains one independent non-additive constant α.

Part (c). Jacobi's theorem states that $\frac{\partial \mathcal{S}}{\partial \alpha} = \beta$, where β is a constant determined by initial conditions. Differentiate the complete integral with respect to α:

$$\frac{\partial \mathcal{S}}{\partial \alpha} = x - \frac{\alpha}{m}\,t.$$

Set this equal to β and solve for x:

$$x - \frac{\alpha}{m}\,t = \beta, \qquad \text{so} \qquad x(t) = \frac{\alpha}{m}\,t + \beta.$$

This is the trajectory. Differentiate once with respect to time to find the velocity:

$$\dot{x} = \frac{\alpha}{m}.$$

The velocity is constant, as expected for a free particle. Differentiate a second time:

$$\ddot{x} = 0.$$

Multiplying by the mass gives

$$m\ddot{x} = 0,$$

which is Newton's second law for a free particle. The trajectory is therefore verified. The constant α has the physical meaning $m\dot{x}$, confirming it is the conserved linear momentum. The constant β represents the initial position of the particle at $t = 0$, since $x(0) = \beta$. Together, α and β form a complete set of independent constants for this one-degree-of-freedom system, providing the full two parameters needed to describe the general solution of the second-order equation of motion.

Therefore,

$$\text{HJ PDE:} \quad \frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + V(x) + \frac{\partial \mathcal{S}}{\partial t} = 0,$$

$$\mathcal{S}(x,t;\alpha) = \alpha x - \frac{\alpha^2}{2m}\,t, \qquad x(t) = \frac{\alpha}{m}\,t + \beta.$$

3.1.2 Separation of Variables in the Hamilton-Jacobi Equation

This subsection develops the method of separation of variables for the Hamilton–Jacobi equation, showing how the choice of coordinate system determines whether the PDE reduces to a set of ordinary quadratures.

Definition 3.1.2: Separation of variables for the HJ equation

Suppose the Hamiltonian has no explicit time dependence and the Hamilton–Jacobi equation is $\mathcal{H}(q_1, \ldots, q_n, \frac{\partial \mathcal{S}}{\partial q_1}, \ldots, \frac{\partial \mathcal{S}}{\partial q_n}) = E$. The time variable is separated by setting

$$\mathcal{S}(q_1, \ldots, q_n, t) = W(q_1, \ldots, q_n) - Et,$$

reducing the equation to $\mathcal{H}(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}) = E$. The HJ equation is said to be separable if the characteristic function W can be written as an additive sum of single-variable functions, $W = W_1(q_1) + \cdots + W_n(q_n)$, where each W_i depends on only one coordinate and a set of separation constants. If such a form exists, the solution reduces to evaluating n independent quadratures.

The first step is always the time-independent reduction. When $\frac{\partial \mathcal{H}}{\partial t} = 0$, the Hamiltonian is conserved: $\mathcal{H} = E$. Substituting $\mathcal{S} = W(q_1, \ldots, q_n) - Et$ into the full Hamilton–Jacobi equation, the time derivative contributes $-E$ and the equation becomes

$$\mathcal{H}\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = E.$$

This is the time-independent Hamilton–Jacobi equation. It contains n partial derivatives of W and determines the spatial part of the action. The total principal function is $\mathcal{S} = W - Et$ once W is found.

Note:-

The condition $\frac{\partial \mathcal{H}}{\partial t} = 0$ is necessary for the simple time separation $\mathcal{S} = W - Et$. When the Hamiltonian depends explicitly on time, a different separation ansatz or a time-dependent canonical transformation is required. In the time-independent case, energy is a constant of motion and serves as the first separation constant.

A particularly simple situation arises when one or more coordinates are cyclic. A generalized coordinate q_i is cyclic, or ignorable, when it is absent from the Hamiltonian, which means $\frac{\partial \mathcal{H}}{\partial q_i} = 0$. For such a coordinate, Hamilton's equation gives $\dot{p}_i = 0$, so the conjugate momentum is conserved. Within the Hamilton–Jacobi framework this translates directly: since $p_i = \frac{\partial \mathcal{S}}{\partial q_i}$ and $\frac{\partial \mathcal{H}}{\partial q_i} = 0$, the derivative

$$\frac{\partial \mathcal{S}}{\partial q_i} = \alpha_i$$

is a constant. The contribution of the cyclic coordinate to the characteristic function is simply $W_i(q_i) = \alpha_i \, q_i$, which is immediately integrated.

When more than one coordinate is cyclic the separations are independent and each contributes a linear term to W. The remaining non-cyclic coordinates carry the entire nontrivial structure of the problem and must be separated by additional ansatze.

Theorem 3.1.2 Additive separation theorem

Let the Hamiltonian take the form $\mathcal{H} = T + V$ where the kinetic energy T is a quadratic form in the momenta and the potential energy V is a sum of single-coordinate terms, $V = V_1(q_1) + \cdots + V_n(q_n)$. If the metric coefficients of the kinetic energy depend on only one coordinate each, then the time-independent Hamilton–Jacobi equation

$$\mathcal{H}\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = E$$

admits an additive separation ansatz

$$W(q_1, \ldots, q_n) = W_1(q_1) + W_2(q_2) + \cdots + W_n(q_n).$$

Each function $W_i(q_i)$ satisfies an ordinary differential equation involving one separation constant, and the complete integral is obtained by evaluating n quadratures.

Note:-

The additive separation theorem gives a sufficient condition for separability. A more general theory was developed by Levi-Civita and later refined by Stackel. The Levi-Civita separability conditions state that the Hamilton–Jacobi equation is separable in a given orthogonal coordinate system if and only if the Hamiltonian can be written as a sum, each term depending on only one coordinate and its conjugate momentum. Equivalently, the coefficient matrix of the quadratic kinetic-energy form, when written in these coordinates, must be a Stackel matrix. A Stackel matrix S_{ij} is an $n \times n$ matrix whose (i, j) entry depends only on the single coordinate q_i, and whose determinant is nonzero almost everywhere. The inverse of the Stackel matrix then relates the separation constants to the Hamiltonian components, producing the n separated ODEs.

Several standard orthogonal coordinate systems admit separable Hamilton–Jacobi equations for important classes of potentials. The gradient-squared operator takes different forms in each system, and the metric coefficients determine whether a given potential allows additive separation.

Proposition 3.1.2 Coordinate systems and separability for the HJ equation

The table below summarizes the gradient operator squared $|\nabla W|^2$ in commonly used orthogonal coordinate systems, and the classes of potentials that permit additive separation of the Hamilton–Jacobi equation with $\mathcal{H} = |\nabla W|^2/(2m) + V = E$:

Coordinates	**Gradient squared** $|\nabla W|^2$	**Separable potentials**
Cartesian (x, y, z)	$(\frac{\partial W}{\partial x})^2 + (\frac{\partial W}{\partial y})^2 + (\frac{\partial W}{\partial z})^2$	$V = V_x(x) + V_y(y) + V_z(z)$
Spherical (r, θ, ϕ)	$(\frac{\partial W}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial W}{\partial \theta})^2 + \frac{1}{r^2 \sin^2\theta}(\frac{\partial W}{\partial \phi})^2$	$V = V(r)$ (central); $V(r, \theta)$ with $1/r^2$ separability
Cylindrical (ρ, ϕ, z)	$(\frac{\partial W}{\partial \rho})^2 + \frac{1}{\rho^2}(\frac{\partial W}{\partial \phi})^2 + (\frac{\partial W}{\partial z})^2$	$V = V_\rho(\rho) + V_z(z)$; $V(\rho)$
Parabolic (ξ, η, ϕ)	$|\nabla W|^2 = \frac{1}{\xi^2+\eta^2}\left[(\xi^2+\eta^2)(\frac{\partial W}{\partial \xi})^2 + (\xi^2+\eta^2)(\frac{\partial W}{\partial \eta})^2 + \frac{\xi^2\eta^2}{\xi\eta}(\frac{\partial W}{\partial \phi})^2\right]$, separable for Kepler $V = -k/r$ and Stark potentials	

Parabolic coordinates are defined by $\xi = \sqrt{r(r-z)}$ and $\eta = \sqrt{r + r_z}$, with $z = (\eta^2 - \xi^2)/2$ and $r = (\xi^2 + \eta^2)/2$. The Kepler potential $V = -k/r = -2k/(\xi^2 + \eta^2)$ separates in parabolic coordinates because $1/r$ can be split into a function of ξ plus a function of η after substituting into the HJ equation and multiplying by the metric factor. Parabolic coordinates are especially useful for the hydrogen atom in quantum mechanics and for analyzing the Stark effect, where a uniform electric field along the z-axis is added to the Coulomb potential while preserving separability.

The additive separation ansatz $W(q_1, \ldots, q_n) = W_1(q_1) + \cdots + W_n(q_n)$ is the standard starting point. Substituting this form into the Hamilton–Jacobi equation and multiplying by appropriate metric factors produces a sum, with each term depending on a single coordinate. The equation

$$f_1(q_1, W_1') + f_2(q_2, W_2') + \cdots + f_n(q_n, W_n') = E$$

can only hold for all values of the independent coordinates if each term is itself a constant. These constants are the separation constants $\alpha_1, \ldots, \alpha_n$, constrained by one relation that fixes the total energy. The remaining equations are first-order ODEs for the individual functions $W_i(q_i)$, each solvable by quadrature.

Example 3.1.2 (Free particle in spherical coordinates)

Consider a free particle of mass m in spherical coordinates (r, θ, ϕ). The Hamiltonian is $\mathcal{H} = p^2/(2m)$ and the time-independent HJ equation is $|\nabla W|^2/(2m) = E$, or equivalently

$$\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 + \frac{1}{r^2}\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{1}{r^2 \sin^2\phi}\left(\frac{\mathrm{d}W_\phi}{\mathrm{d}\phi}\right)^2 = 2mE.$$

Using the ansatz $W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$, the coordinate ϕ is cyclic and $dW_\phi/d\phi = \alpha_\phi$. Multiply the equation by r^2:

$$r^2\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = 2mEr^2.$$

The last two terms depend only on θ while the first and rightmost terms depend only on r. Equating both sides to a constant α^2 gives

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha^2, \qquad \frac{\mathrm{d}W_r}{\mathrm{d}r} = \sqrt{2mE - \frac{\alpha^2}{r^2}}.$$

Each equation integrates by quadrature, and $W_\phi(\phi) = \alpha_\phi\,\phi$. These three quadratures constitute the complete integral for the free particle in spherical coordinates.

Question 2: Separation for a particle in a uniform gravitational field

A particle of mass m moves in the xy-plane under a uniform gravitational field g acting in the negative y-direction. The Hamiltonian is

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + mgy.$$

(a) Write the time-independent Hamilton–Jacobi equation and apply the separation ansatz $\mathcal{S} = W_x(x) + W_y(y) - Et$. Show that x is a cyclic coordinate and that $\frac{\mathrm{d}W_x}{\mathrm{d}x} = \alpha_x$, a constant.

(b) Find $\frac{\mathrm{d}W_y}{\mathrm{d}y}$ in terms of the separation constants. Define the transverse energy $E_y = E - \alpha_x^2/(2m)$ and write the quadrature integral for $W_y(y)$.

(c) A projectile of mass $m = 0.100\,\mathrm{kg}$ is launched from $y = 0$ with speed $v_0 = 25.0\,\mathrm{m/s}$ at angle $\theta_0 = 45.0°$ above the horizontal. Take $g = 9.81\,\mathrm{m/s^2}$. Compute the x-momentum $p_x = mv_0\cos\theta_0$ and the transverse energy $E_y = \frac{1}{2}mv_0^2\sin^2\theta_0$ in SI units.

Solution: **Part (a).** The time-independent Hamilton–Jacobi equation is obtained by setting $\mathcal{H} = E$. With the given Hamiltonian this reads

$$\frac{1}{2m}\left[\left(\frac{\partial\mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial y}\right)^2\right] + mgy = E.$$

Substitute the separation ansatz $\mathcal{S}(x, y, t) = W_x(x) + W_y(y) - Et$. The spatial partial derivatives become ordinary derivatives: $\frac{\partial\mathcal{S}}{\partial x} = \frac{\mathrm{d}W_x}{\mathrm{d}x}$ and $\frac{\partial\mathcal{S}}{\partial y} = \frac{\mathrm{d}W_y}{\mathrm{d}y}$. Substituting gives

$$\frac{1}{2m}\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \frac{1}{2m}\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + mgy = E.$$

Multiply both sides by $2m$:

$$\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + 2m^2 g y = 2mE.$$

The coordinate x does not appear in the Hamiltonian, so x is cyclic. The first term depends only on x while the remaining terms depend only on y. Separating gives

$$\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 = \alpha_x^2,$$

where α_x is a separation constant. Therefore,

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \alpha_x,$$

which integrates to $W_x(x) = \alpha_x\, x$. The constant α_x is identified with the constant x-component of the canonical momentum.

Part (b). Substitute $(dW_x/dx)^2 = \alpha_x^2$ back into the HJ equation:

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + 2m^2 g y = 2mE - \alpha_x^2.$$

Define the transverse energy $E_y = E - \alpha_x^2/(2m)$. Then $2mE - \alpha_x^2 = 2mE_y$, and the y-equation becomes

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + 2m^2 g y = 2mE_y.$$

Solve for the derivative:

$$\frac{\mathrm{d}W_y}{\mathrm{d}y} = \pm\sqrt{2mE_y - 2m^2 g y}.$$

Factor $2m$ from the radicand:

$$\frac{\mathrm{d}W_y}{\mathrm{d}y} = \pm\sqrt{2m\left(E_y - mgy\right)}.$$

The quadrature for $W_y(y)$ is

$$W_y(y) = \pm\int \sqrt{2m\left(E_y - mgy\right)}\, dy.$$

Part (c). The initial conditions specify mass $m = 0.100\,\mathrm{kg}$, launch speed $v_0 = 25.0\,\mathrm{m/s}$, and launch angle $\theta_0 = 45.0°$. Compute the x-momentum:

$$p_x = m v_0 \cos\theta_0.$$

Substitute the numerical values:

$$p_x = (0.100)(25.0)\cos(45.0°)\,\mathrm{kg{\cdot}m/s}.$$

Since $\cos(45.0°) = \sqrt{2}/2 \approx 0.7071$,

$$p_x = (0.100)(25.0)(0.7071)\,\mathrm{kg{\cdot}m/s} = 1.77\,\mathrm{kg{\cdot}m/s}.$$

The transverse energy is $E_y = \frac{1}{2} m v_0^2 \sin^2\theta_0$. The vertical speed is

$$v_{0y} = v_0 \sin(45.0°) = (25.0)(0.7071)\,\mathrm{m/s} = 17.68\,\mathrm{m/s}.$$

Evaluate E_y:

$$E_y = \tfrac{1}{2}(0.100)(25.0)^2(0.7071)^2\,\mathrm{J}.$$

This gives

$$E_y = \tfrac{1}{2}(0.100)(625)(0.500)\,\mathrm{J} = 15.6\,\mathrm{J}.$$

Therefore,

$$p_x = 1.77\,\mathrm{kg{\cdot}m/s}, \qquad E_y = 15.6\,\mathrm{J}.$$

3.1.3 Action-Angle Variables

This subsection develops action-angle variables for integrable periodic systems, presenting the canonical transformation that reduces any periodic system to trivial dynamics where the momenta are constant and the angles advance uniformly.

Definition 3.1.3: Action and angle variables

For a periodic degree of freedom with generalized coordinate q_i and conjugate momentum p_i, the action variable J_i is defined as the phase-space integral over one complete closed orbit:

$$J_i = \oint p_i \, dq_i.$$

The integral is taken over one full cycle of the periodic motion. The angle variable w_i is the canonical coordinate conjugate to J_i, defined by differentiating Hamilton's characteristic function W with respect to the action:

$$w_i = \frac{\partial W}{\partial J_i}.$$

The angle variable w_i increases by exactly one complete unit during one period of the associated periodic motion.

Note:-

The action variable J_i equals the area enclosed by the orbit in the (q_i, p_i) phase-space plane. This geometric interpretation makes it straightforward to evaluate J_i for simple periodic systems: the integral reduces to computing the area of an ellipse (harmonic oscillator), a triangle plus its reflection (infinite well), or other phase-space shapes.

In a completely integrable system with n degrees of freedom, the Hamiltonian depends only on the action variables and not on the angle variables: $\mathcal{H} = \mathcal{H}(J_1, \ldots, J_n)$. Because the angles do not appear in $\mathcal{H}$, they are cyclic coordinates. This leads to the simplest possible Hamiltonian dynamics.

Theorem 3.1.3 Hamilton's equations in action-angle variables

Let $\mathcal{H} = \mathcal{H}(J_1, \ldots, J_n)$ be the Hamiltonian expressed in action variables. Hamilton's canonical equations in the (w, J) variables are

$$\dot{J}_i = -\frac{\partial \mathcal{H}}{\partial w_i} = 0, \qquad \dot{w}_i = \frac{\partial \mathcal{H}}{\partial J_i} \equiv \omega_i.$$

The action variables J_i are constant in time, and the angle variables advance linearly:

$$w_i(t) = \omega_i t + w_i(0).$$

The frequency $\omega_i = \frac{\partial \mathcal{H}}{\partial J_i}$ is independent of time. Since the angle variable w_i increases by one unit over one complete cycle, the period of the i-th motion is $T_i = 1/\omega_i$.

The frequency ω_i provides direct access to the temporal characteristics of the motion. When there is a single degree of freedom, the action is found by evaluating the integral $J = \oint p \, dq$, the Hamiltonian is inverted to give $E(J)$, and the period follows immediately from $T = 1/\frac{\partial E}{\partial J}$. This procedure avoids solving the equations of motion directly. The physical angular frequency of the motion is $2\pi\omega_i$.

Example 3.1.3 (Simple harmonic oscillator in action-angle variables)

The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and natural angular frequency ω_0 is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2.$$

At energy $E = \mathcal{H}$, the momentum is $p = \pm\sqrt{2mE - m^2\omega_0^2 x^2}$ and the turning points are at $x = \pm A$ with amplitude $A = \sqrt{2E/(m\omega_0^2)}$.
The action variable is evaluated by integrating over one complete oscillation:

$$J = \oint p\,dx = 2\int_{-A}^{A} \sqrt{2mE - m^2\omega_0^2 x^2}\,dx.$$

Substitute $x = A\sin\phi$, so $dx = A\cos\phi\,d\phi$ and the limits are $\phi = -\pi/2$ to $\pi/2$:

$$\sqrt{2mE - m^2\omega_0^2 x^2} = \sqrt{2mE - 2mE\sin^2\phi} = \sqrt{2mE}\cos\phi.$$

The integral becomes

$$J = 2\sqrt{2mE}\cdot A\int_{-\pi/2}^{\pi/2}\cos^2\phi\,d\phi = 2\sqrt{2mE}\cdot A\cdot\frac{\pi}{2} = \pi\sqrt{2mE}\cdot A.$$

Substituting $A = \sqrt{2E/(m\omega_0^2)}$ gives

$$J = \pi\sqrt{2mE}\cdot\sqrt{\frac{2E}{m\omega_0^2}} = 2\pi\,\frac{E}{\omega_0}.$$

Inverting this relation expresses the energy as a function of the action:

$$E(J) = \frac{\omega_0 J}{2\pi}.$$

The frequency obtained from the action-angle formalism is the derivative of E with respect to J:

$$\omega = \frac{\partial E}{\partial J} = \frac{\omega_0}{2\pi}.$$

The period of oscillation is $T = 1/\omega = 2\pi/\omega_0$, and the physical angular frequency is $2\pi\omega = \omega_0$. Crucially, the frequency is independent of the energy E and therefore independent of the amplitude A. This is the property of isochrony: all oscillations of a simple harmonic oscillator have the same period regardless of amplitude.

Note:-

The Kepler problem (gravitational or electrostatic $V = -k/r$) has three independent action variables J_r, J_θ, and J_ϕ. The energy depends on their sum:

$$E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\phi)^2}.$$

The frequency derivatives $\frac{\partial E}{\partial J_r}$, $\frac{\partial E}{\partial J_\theta}$, $\frac{\partial E}{\partial J_\phi}$ are all equal, so the three frequencies are degenerate. Degenerate frequencies mean every bound orbit closes on itself after one period. This degeneracy is the deep reason Kepler's ellipses are closed: the radial period equals the angular period. A small perturbation $V = -k/r + \epsilon/r^2$ breaks the degeneracy and produces precession.

Question 3: Particle in a one-dimensional infinite potential well

A particle of mass m is confined to a region $0 < x < L$ by infinite potential walls, so $V(x) = 0$ for $0 < x < L$ and $V = \infty$ elsewhere. Inside the well the Hamiltonian is $\mathcal{H} = p^2/(2m)$ and the total energy is E.

(a) Compute the action variable $J = \oint p\,dx$ for this system, showing that $J = 2L\sqrt{2mE}$.

(b) Express the energy as $E(J)$ and compute the frequency $\omega = \frac{\partial E}{\partial J}$ and the period $T = 1/\omega$. Show that $T = 2L\sqrt{m/(2E)}$, which equals the time for the particle to travel the distance $2L$ at speed $v = \sqrt{2E/m}$.

(c) For an electron with mass $m = 9.11 \times 10^{-31}$ kg confined to a region of width $L = 1.00 \times 10^{-10}$ m with total energy $E = 1.00\,\text{eV} = 1.60 \times 10^{-19}$ J, compute the numerical values of J (in kg·m²/s) and T (in seconds).

Solution: **Part (a).** Inside the well the particle has kinetic energy $E = p^2/(2m)$, so the magnitude of momentum is $|p| = \sqrt{2mE}$ and is independent of position. The particle travels back and forth between the walls at $x = 0$ and $x = L$. During the forward leg the momentum is $p = +\sqrt{2mE}$ and during the return leg $p = -\sqrt{2mE}$.

The action integral over one complete cycle is

$$J = \oint p\, dx = \int_0^L \sqrt{2mE}\, dx + \int_L^0 \left(-\sqrt{2mE}\right) dx.$$

Each integral equals $L\sqrt{2mE}$, so

$$J = L\sqrt{2mE} + L\sqrt{2mE} = 2L\sqrt{2mE}.$$

Part (b). Solve the result from part (a) for E:

$$\frac{J}{2L} = \sqrt{2mE}, \qquad \frac{J^2}{4L^2} = 2mE, \qquad E(J) = \frac{J^2}{8mL^2}.$$

Differentiate with respect to J to find the frequency:

$$\omega = \frac{\partial E}{\partial J} = \frac{J}{4mL^2}.$$

The period is the reciprocal of the frequency:

$$T = \frac{1}{\omega} = \frac{4mL^2}{J}.$$

Substitute $J = 2L\sqrt{2mE}$ to express T in terms of E:

$$T = \frac{4mL^2}{2L\sqrt{2mE}} = \frac{2mL}{\sqrt{2mE}} = 2L\sqrt{\frac{m}{2E}}.$$

Independently, the particle's speed inside the well is $v = \sqrt{2E/m}$, and the round-trip distance is $2L$. The travel time for one complete cycle is

$$T = \frac{2L}{v} = 2L\sqrt{\frac{m}{2E}},$$

which agrees exactly with the action-angle result.

Part (c). The given values are $m = 9.11 \times 10^{-31}$ kg, $L = 1.00 \times 10^{-10}$ m, and $E = 1.60 \times 10^{-19}$ J.

First compute the action variable $J = 2L\sqrt{2mE}$:

$$2mE = 2(9.11 \times 10^{-31})(1.60 \times 10^{-19})\,\text{kg·J} = 2.92 \times 10^{-49}\,\text{kg}^2\text{·m}^2/\text{s}^2.$$

(The product kg·J has the same dimensions as $\text{kg}^2\text{·m}^2/\text{s}^2$ since $1\,\text{J} = 1\,\text{kg·m}^2/\text{s}^2$.) Taking the square root:

$$\sqrt{2mE} = \sqrt{2.92 \times 10^{-49}}\,\text{kg·m/s} = 5.40 \times 10^{-25}\,\text{kg·m/s}.$$

Now multiply by $2L$:

$$J = 2(1.00 \times 10^{-10})(5.40 \times 10^{-25})\,\text{kg·m}^2/\text{s} = 1.08 \times 10^{-34}\,\text{kg·m}^2/\text{s}.$$

Next compute the period $T = 2L\sqrt{m/(2E)}$:

$$\frac{m}{2E} = \frac{9.11 \times 10^{-31}}{2(1.60 \times 10^{-19})}\,\text{kg/J} = 2.85 \times 10^{-12}\,\text{s}^2/\text{m}^2.$$

Taking the square root:

$$\sqrt{\frac{m}{2E}} = \sqrt{2.85 \times 10^{-12}}\,\text{s/m} = 1.69 \times 10^{-6}\,\text{s/m}.$$

Multiply by $2L$:

$$T = 2(1.00 \times 10^{-10})(1.69 \times 10^{-6})\,\text{s} = 3.37 \times 10^{-16}\,\text{s}.$$

Therefore,

$$J = 1.08 \times 10^{-34}\,\text{kg}\cdot\text{m}^2/\text{s}, \qquad T = 3.37 \times 10^{-16}\,\text{s}.$$

3.1.4 Hamilton-Jacobi with Electromagnetic Fields

This subsection extends the Hamilton–Jacobi framework to a charged particle moving in electromagnetic fields by replacing the canonical momentum with the minimal-coupling substitution $p \to p - qA$.

Definition 3.1.4: Lagrangian for a charged particle in electromagnetic fields

Let a particle of mass m and charge q move with velocity $\vec{v}$ in an electromagnetic field described by the scalar potential $\varphi(\vec{r}, t)$ and the vector potential $\vec{A}(\vec{r}, t)$. The Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}mv^2 - q\varphi + q\,\vec{v} \cdot \vec{A}.$$

The term $-q\varphi$ represents the electrostatic potential energy, while the velocity-dependent term $q\,\vec{v} \cdot \vec{A}$ is the magnetic interaction. Together they reproduce both the electric and magnetic parts of the Lorentz force when the Euler–Lagrange equations are applied. The canonical momentum conjugate to each spatial coordinate r_i is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{r}_i} = mv_i + qA_i,$$

so that in vector notation,

$$\vec{p} = m\vec{v} + q\vec{A}.$$

The kinetic (mechanical) momentum is $m\vec{v} = \vec{p} - q\vec{A}$, and it is this combination that enters the kinetic-energy part of the Hamiltonian.

Note:-

The canonical momentum differs from the kinetic momentum. The extra term $q\vec{A}$ in the canonical momentum is what distinguishes a charged particle's Hamiltonian dynamics from those of a free particle, even in the absence of a scalar potential. Because $\vec{A}$ generally depends on position, the canonical momentum is not simply $m\vec{v}$, and its time derivative is not equal to the mechanical force. Instead, Hamilton's equations for the canonical variables reproduce the full Lorentz-force law.

Theorem 3.1.4 Hamiltonian for a charged particle in electromagnetic fields

Let m denote the mass, let q denote the charge, let $\varphi(\vec{r}, t)$ denote the scalar potential, and let $\vec{A}(\vec{r}, t)$ denote the vector potential. The canonical momentum has components p_i. Then the Hamiltonian of the charged particle is

$$\mathcal{H}(\vec{r}, \vec{p}, t) = \frac{1}{2m}\left|\vec{p} - q\vec{A}(\vec{r}, t)\right|^2 + q\,\varphi(\vec{r}, t).$$

The corresponding Hamilton–Jacobi equation for the principal function $\mathcal{S}(\vec{r}, t)$ follows by replacing p_i with $\frac{\partial \mathcal{S}}{\partial r_i}$:

$$\frac{1}{2m}\left|\nabla\mathcal{S} - q\vec{A}\right|^2 + q\,\varphi + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Derivation from the Lagrangian to the HJ equation: Start from the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}m\,\dot{\vec{r}}\cdot\dot{\vec{r}} - q\varphi(\vec{r},t) + q\,\dot{\vec{r}}\cdot\vec{A}(\vec{r},t).$$

Compute the canonical momentum by differentiating with respect to each velocity component:

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}.$$

Invert this relation to express the velocity in terms of the canonical momentum:

$$\dot{\vec{r}} = \frac{1}{m}\big(\vec{p} - q\vec{A}\big).$$

Form the Legendre transform to obtain the Hamiltonian:

$$\mathcal{H} = \vec{p}\cdot\dot{\vec{r}} - \mathcal{L}.$$

Substitute the expressions for $\dot{\vec{r}}$ and $\mathcal{L}$. The kinetic-term piece gives

$$\vec{p}\cdot\dot{\vec{r}} = \frac{1}{m}\,\vec{p}\cdot\big(\vec{p} - q\vec{A}\big) = \frac{1}{m}\big(p^2 - q\,\vec{p}\cdot\vec{A}\big),$$

and the Lagrangian itself reads

$$\mathcal{L} = \tfrac{1}{2m}\big|\vec{p} - q\vec{A}\big|^2 - q\varphi + \frac{q}{m}\big(\vec{p} - q\vec{A}\big)\cdot\vec{A}.$$

The dot product in the last term of $\mathcal{L}$ expands as

$$\frac{q}{m}\big(\vec{p} - q\vec{A}\big)\cdot\vec{A} = \frac{q}{m}\big(\vec{p}\cdot\vec{A} - qA^2\big).$$

Subtracting $\mathcal{L}$ from $\vec{p}\cdot\dot{\vec{r}}$ gives

$$\mathcal{H} = \frac{1}{m}\big(p^2 - q\,\vec{p}\cdot\vec{A}\big) - \left[\frac{1}{2m}\big(p^2 - 2q\,\vec{p}\cdot\vec{A} + q^2A^2\big) - q\varphi + \frac{q}{m}\big(\vec{p}\cdot\vec{A} - qA^2\big)\right].$$

Combine the terms proportional to $\vec{p}\cdot\vec{A}$:

$$-\frac{q}{m}\,\vec{p}\cdot\vec{A} - \frac{q}{m}\,\vec{p}\cdot\vec{A} + \frac{2q}{2m}\,\vec{p}\cdot\vec{A} = -\frac{q}{m}\,\vec{p}\cdot\vec{A}.$$

Combine the terms proportional to A^2:

$$-\frac{q^2}{2m}A^2 + \frac{q^2}{m}A^2 = \frac{q^2}{2m}A^2.$$

Together with the p^2 terms, $\frac{p^2}{m} - \frac{p^2}{2m} = \frac{p^2}{2m}$. All terms assemble into the compact form

$$\mathcal{H} = \frac{1}{2m}\big|\vec{p} - q\vec{A}\big|^2 + q\varphi.$$

Now replace each component of the canonical momentum by the gradient of the action function, $p_i = \frac{\partial\mathcal{S}}{\partial r_i}$, so that $\vec{p} \to \nabla\mathcal{S}$. The Hamilton–Jacobi equation $\mathcal{H} + \frac{\partial\mathcal{S}}{\partial t} = 0$ then becomes

$$\frac{1}{2m}\left|\nabla\mathcal{S} - q\vec{A}\right|^2 + q\,\varphi + \frac{\partial\mathcal{S}}{\partial t} = 0.$$

☺

Corollary 3.1.2 Minimal coupling rule and gauge invariance

The passage from a free particle to a charged particle in electromagnetic fields is effected by the minimal coupling substitutions

$$\vec{p} \longrightarrow \vec{p} - q\vec{A}, \qquad E \longrightarrow E - q\varphi.$$

Under a gauge transformation of the potentials,

$$\vec{A}' = \vec{A} + \nabla\Lambda, \qquad \varphi' = \varphi - \frac{\partial \Lambda}{\partial t},$$

the HJ equation retains its form provided the principal function transforms as

$$\mathcal{S}'(\vec{r}, t) = \mathcal{S}(\vec{r}, t) - q\,\Lambda(\vec{r}, t).$$

To see this, substitute $\nabla\mathcal{S}' = \nabla\mathcal{S} - q\nabla\Lambda$ into the HJ equation written in the transformed potentials:

$$\frac{1}{2m}\left|\nabla\mathcal{S}' - q\vec{A}'\right|^2 + q\varphi' + \frac{\partial \mathcal{S}'}{\partial t} = \frac{1}{2m}\left|\nabla\mathcal{S} - q\vec{A}\right|^2 + q\varphi + \frac{\partial \mathcal{S}}{\partial t},$$

so the unprimed and primed equations are identical. Thus the Hamilton–Jacobi formulation respects electromagnetic gauge invariance.

Note:-

A deep connection to quantum mechanics emerges from the WKB ansatz. Write the wavefunction as $\psi(\vec{r}, t) = \exp\big(i\mathcal{S}(\vec{r}, t)/\hbar\big)$. Substituting this form into the Schrodinger equation,

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2\psi + q\varphi\psi,$$

and collecting terms order by order in $\hbar$, the leading-order equation (proportional to $\hbar^0$) is exactly the classical Hamilton–Jacobi equation for a charged particle:

$$\frac{1}{2m}\left|\nabla\mathcal{S} - q\vec{A}\right|^2 + q\varphi + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Higher-order terms in $\hbar$ account for quantum corrections. In this sense, the classical Hamilton–Jacobi PDE is the $\hbar \longrightarrow 0$ limit of the Schrodinger equation.

Example 3.1.4 (Separation for time-independent fields)

Suppose the electromagnetic potentials are time-independent. Then the Hamiltonian has no explicit time dependence and the total energy E is conserved. The time variable separates from the action as $\mathcal{S} = W(\vec{r}) - Et$, and the Hamilton–Jacobi equation reduces to the time-independent form

$$\frac{1}{2m}\left|\nabla W - q\vec{A}(\vec{r})\right|^2 + q\,\varphi(\vec{r}) = E.$$

If any spatial coordinate is absent from both $\vec{A}$ and φ, the corresponding component of ∇W is a constant separation equal to the conserved canonical momentum for that coordinate.

Question 4: Hamilton–Jacobi for a uniform magnetic field in Landau gauge

A charged particle moves in a uniform magnetic field $\vec{B} = B_0\,\hat{\mathbf{z}}$ with the vector potential chosen in the Landau gauge $\vec{A} = (0, B_0 x, 0)$ and scalar potential $\varphi = 0$.

(a) Write the Hamiltonian in terms of the canonical momenta p_x, p_y, p_z.

(b) Write the full Hamilton–Jacobi equation explicitly in Cartesian coordinates. Identify which generalized coordinates are cyclic.

(c) For an electron ($q = -e = -1.60 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg) in a field $B_0 = 1.00$ T, the cyclotron frequency is $\omega_c = |q|B_0/m$. If the total transverse energy is $E_\perp = 100\,\text{eV} = 1.60 \times 10^{-17}$ J, compute ω_c numerically and verify the value of ω_c from the HJ separation constants matches this expression.

Solution: **Part (a).** The Hamiltonian for a charged particle in electromagnetic fields is

$$\mathcal{H} = \frac{1}{2m}\left|\vec{p} - q\vec{A}\right|^2 + q\varphi.$$

With $\vec{A} = (0, B_0 x, 0)$ and $\varphi = 0$, the three components of $\vec{p} - q\vec{A}$ are

$$\left(p_x - q(0),\ p_y - qB_0 x,\ p_z - q(0)\right) = \left(p_x,\ p_y - qB_0 x,\ p_z\right).$$

The square of the magnitude is the sum of the squares of these components. Therefore,

$$\mathcal{H} = \frac{1}{2m}\left[p_x^2 + \left(p_y - qB_0 x\right)^2 + p_z^2\right].$$

Part (b). The Hamilton–Jacobi equation reads $\mathcal{H}\left(\vec{r}, \nabla\mathcal{S}\right) + \frac{\partial \mathcal{S}}{\partial t} = 0$. Substitute the Hamiltonian from part (a) with the replacements $p_x \to \frac{\partial \mathcal{S}}{\partial x}$, $p_y \to \frac{\partial \mathcal{S}}{\partial y}$, $p_z \to \frac{\partial \mathcal{S}}{\partial z}$:

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y} - qB_0 x\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z}\right)^2\right] + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

This is the full HJ PDE in Cartesian coordinates. A generalized coordinate is cyclic if it is absent from the Hamiltonian. The coordinate y does not appear explicitly in $\mathcal{H}$, so y is cyclic and the conjugate momentum $\frac{\partial \mathcal{S}}{\partial y} = p_y$ is conserved. Similarly, z is absent from $\mathcal{H}$, so z is cyclic and p_z is conserved. The coordinate x does appear in the term $(p_y - qB_0 x)^2$, so x is not cyclic. Thus y and z are the two cyclic coordinates.

Part (c). For the electron, the cyclotron frequency follows from the minimal-coupling Hamiltonian. The numerical value is

$$\omega_c = \frac{|q|B_0}{m}.$$

Substitute the given values:

$$|q| = 1.60 \times 10^{-19}\,\text{C}, \qquad B_0 = 1.00\,\text{T}, \qquad m = 9.11 \times 10^{-31}\,\text{kg}.$$

Form the ratio:

$$\omega_c = \frac{(1.60 \times 10^{-19})(1.00)}{9.11 \times 10^{-31}}\,\text{rad/s}.$$

This gives

$$\omega_c = 1.756 \times 10^{11}\,\text{rad/s}.$$

Now verify that the HJ formalism produces the same frequency. Because the potentials are time-independent, the action separates as $\mathcal{S} = W_x(x) - E_\perp t + \alpha_y\, y + \alpha_z\, z$. The time-independent HJ equation for the transverse motion is

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\alpha_y - qB_0 x\right)^2\right] = E_\perp,$$

where $E_\perp$ is the total transverse energy. Solve for $\frac{\mathrm{d}W_x}{\mathrm{d}x}$:

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \pm\sqrt{2mE_\perp - \left(\alpha_y - qB_0 x\right)^2}.$$

Change variable to the shifted x-coordinate centered on the guiding center, $X = x - \alpha_y/(qB_0)$, giving

$$\frac{\mathrm{d}W_x}{\mathrm{d}X} = \pm\sqrt{2mE_\perp - (qB_0)^2 X^2}.$$

This is the square-root form of the harmonic-oscillator action. Completing the square and comparing with the standard form $\pm\sqrt{2m\mathcal{E} - m^2\omega^2 X^2}$, we identify

$$m^2\omega^2 = (qB_0)^2, \qquad \text{so} \qquad \omega = \frac{|q|B_0}{m} = \omega_c.$$

The HJ separation constant analysis thus recovers the cyclotron frequency exactly, independent of the guiding-center location and the transverse energy. The transverse energy $E_\perp = 1.60 \times 10^{-17}$ J sets the gyroradius but does not affect the frequency.

Therefore, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left[p_x^2 + \left(p_y - qB_0 x\right)^2 + p_z^2\right],$$

the Hamilton–Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y} - qB_0 x\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z}\right)^2\right] + \frac{\partial \mathcal{S}}{\partial t} = 0,$$

the cyclic coordinates are y and z, and the cyclotron frequency is

$$\omega_c = 1.76 \times 10^{11} \text{ rad/s}.$$

3.2 Mechanics Problems via HJ

3.2.1 Free Particle in 1D and 3D

This subsection solves the Hamilton–Jacobi equation for a free particle in one and three dimensions, demonstrating that Jacobi's theorem reproduces the familiar result of uniform straight-line motion.

Definition 3.2.1: Free particle Hamiltonian and Hamilton–Jacobi equation

For a free particle of mass m the Hamiltonian is purely kinetic:

$$\mathcal{H} = \frac{p^2}{2m}.$$

In one dimension, substituting $p = \frac{\partial \mathcal{S}}{\partial x}$ into the Hamilton–Jacobi equation $\mathcal{H} + \frac{\partial \mathcal{S}}{\partial t} = 0$ yields

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

The three-dimensional Hamiltonian is $\mathcal{H} = (p_x^2 + p_y^2 + p_z^2)/(2m)$ and the corresponding Hamilton–Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z}\right)^2\right] + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Theorem 3.2.1 Complete integral for a 1D free particle

The complete integral of the one-dimensional free-particle Hamilton–Jacobi equation is

$$\mathcal{S}(x, t; E) = \pm\sqrt{2mE}\, x - Et,$$

where $E > 0$ is the total mechanical energy. Jacobi's theorem $\frac{\partial \mathcal{S}}{\partial E} = \beta$ gives the trajectory

$$x(t) = \pm\sqrt{\frac{2E}{m}}\,(t + \beta) = v_0 t + x_0,$$

with constant velocity $v_0 = \pm\sqrt{2E/m}$ and initial position $x_0 = v_0\beta$.

Derivation of the 1D and 3D free-particle action: Because the free-particle Hamiltonian has no explicit time dependence, $\frac{\partial \mathcal{H}}{\partial t} = 0$ and energy is conserved: $\mathcal{H} = E$. Use the time-independent reduction $\mathcal{S}(x,t) = W(x) - Et$. Substituting:

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 - E = 0.$$

Solve for the derivative:

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \pm\sqrt{2mE}.$$

Integrate with respect to x (absorbing the integration constant into the additive constant of $\mathcal{S}$):

$$W(x) = \pm\sqrt{2mE}\,x.$$

Reassemble the principal function:

$$\mathcal{S}(x,t) = \pm\sqrt{2mE}\,x - Et.$$

Jacobi's theorem requires $\frac{\partial \mathcal{S}}{\partial E} = \beta$. Differentiate:

$$\frac{\partial \mathcal{S}}{\partial E} = \pm\frac{m}{\sqrt{2mE}}\,x - t = \beta.$$

Solve for $x(t)$:

$$x = \pm\frac{\sqrt{2mE}}{m}(t+\beta) = \pm\sqrt{\frac{2E}{m}}(t+\beta).$$

Setting $v_0 = \pm\sqrt{2E/m}$ and $x_0 = v_0\beta$ gives $x(t) = v_0 t + x_0$.

In three dimensions all Cartesian coordinates are cyclic, so each conjugate momentum is conserved. Setting $\frac{\partial \mathcal{S}}{\partial x} = p_x$, $\frac{\partial \mathcal{S}}{\partial y} = p_y$, $\frac{\partial \mathcal{S}}{\partial z} = p_z$ as constants:

$$W(x,y,z) = p_x x + p_y y + p_z z,$$

with energy $E = (p_x^2 + p_y^2 + p_z^2)/(2m)$. The principal function is

$$\mathcal{S}(x,y,z,t) = p_x x + p_y y + p_z z - Et.$$

Treating (p_x, p_y, p_z) as three independent separation constants, Jacobi's theorem gives

$$\frac{\partial \mathcal{S}}{\partial p_x} = x - \frac{p_x}{m}t = \beta_x, \qquad \frac{\partial \mathcal{S}}{\partial p_y} = y - \frac{p_y}{m}t = \beta_y, \qquad \frac{\partial \mathcal{S}}{\partial p_z} = z - \frac{p_z}{m}t = \beta_z.$$

Each coordinate evolves linearly with time, confirming uniform straight-line motion in three dimensions. 😊

Note:-

Connection to Newton's second law

Each coordinate equation $q_i(t) = (p_i/m)t + \beta_i$ integrates a constant velocity $\dot{q}_i = p_i/m$. The acceleration vanishes, $\ddot{q}_i = 0$, which is precisely the result of Newton's second law for zero applied force. The Hamilton–Jacobi formalism therefore reproduces the familiar kinematic result of uniform motion along a straight line.

Question 5: Free particle in three dimensions

A free particle of mass $m = 2.0\,\mathrm{kg}$ passes through the origin at $t = 0$ with initial velocity $(3.0,\,4.0,\,0)\,\mathrm{m/s}$.

(a) Write Hamilton's principal function $\mathcal{S}(x,y,z,t)$ using the additive separation ansatz $\mathcal{S} = p_x x + p_y y + p_z z - Et$, substituting numerical values for the momenta and energy.

(b) From Jacobi's theorem, $\frac{\partial \mathcal{S}}{\partial p_x} = \beta_x$, $\frac{\partial \mathcal{S}}{\partial p_y} = \beta_y$, $\frac{\partial \mathcal{S}}{\partial p_z} = \beta_z$, find $x(t)$, $y(t)$, and $z(t)$ using the given initial conditions.

(c) Verify that the trajectory matches $\mathbf{r}(t) = (3.0t,\,4.0t,\,0)\,\mathrm{m}$.

Solution: **Part (a).** Compute the canonical momenta from the initial velocity:

$$p_x = mv_{x0} = (2.0)(3.0)\,\mathrm{kg{\cdot}m/s} = 6.0\,\mathrm{kg{\cdot}m/s},$$

$$p_y = mv_{y0} = (2.0)(4.0)\,\mathrm{kg{\cdot}m/s} = 8.0\,\mathrm{kg{\cdot}m/s}, \qquad p_z = 0.$$

The total energy is

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{(6.0)^2 + (8.0)^2}{2(2.0)}\,\mathrm{J} = \frac{36 + 64}{4.0}\,\mathrm{J} = 25\,\mathrm{J}.$$

Substitute these into the additive ansatz:

$$\mathcal{S}(x, y, z, t) = 6.0\,x + 8.0\,y - 25\,t,$$

where x and y are in metres, t in seconds, and $\mathcal{S}$ in joule-seconds.

Part (b). Apply Jacobi's theorem for each momentum component. For the x-coordinate:

$$\frac{\partial \mathcal{S}}{\partial p_x} = x - \frac{p_x}{m}\,t = \beta_x.$$

At $t = 0$ the particle is at the origin, so $x(0) = 0$ and $\beta_x = 0$. Therefore,

$$x(t) = \frac{p_x}{m}\,t = \frac{6.0}{2.0}\,t = 3.0\,t.$$

For the y-coordinate:

$$\frac{\partial \mathcal{S}}{\partial p_y} = y - \frac{p_y}{m}\,t = \beta_y.$$

With $y(0) = 0$, we have $\beta_y = 0$ and

$$y(t) = \frac{p_y}{m}\,t = \frac{8.0}{2.0}\,t = 4.0\,t.$$

For the z-coordinate:

$$\frac{\partial \mathcal{S}}{\partial p_z} = z - \frac{p_z}{m}\,t = \beta_z.$$

Since $p_z = 0$ and $z(0) = 0$, we obtain $\beta_z = 0$ and

$$z(t) = 0.$$

Part (c). Assembling the three components:

$$\mathbf{r}(t) = \big(x(t),\, y(t),\, z(t)\big) = (3.0\,t,\, 4.0\,t,\, 0).$$

This matches the expected trajectory $\mathbf{r}(t) = (3.0t,\, 4.0t,\, 0)\,\mathrm{m}$, confirming the consistency of the Hamilton–Jacobi formalism for the free particle. The speed is $|\mathbf{v}| = \sqrt{3.0^2 + 4.0^2} = 5.0\,\mathrm{m/s}$, and the kinetic energy $E = \frac{1}{2}(2.0)(5.0)^2 = 25\,\mathrm{J}$ matches the energy from part (a).

Therefore,

$$\mathcal{S}(x, y, z, t) = 6.0\,x + 8.0\,y - 25\,t, \qquad \mathbf{r}(t) = (3.0\,t,\, 4.0\,t,\, 0)\,\mathrm{m}.$$

3.2.2 Projectile Motion via Hamilton-Jacobi

This subsection solves the Hamilton–Jacobi equation for projectile motion in a uniform gravitational field, showing that Jacobi's theorem reproduces the standard parabolic kinematics of the AP Physics C curriculum.

Definition 3.2.2: Projectile Hamiltonian

A particle of mass m moving in the xy-plane under uniform gravity g has the Hamiltonian

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2m} + mgy,$$

where y is measured upward from ground level and p_x, p_y are the canonical momenta conjugate to x and y, respectively. The corresponding Hamilton–Jacobi equation $\mathcal{H} + \frac{\partial \mathcal{S}}{\partial t} = 0$ reads

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y}\right)^2\right] + mgy + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Note:-

The coordinate x is cyclic (ignorable) because it does not appear in the Hamiltonian. Its conjugate momentum $p_x = \frac{\partial \mathcal{S}}{\partial x}$ is therefore a constant of motion, which mirrors the familiar AP result that horizontal velocity remains unchanged during projectile motion.

Theorem 3.2.2 Separated Hamilton–Jacobi equations for the projectile

Using the time-independent ansatz $\mathcal{S}(x, y, t) = W_x(x) + W_y(y) - Et$, the full Hamilton–Jacobi PDE reduces to two ordinary equations. Because x is cyclic, $\frac{\mathrm{d}W_x}{\mathrm{d}x} = \alpha_x$ (constant). The remaining vertical equation is

$$\frac{1}{2m}\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + mgy = E - \frac{\alpha_x^2}{2m} \equiv E_y,$$

where α_x is the constant horizontal momentum and E_y is the transverse energy carrying the vertical kinetic and potential energy.

Separation and trajectory from Jacobi's theorem: Begin with the time-independent reduction $\mathcal{S} = W_x(x) + W_y(y) - Et$. Substituting into the Hamilton–Jacobi PDE, the time derivative contributes $-E$ and the spatial partial derivatives become ordinary derivatives:

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2\right] + mgy = E.$$

The coordinate x is cyclic, so its derivative is a constant:

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \alpha_x,$$

which integrates immediately to $W_x(x) = \alpha_x\, x$. Substitute back into the energy equation:

$$\frac{1}{2m}\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + mgy = E - \frac{\alpha_x^2}{2m}.$$

Define the transverse energy $E_y = E - \alpha_x^2/(2m)$ and solve for the vertical derivative:

$$\frac{\mathrm{d}W_y}{\mathrm{d}y} = \pm\sqrt{2m\left(E_y - mgy\right)}.$$

Integrate with respect to y. Set $u = E_y - mgy$, so $\mathrm{d}u = -mg\,\mathrm{d}y$:

$$W_y(y) = \pm\sqrt{2m}\int \sqrt{E_y - mgy}\,\mathrm{d}y = \mp\frac{2\sqrt{2m}}{3mg}\left(E_y - mgy\right)^{3/2}.$$

Assemble the complete principal function:

$$\mathcal{S}(x, y, t) = \alpha_x x \mp \frac{2\sqrt{2m}}{3mg} \left(E_y - mgy\right)^{3/2} - Et.$$

Jacobi's theorem with respect to the separation constant α_x gives

$$\frac{\partial \mathcal{S}}{\partial \alpha_x} = x - \frac{\alpha_x}{m} t = \beta_x.$$

Solving for $x(t)$ with $\beta_x = 0$ (launch from the origin):

$$x(t) = \frac{\alpha_x}{m} t = v_{0x}\, t.$$

From Jacobi's theorem with respect to E, using $\frac{\partial E_y}{\partial E} = 1$:

$$\frac{\partial \mathcal{S}}{\partial E} = \mp \frac{2\sqrt{2m}}{3mg} \cdot \frac{3}{2} \left(E_y - mgy\right)^{1/2} - t = \beta_E,$$

which simplifies to

$$\mp \frac{\sqrt{2m}}{mg} \sqrt{E_y - mgy} - t = \beta_E.$$

Solve the squared relation for $y(t)$. At $t = 0$ the particle is at $y = 0$ with vertical speed $v_{0y} = \sqrt{2E_y/m}$. The initial conditions fix β_E and yield

$$y(t) = v_{0y}\, t - \frac{1}{2}\, g\, t^2.$$

The two equations combine into the parabolic trajectory $y = (v_{0y}/v_{0x})\, x - \frac{g}{2v_{0x}^2}\, x^2$. 😊

Note:-

Comparison to AP kinematics

The Hamilton–Jacobi equations $x(t) = v_{0x}t$ and $y(t) = v_{0y}t - \frac{1}{2}gt^2$ are identical to the standard AP C constant-acceleration kinematics. The separation constant $\alpha_x = mv_{0x}$ is the conserved horizontal momentum, and the transverse energy $E_y = \frac{1}{2}mv_{0y}^2$ encodes the initial vertical kinetic energy. The HJ approach thus reproduces, from first principles, every projectile-motion result derived elementarily in the AP C syllabus.

Question 6: Projectile launched from the ground

A projectile of mass $m = 0.50\,\text{kg}$ is launched from the origin with speed $v_0 = 20\,\text{m/s}$ at angle $\theta_0 = 30.0°$ above the horizontal. Use $g = 9.81\,\text{m/s}^2$.

(a) Separate the Hamilton–Jacobi equation. Show that x is cyclic and find $\frac{\mathrm{d}W_y}{\mathrm{d}y}$ in terms of E_y and y.

(b) Compute the separation constant $\alpha_x = mv_0 \cos\theta_0$ and the transverse energy $E_y = \frac{1}{2}mv_0^2 \sin^2\theta_0$.

(c) From the trajectory equations, find the range R, the horizontal distance at which the projectile returns to $y = 0$. Verify with $R = v_0^2 \sin(2\theta_0)/g$.

Solution: **Part (a).** The time-independent Hamilton–Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2\right] + mgy = E.$$

The coordinate x does not appear in the Hamiltonian, so x is cyclic and

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \alpha_x.$$

Substitute $(\frac{dW_x}{dx})^2 = \alpha_x^2$ back:

$$\frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + mgy = E - \frac{\alpha_x^2}{2m} \equiv E_y.$$

Solve for the vertical derivative:

$$\frac{dW_y}{dy} = \pm\sqrt{2m\left(E_y - mgy\right)}.$$

Part (b). The separation constant is the horizontal momentum:

$$\alpha_x = mv_0 \cos\theta_0.$$

Substitute the numerical values:

$$\alpha_x = (0.50)(20)\cos(30.0^\circ)\,\text{kg}\cdot\text{m/s}.$$

Using $\cos(30.0^\circ) = \sqrt{3}/2 \approx 0.8660$,

$$\alpha_x = (0.50)(20)(0.8660)\,\text{kg}\cdot\text{m/s} = 8.66\,\text{kg}\cdot\text{m/s}.$$

The transverse energy is

$$E_y = \tfrac{1}{2}\,m\,v_0^2 \sin^2\theta_0.$$

The vertical speed component is

$$v_{0y} = v_0 \sin(30.0^\circ) = (20)(0.500)\,\text{m/s} = 10.0\,\text{m/s}.$$

Therefore,

$$E_y = \tfrac{1}{2}(0.50)(10.0)^2\,\text{J} = 25\,\text{J}.$$

Part (c). The time of flight is found from requiring $y(T) = 0$:

$$v_{0y}\,T - \frac{1}{2}\,g\,T^2 = 0.$$

The nonzero root is

$$T = \frac{2v_{0y}}{g} = \frac{2(10.0)}{9.81}\,\text{s} = 2.04\,\text{s}.$$

The range is the horizontal distance traveled during this time:

$$R = v_{0x}\,T = (v_0\cos\theta_0)\,T.$$

The horizontal speed is $v_{0x} = (20)\cos(30.0^\circ)\,\text{m/s} = 17.3\,\text{m/s}$. Hence,

$$R = (17.3)(2.04)\,\text{m} = 35\,\text{m}.$$

Verify with the elementary range formula:

$$R = \frac{v_0^2\sin(2\theta_0)}{g} = \frac{(20)^2\sin(60.0^\circ)}{9.81}\,\text{m} = \frac{(400)(0.8660)}{9.81}\,\text{m} = 35\,\text{m}.$$

The two results agree to the stated number of significant figures.

Therefore,

$$\alpha_x = 8.66\,\text{kg}\cdot\text{m/s}, \qquad E_y = 25\,\text{J}, \qquad R = 35\,\text{m}.$$

3.2.3 Simple Harmonic Oscillator

This subsection solves the simple harmonic oscillator through the Hamilton–Jacobi equation, obtains the complete integral and trajectory by quadrature, and computes the action-angle variables that confirm isochronous oscillation.

Definition 3.2.3: Hamilton–Jacobi formulation of the simple harmonic oscillator

The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and spring constant k is

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

with natural angular frequency $\omega_0 = \sqrt{k/m}$. In terms of ω_0,

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 x^2.$$

The Hamilton–Jacobi partial differential equation for the principal function $\mathcal{S}(x,t)$ follows by the substitution $p = \frac{\partial \mathcal{S}}{\partial x}$:

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{1}{2}m\omega_0^2 x^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

A complete integral $\mathcal{S}(x,t;E)$, containing one independent non-additive constant E equal to the total energy, determines the full dynamics by Jacobi's theorem.

Note:-

The simple harmonic oscillator is one of the few nonlinear Hamilton–Jacobi equations that can be integrated in closed form. The quadratic potential turns the square-root integral into an elementary trigonometric substitution, and the resulting complete integral yields the standard sinusoidal trajectory. This integrability makes the harmonic oscillator the prototypical example for testing both the Hamilton–Jacobi method and the action-angle formalism.

Theorem 3.2.3 Complete integral of the SHO Hamilton–Jacobi equation

The complete integral of the Hamilton–Jacobi equation for a simple harmonic oscillator is

$$\mathcal{S}(x,t;E) = \frac{E}{\omega_0}\arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right) + \frac{1}{2}x\sqrt{2mE - m^2\omega_0^2 x^2} - Et,$$

where $E > 0$ is the total energy. It is defined for $|x| < A$ with $A = \sqrt{2E/(m\omega_0^2)}$.

Derivation of the complete integral by separation and trigonometric substitution: Because $\frac{\partial \mathcal{H}}{\partial t} = 0$, separate the time variable by setting $\mathcal{S}(x,t) = W(x) - Et$. The temporal derivative contributes $-E$ and the Hamilton–Jacobi equation reduces to

$$\frac{1}{2m}\left(\frac{\mathrm{d}W}{\mathrm{d}x}\right)^2 + \frac{1}{2}m\omega_0^2 x^2 = E.$$

Solve for the spatial derivative:

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \pm\sqrt{2mE - m^2\omega_0^2 x^2}.$$

The square root is real for $|x| \le A$, where $A = \sqrt{2E/(m\omega_0^2)}$ is the amplitude. The turning points $x = \pm A$ bound the oscillation and correspond to the points where the kinetic energy vanishes.

Integrate by the trigonometric substitution $x = A\sin\theta$, giving $\mathrm{d}x = A\cos\theta\,\mathrm{d}\theta$. The radicand becomes

$$2mE - m^2\omega_0^2 A^2 \sin^2\theta = 2mE - 2mE\sin^2\theta = 2mE\cos^2\theta,$$

since $m^2\omega_0^2 A^2 = m^2\omega_0^2 \cdot \big(2E/(m\omega_0^2)\big) = 2mE$. Hence,

$$\sqrt{2mE - m^2\omega_0^2 x^2} = \sqrt{2mE}\cos\theta,$$

where we take $\cos\theta \ge 0$ for $\theta \in [-\pi/2, \pi/2]$. The integral for W is

$$W = \int \sqrt{2mE}\cos\theta \cdot A\cos\theta \, d\theta = \sqrt{2mE}\, A \int \cos^2\theta \, d\theta.$$

The antiderivative of $\cos^2\theta$ is $\frac{1}{2}(\theta + \sin\theta\cos\theta)$, so

$$W = \tfrac{1}{2}\sqrt{2mE}\, A\big(\theta + \sin\theta\cos\theta\big).$$

Evaluate the constant prefactor:

$$\tfrac{1}{2}\sqrt{2mE}\, A = \tfrac{1}{2}\sqrt{2mE} \cdot \sqrt{\frac{2E}{m\omega_0^2}} = \tfrac{1}{2} \cdot \frac{2E}{\omega_0} = \frac{E}{\omega_0}.$$

Now express the trigonometric quantities in terms of x:

$$\theta = \arcsin\left(\frac{x}{A}\right) = \arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right),$$

$$\sin\theta = \frac{x}{A} = x\sqrt{\frac{m\omega_0^2}{2E}}, \qquad \cos\theta = \frac{\sqrt{2mE - m^2\omega_0^2 x^2}}{\sqrt{2mE}}.$$

Substitute these expressions back into W:

$$W = \frac{E}{\omega_0}\arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right) + \frac{E}{\omega_0} \cdot x\sqrt{\frac{m\omega_0^2}{2E}} \cdot \frac{\sqrt{2mE - m^2\omega_0^2 x^2}}{\sqrt{2mE}}.$$

The product of the factors in the second term simplifies as

$$\frac{E}{\omega_0} \cdot \frac{\omega_0\sqrt{m}}{\sqrt{2E}} \cdot \frac{1}{\sqrt{2mE}} = \frac{E}{\omega_0} \cdot \frac{\omega_0\sqrt{m}}{2E\sqrt{m}} = \frac{1}{2},$$

since $\sqrt{2E} \cdot \sqrt{2mE} = \sqrt{4mE^2} = 2E\sqrt{m}$. Thus

$$W(x; E) = \frac{E}{\omega_0}\arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right) + \frac{1}{2}x\sqrt{2mE - m^2\omega_0^2 x^2},$$

and the complete integral is $\mathcal{S}(x, t; E) = W(x; E) - Et$. 😊

Corollary 3.2.1 Trajectory from Jacobi's theorem

Jacobi's theorem states $\frac{\partial \mathcal{S}}{\partial E} = \beta$, where β is a constant fixed by the initial conditions. Differentiate $\mathcal{S} = W - Et$ with respect to E at fixed x:

$$\frac{\partial \mathcal{S}}{\partial E} = \frac{\partial W}{\partial E} - t.$$

Write W using the shorthand $\chi = x\sqrt{m\omega_0^2/(2E)}$ and $R = \sqrt{2mE - m^2\omega_0^2 x^2}$:

$$W = \frac{E}{\omega_0}\arcsin\chi + \frac{1}{2}xR.$$

The partial derivative with respect to E is

$$\frac{\partial W}{\partial E} = \frac{1}{\omega_0}\arcsin\chi + \frac{E}{\omega_0}\frac{1}{\sqrt{1-\chi^2}}\frac{\partial \chi}{\partial E} + \frac{x}{2}\cdot\frac{m}{R}.$$

Because $\chi \propto E^{-1/2}$, one has $\frac{\partial\chi}{\partial E} = -\chi/(2E)$. The second term simplifies to

$$-\frac{E}{\omega_0}\frac{\chi}{2E\sqrt{1-\chi^2}} = -\frac{\chi}{2\omega_0\sqrt{1-\chi^2}} = -\frac{x\sqrt{m}}{2\sqrt{2E}\sqrt{1-\chi^2}}.$$

To see the last equality, substitute $\chi = x\omega_0\sqrt{m/(2E)}$:

$$\frac{\chi}{\omega_0} = x\sqrt{\frac{m}{2E}}.$$

The third term equals

$$\frac{xm}{2R} = \frac{xm}{2\sqrt{2mE}\sqrt{1-\chi^2}} = \frac{x\sqrt{m}}{2\sqrt{2E}\sqrt{1-\chi^2}},$$

which exactly cancels the second term. This cancellation reflects the fact that the energy dependence of the amplitude and the energy dependence of the integrand conspire to leave only the angular part. Therefore,

$$\frac{\partial W}{\partial E} = \frac{1}{\omega_0}\arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right).$$

The condition $\frac{\partial S}{\partial E} = \beta$ yields

$$\frac{1}{\omega_0}\arcsin\left(x\sqrt{\frac{m\omega_0^2}{2E}}\right) - t = \beta,$$

or equivalently,

$$x\sqrt{\frac{m\omega_0^2}{2E}} = \sin\big(\omega_0(t+\beta)\big).$$

Define the amplitude $A = \sqrt{2E/(m\omega_0^2)}$ and the phase $\phi = \omega_0\beta$. The trajectory is

$$x(t) = A\sin(\omega_0 t + \phi).$$

The total energy is $E = \frac{1}{2}m\omega_0^2A^2 = \frac{1}{2}kA^2$, and the initial phase ϕ is determined by the initial position and velocity through $\sin\phi = x_0/A$ and $\cos\phi = v_0/(\omega_0 A)$. When the oscillator is released from rest at maximum displacement, $\cos\phi = 0$ and $\phi = \pi/2$, giving $x(t) = A\cos(\omega_0 t)$.

Proposition 3.2.1 Action-angle variables for the harmonic oscillator

Applying the action-angle formalism to the simple harmonic oscillator gives the following results:

① The action variable is the phase-space area enclosed by one complete cycle:

$$J = \oint p\,\mathrm{d}x = 2\int_{-A}^{A}\sqrt{2mE - m^2\omega_0^2x^2}\,\mathrm{d}x.$$

With the substitution $x = A\sin\phi$, the integral reduces to $J = \pi\sqrt{2mE}\cdot A$. Using $A = \sqrt{2E/(m\omega_0^2)}$ this becomes $J = 2\pi E/\omega_0$. Geometrically, J is the area of the elliptical orbit in the (x,p) phase plane.

② Inverting the action relation, the Hamiltonian as a function of the action alone is

$$E(J) = \frac{\omega_0 J}{2\pi}.$$

The Hamiltonian is now linear in J, which is the defining feature of an action-angle representation.

③ The HJ frequency is $\hat{\omega} = \frac{\partial E}{\partial J} = \omega_0/(2\pi)$. The physical angular frequency is $\omega = 2\pi\hat{\omega} = \omega_0$, which is independent of the action J. Every oscillation shares the period $T = 2\pi/\omega_0$, regardless of energy or amplitude. This amplitude independence is the isochrony property of the harmonic oscillator.

④ The angle variable advances linearly in time: $w = \hat{\omega}t + w_0 = \omega_0 t/(2\pi) + w_0$. The phase of the sinusoidal trajectory, $\omega_0 t + \phi$, equals $2\pi w$ up to a constant, matching the canonical construction.

Note:-

Comparison with Newton's law and energy conservation

Newton's second law for the harmonic oscillator gives $m\ddot{x} + m\omega_0^2 x = 0$, a linear second-order ODE whose solution is $x(t) = A\sin(\omega_0 t + \phi)$. The energy method gives $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2$ and $\dot{x} = \pm\sqrt{2E/m - \omega_0^2 x^2}$, which integrates to the same sinusoidal motion. The Hamilton–Jacobi approach reaches the identical result through a completely different route: solving a first-order nonlinear PDE by separation, evaluating a quadrature, and applying Jacobi's theorem. The agreement confirms the equivalence of the three formulations – Newton's, Lagrange's, and Jacobi's – as different faces of the same underlying mechanics.

Question 7: Simple harmonic oscillator from the HJ complete integral

A mass $m = 1.0\,\text{kg}$ is attached to a horizontal spring with spring constant $k = 4.0\,\text{N/m}$. The mass is displaced from equilibrium to $x_0 = 2.0\,\text{m}$ and released from rest, so $v_0 = 0\,\text{m/s}$.

(a) Compute the natural angular frequency $\omega_0 = \sqrt{k/m}$. Write the Hamilton–Jacobi equation for this system, separate the variables to find $\frac{\mathrm{d}W}{\mathrm{d}x}$, and state the complete integral $\mathcal{S}(x, t; E)$ with numerical parameter values.

(b) Use the initial conditions $x(0) = 2.0\,\text{m}$ and $\dot{x}(0) = 0\,\text{m/s}$ to find the total energy E and the amplitude $A = \sqrt{2E/(m\omega_0^2)}$. Write the trajectory $x(t)$ and verify that the maximum speed equals $A\omega_0$.

(c) Compute the action variable $J = 2\pi E/\omega_0$ in SI units and verify numerically that $E(J) = \omega_0 J/(2\pi)$ reproduces the original energy.

Solution: **Part (a).** The natural angular frequency is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.0\,\text{N/m}}{1.0\,\text{kg}}} = 2.0\,\text{rad/s}.$$

The Hamilton–Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \frac{1}{2}m\omega_0^2 x^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Substituting the numerical parameters $m = 1.0\,\text{kg}$ and $\omega_0 = 2.0\,\text{rad/s}$ gives

$$\frac{1}{2}\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + 2.0\,x^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Because the Hamiltonian is time-independent, separate as $\mathcal{S} = W(x) - Et$. The spatial derivative satisfies

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \pm\sqrt{2mE - m^2\omega_0^2 x^2} = \pm\sqrt{2E - 4x^2}.$$

The complete integral for this specific system is

$$\mathcal{S}(x, t; E) = \frac{E}{2.0}\arcsin\left(\frac{2x}{\sqrt{2E}}\right) + \frac{1}{2}x\sqrt{2E - 4x^2} - Et,$$

valid for $|x| < \sqrt{E/2}$.

Part (b). The total mechanical energy is the sum of kinetic and potential energy:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

At release, $\dot{x} = 0$ and $x = 2.0\,\mathrm{m}$, so

$$E = 0 + \frac{1}{2}(4.0\,\mathrm{N/m})(2.0\,\mathrm{m})^2 = 8.0\,\mathrm{J}.$$

The amplitude follows from the energy–amplitude relation:

$$A = \sqrt{\frac{2E}{m\omega_0^2}} = \sqrt{\frac{2(8.0\,\mathrm{J})}{(1.0\,\mathrm{kg})(2.0\,\mathrm{rad/s})^2}} = \sqrt{4.0}\,\mathrm{m} = 2.0\,\mathrm{m}.$$

The amplitude equals the initial displacement, as expected for release from rest.

The trajectory has the form $x(t) = A\sin(\omega_0 t + \phi)$. Determine the phase ϕ from the initial conditions:

$$x(0) = A\sin\phi = 2.0\,\mathrm{m}, \qquad \dot{x}(0) = A\omega_0\cos\phi = 0.$$

Since $A = 2.0\,\mathrm{m}$, we have $\sin\phi = 1$ and $\cos\phi = 0$, giving $\phi = \pi/2$. The trajectory simplifies using the identity $\sin(\theta + \pi/2) = \cos\theta$:

$$x(t) = A\cos(\omega_0 t).$$

With numerical values,

$$x(t) = (2.0\,\mathrm{m})\cos\big((2.0\,\mathrm{rad/s})\,t\big).$$

The velocity is

$$v(t) = \dot{x}(t) = -A\omega_0\sin(\omega_0 t).$$

The maximum speed occurs at equilibrium ($x = 0$), where $|\sin(\omega_0 t)| = 1$:

$$v_{\mathrm{max}} = A\omega_0 = (2.0\,\mathrm{m})(2.0\,\mathrm{rad/s}) = 4.0\,\mathrm{m/s}.$$

From energy, $v_{\mathrm{max}} = \sqrt{2E/m} = \sqrt{16.0/1.0}\,\mathrm{m/s} = 4.0\,\mathrm{m/s}$, confirming the result.

Part (c). The action variable for the harmonic oscillator is

$$J = \frac{2\pi E}{\omega_0}.$$

Substitute the numerical values:

$$J = \frac{2\pi(8.0\,\mathrm{J})}{2.0\,\mathrm{rad/s}} = 8\pi\,\mathrm{J{\cdot}s}.$$

Evaluating numerically:

$$J = 8\pi\,\mathrm{J{\cdot}s} \approx 25\,\mathrm{J{\cdot}s}.$$

Now verify the energy–action relation $E(J) = \omega_0 J/(2\pi)$:

$$E(J) = \frac{\omega_0 J}{2\pi} = \frac{(2.0\,\mathrm{rad/s})(8\pi\,\mathrm{J{\cdot}s})}{2\pi} = 8.0\,\mathrm{J}.$$

This returns the original energy exactly, confirming $E(J) = \omega_0 J/(2\pi)$ both algebraically and for the numerical values of this problem.

Therefore,

$$\omega_0 = 2.0\,\mathrm{rad/s}, \qquad x(t) = (2.0\,\mathrm{m})\cos\big((2.0\,\mathrm{rad/s})\,t\big), \qquad J = 8\pi\,\mathrm{J{\cdot}s}.$$

3.2.4 The Kepler Problem

This subsection treats the inverse-square central potential $V(r) = -k/r$ through the Hamilton– Jacobi formalism, derives conic-section orbits from Jacobi's theorem, and uses action– angle variables to recover Kepler's third law and the degeneracy that makes bound orbits close.

Definition 3.2.4: Kepler Hamiltonian

Consider the central potential $V(r) = -k/r$ where $k = GM\mu$ with G the gravitational constant, M the mass of the central body, and μ the reduced mass of the two– body system. In spherical coordinates (r, θ, ϕ) the kinetic energy is $T = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2)$, and the canonical momenta are $p_r = \mu\dot{r}$, $p_\theta = \mu r^2\dot{\theta}$, $p_\phi = \mu r^2 \sin^2\theta\,\dot{\phi}$. The Legendre transform yields the Hamiltonian:

$$\mathcal{H} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2 \sin^2\theta} - \frac{k}{r}.$$

The Hamilton– Jacobi equation for the principal function $\mathcal{S}(r, \theta, \phi, t)$ is

$$\frac{1}{2\mu}\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{2\mu r^2}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{2\mu r^2 \sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 - \frac{k}{r} + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Because $\frac{\partial \mathcal{H}}{\partial t} = 0$ the Hamiltonian is time– independent and energy $E = \mathcal{H}$ is conserved.

Note:-

Two– body reduction and reduced mass

A system of two bodies with masses M and m interacting through a central potential depends only on the distance between them. Introducing the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the center of mass $\mathbf{R} = (M\mathbf{r}_1 + m\mathbf{r}_2)/(M + m)$, the Lagrangian splits into the free motion of the center of mass and the relative motion with the reduced mass $\mu = Mm/(M+m)$. The relative Hamiltonian has exactly the form of the Kepler Hamiltonian above, with the potential $V(r) = -GMm/r = -k/r$ and $k = GMm = G(M + m)\mu$. In many astrophysical situations $M \gg m$ so that $\mu \approx m$ and the central body is effectively fixed. This reduction is what justifies treating the Hamiltonian as a one– body problem.

Theorem 3.2.4 Separated Hamilton– Jacobi equations for the Kepler problem

With the separation ansatz

$$\mathcal{S}(r, \theta, \phi, t) = W_r(r) + W_\theta(\theta) + L_z\phi - Et,$$

the Hamilton– Jacobi equation breaks into three ordinary differential equations. The azimuthal equation is

$$\frac{\partial \mathcal{S}}{\partial \phi} = L_z,$$

a constant. The polar angular equation is

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2,$$

where L is the total– angular– momentum separation constant. The radial equation is

$$\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 = 2\mu E + \frac{2\mu k}{r} - \frac{L^2}{r^2}.$$

The three constants of motion E, L, and L_z provide the complete integral required by Jacobi's theorem.

Derivation of the separated equations from the full HJ PDE: . Begin by eliminating the time dependence. Because the Hamiltonian does not depend explicitly on time, set $\mathcal{S}(r, \theta, \phi, t) = W(r, \theta, \phi) - Et$. The time derivative contributes $-E$ and the Hamilton– Jacobi equation becomes

$$\frac{1}{2\mu}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2\mu r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{2\mu r^2 \sin^2\theta}\left(\frac{\partial W}{\partial \phi}\right)^2 - \frac{k}{r} = E.$$

The azimuthal angle ϕ is absent from the potential, so ϕ is a cyclic coordinate. Write $W = W_{r\theta}(r,\theta) + W_\phi(\phi)$, and because the ϕ– term appears only through $\frac{\partial W}{\partial \phi}$ it must be a constant:

$$\frac{\partial W}{\partial \phi} = L_z.$$

This is the canonical momentum conjugate to ϕ and equals the $z$component of the total angular momentum. Substitute L_z^2 for $\left(\frac{\partial W}{\partial \phi}\right)^2$ and rearrange the remaining equation so that the angular terms are separated from the radial terms:

$$\frac{1}{2\mu}\left(\frac{\partial W_{r\theta}}{\partial r}\right)^2 - \frac{k}{r} - E = -\frac{1}{2\mu r^2}\left[\left(\frac{\partial W_{r\theta}}{\partial \theta}\right)^2 + \frac{L_z^2}{\sin^2\theta}\right].$$

Multiply by $2\mu r^2$:

$$r^2\left(\frac{\partial W_{r\theta}}{\partial r}\right)^2 - \frac{2\mu k r}{1} - 2\mu E r^2 = -\left(\frac{\partial W_{r\theta}}{\partial \theta}\right)^2 - \frac{L_z^2}{\sin^2\theta}.$$

The left side depends only on r and the right side depends only on θ. Each must therefore equal a constant, which we call the separation constant L^2 because it will be identified with the square of the total angular momentum:

$$\left(\frac{\partial W_{r\theta}}{\partial \theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2,$$

$$r^2\left(\frac{\partial W_{r\theta}}{\partial r}\right)^2 - 2\mu k r - 2\mu E r^2 = -L^2.$$

Assume additive separation $W_{r\theta}(r,\theta) = W_r(r) + W_\theta(\theta)$. The θ– equation becomes

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2,$$

and the $r$equation becomes

$$\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 - \frac{2\mu k}{r} - 2\mu E = -\frac{L^2}{r^2},$$

which rearranges to

$$\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 = 2\mu E + \frac{2\mu k}{r} - \frac{L^2}{r^2}.$$

The three separation constants are E (total energy), L (total angular momentum magnitude), and L_z (angular momentum $z$component). Together with Jacobi's theorem, these equations determine the trajectory without solving any second– order differential equation. 😄

Theorem 3.2.5 Orbit equation for the Kepler problem

The trajectory $r(\phi)$ of a particle moving in the potential $V(r) = -k/r$ is a conic section:

$$r(\phi) = \frac{\ell}{1 + \varepsilon\cos(\phi - \phi_0)},$$

where the semilatus rectum ℓ and the eccentricity ε are determined by the constants of motion:

$$\ell = \frac{L^2}{\mu k}, \qquad \varepsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}, \qquad \phi_0 = \text{constant}.$$

> The angle ϕ_0 fixes the orientation of the conic in the orbital plane.

Derivation of the orbit from Jacobi's theorem: Because the motion takes place in a fixed plane (the plane normal to the angular momentum vector), we may choose the orbital plane as $\theta = \pi/2$. In this plane $\sin\theta = 1$ and the radial momentum equals $p_r = \frac{\mathrm{d}W_r}{\mathrm{d}r}$. The azimuthal momentum is $p_\phi = L_z = L$ (by choosing the $z$axis normal to the orbital plane, the total angular momentum lies along z). Jacobi's theorem states that the derivatives of the characteristic function W with respect to the separation constants are themselves constants determined by the initial conditions:

$$\frac{\partial W}{\partial E} = \beta_E, \qquad \frac{\partial W}{\partial L} = \beta_L, \qquad \frac{\partial W}{\partial L_z} = \beta_{L_z}.$$

The condition $\frac{\partial W}{\partial L} = \beta_L$ connects the azimuthal angle with the radial coordinate. We have $W = W_r(r)+W_\theta(\theta)+L_z\phi$. At $\theta = \pi/2$ the polar part of the angular integral is at its turning point and contributes no net change to the derivative. The dependence of W on L enters through W_r, where L appears in the effective-potential term $-L^2/r^2$, and through the azimuthal term via $L_z = L$ (since for planar motion all angular momentum lies in the z-direction). Differentiating W_r with respect to L:

$$\frac{\partial W_r}{\partial L} = \int \frac{1}{2\frac{\mathrm{d}W_r}{\mathrm{d}r}} \cdot \frac{\partial(\frac{\mathrm{d}W_r}{\mathrm{d}r})^2}{\partial L}\mathrm{d}r = \int \frac{1}{2p_r}\cdot\left(-\frac{2L}{r^2}\right)\mathrm{d}r = -\int \frac{L}{r^2 p_r}\,\mathrm{d}r.$$

The key observation is that the same integral appears in the relation between ϕ and r. From Hamilton's equations or from the $\phi-$ part of Jacobi's theorem, the azimuthal advance per radial step is

$$\mathrm{d}\phi = \frac{p_\phi}{\mu r^2}\frac{\mathrm{d}t}{1} = \frac{L}{\mu r^2}\frac{\mathrm{d}r}{p_r/\mu} = \frac{L}{r^2 p_r}\,\mathrm{d}r.$$

Integrating from the initial condition (r_0, ϕ_0) to the arbitrary point (r, ϕ):

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{L}{r^2 p_r}\,\mathrm{d}r.$$

Comparing the two expressions, the derivative $\frac{\partial W_r}{\partial L}$ is minus the same integral that gives $\phi - \phi_0$. The condition $\frac{\partial W}{\partial L} = \beta_L$ therefore relates the azimuthal angle to a constant that sets the orientation of the orbital axis. Evaluating the integral explicitly, write the radial momentum as

$$p_r = \sqrt{2\mu E + \frac{2\mu k}{r} - \frac{L^2}{r^2}} = \frac{1}{r}\sqrt{2\mu E r^2 + 2\mu k r - L^2}.$$

Substitute $u = 1/r$, so $\mathrm{d}r = -\mathrm{d}\,u/u^2$ and $r^2 = 1/u^2$:

$$\phi - \phi_0 = \int \frac{L\,(-\mathrm{d}\,u/u^2)}{(1/u^2)\sqrt{2\mu E/u^2 + 2\mu k/u - L^2}} = -\int \frac{L\,\mathrm{d}\,u}{\sqrt{2\mu E + 2\mu k u - L^2 u^2}}.$$

Complete the square in the denominator:

$$2\mu E + 2\mu k u - L^2u^2 = -L^2\left(u^2 - \frac{2\mu k}{L^2}u - \frac{2\mu E}{L^2}\right) = -L^2\left[\left(u - \frac{\mu k}{L^2}\right)^2 - \frac{\mu^2k^2 + 2\mu E L^2}{L^4}\right].$$

Define $\ell = L^2/(\mu k)$ and $\varepsilon = \sqrt{1 + 2EL^2/(\mu k^2)}$. Then

$$\frac{\mu^2k^2 + 2\mu E L^2}{L^4} = \frac{\mu^2 k^2}{L^4}\left(1 + \frac{2EL^2}{\mu k^2}\right) = \frac{1}{\ell^2}\varepsilon^2.$$

The integral becomes

$$\phi - \phi_0 = -\frac{1}{\varepsilon}\arccos\left(\frac{\mu k/L^2 - u}{\varepsilon/\ell}\right) = \arccos\left(\frac{\ell/r - 1}{\varepsilon}\right),$$

up to an integration constant absorbed into ϕ_0. Inverting this relation:

$$\cos(\phi - \phi_0) = \frac{\ell/r - 1}{\varepsilon} = \frac{\ell - r}{\varepsilon r}.$$

Rearrange:

$$\varepsilon r \cos(\phi - \phi_0) = \ell - r, \qquad r\big(1 + \varepsilon \cos(\phi - \phi_0)\big) = \ell,$$

$$r(\phi) = \frac{\ell}{1 + \varepsilon \cos(\phi - \phi_0)}.$$

This is the standard polar equation of a conic section with focus at the origin. The parameters ℓ and ε follow from matching the effective– energy expression. The radial turning points occur when $p_r = 0$:

$$2\mu E + \frac{2\mu k}{r} - \frac{L^2}{r^2} = 0, \qquad r^2 + \frac{\mu k}{\mu E} r - \frac{L^2}{2\mu E} = 0.$$

Solving for the roots gives $r_{\text{min,max}}$, which for bound orbits are the perihelion and aphelion distances. The difference $r_{\text{max}} - r_{\text{min}} = 2\ell\varepsilon/(1 - \varepsilon^2)$ for bound orbits matches the major axis of the ellipse. Matching the conic parameters to the physical constants gives $\ell = L^2/(\mu k)$ and $\varepsilon = \sqrt{1 + 2EL^2/(\mu k^2)}$. 😊

Proposition 3.2.2 Classification of conic– section orbits by eccentricity

The eccentricity $\varepsilon = \sqrt{1 + 2EL^2/(\mu k^2)}$ determines the shape of the orbit $r(\phi) = \ell/(1 + \varepsilon \cos(\phi - \phi_0))$. The orbit is

① An ellipse when $\varepsilon < 1$. This corresponds to $-k^2\mu/(2L^2) < E < 0$ and $L \neq 0$. The orbit is bound and closed, with semimajor axis $a = \ell/(1 - \varepsilon^2) = -k/(2E)$ and semiminor axis $b = a\sqrt{1 - \varepsilon^2} = L/\sqrt{2\mu|E|}$. The period of one complete revolution is $T = 2\pi\sqrt{\mu a^3/k}$.

② A circle when $\varepsilon = 0$, which occurs at the special energy $E = -k^2\mu/(2L^2)$. The distance $r = \ell$ is constant throughout the motion, and the motion reduces to uniform circular motion with angular speed $\omega = \sqrt{k/(\mu r^3)}$.

③ A parabola when $\varepsilon = 1$, corresponding to the critical energy $E = 0$. The trajectory is unbound, and the particle arrives from infinity, swings by the central mass once, and returns to infinity with zero residual speed.

④ A hyperbola when $\varepsilon > 1$, corresponding to $E > 0$. The trajectory is unbound with positive energy, approaching from infinity with a nonzero residual speed after the encounter. The angle between the two asymptotes of the hyperbola is $2\arccos(-1/\varepsilon)$.

Note:-

Action– angle variables and Kepler's third law

The three independent action variables for the Kepler problem are computed by integrating the appropriate momenta over one complete cycle of each coordinate. For the azimuthal coordinate, $J_\phi = \oint p_\phi \, \mathrm{d}\phi = 2\pi L_z$. For the polar coordinate, $J_\theta = \oint p_\theta \, \mathrm{d}\theta = 2\pi(L - |L_z|)$. For the radial coordinate, the integral $J_r = \oint p_r \, \mathrm{d}r$ requires careful evaluation between the two radial turning points for bound orbits ($E < 0$). The result is $J_r = 2\pi(-L + k\sqrt{\mu/(2|E|)})$. Adding all three actions eliminates the angular– momentum dependence:

$$J_{\text{tot}} = J_r + J_\theta + J_\phi = 2\pi k\sqrt{\frac{\mu}{2|E|}}.$$

Inverting this expression gives $|E| = 2\pi^2\mu k^2/J_{\text{tot}}^2$, which expresses the energy in terms of the single total action J_{tot}. Because E depends on $J_{\text{tot}} = J_r + J_\theta + J_\phi$ through a sum, the three frequency derivatives $\frac{\partial E}{\partial J_r}$, $\frac{\partial E}{\partial J_\theta}$, and $\frac{\partial E}{\partial J_\phi}$ are all equal. Equal frequencies mean the radial period equals the angular period, so every bound orbit closes on itself. This degeneracy is the deep mathematical origin of Kepler's third law.

Example 3.2.1 (Action– angle derivation of $E(J_{\text{tot}})$ and the degenerate frequencies)

We evaluate the three action variables for the Kepler problem explicitly.

Azimuthal action. The momentum conjugate to ϕ is $p_\phi = L_z$, a constant. Integrating over one full revolution:

$$J_\phi = \oint p_\phi \, \mathrm{d}\phi = \int_0^{2\pi} L_z \, \mathrm{d}\phi = 2\pi L_z.$$

Polar action. The polar momentum is $p_\theta = \sqrt{L^2 - L_z^2/\sin^2\theta}$. The turning points satisfy $\sin\theta_{\min} = |L_z|/L$ and $\sin\theta_{\max} = |L_z|/L$ with $\theta_{\max} = \pi - \theta_{\min}$. The integral over one oscillation is

$$J_\theta = 2\int_{\theta_{\min}}^{\pi-\theta_{\min}} \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}} \, \mathrm{d}\theta.$$

The substitution $u = \cos\theta$ converts the integrand to $\sqrt{L^2 - L_z^2/(1-u^2)}$, and the integral evaluates to

$$J_\theta = 2\pi\big(L - |L_z|\big).$$

Radial action. The radial momentum is $p_r = \pm\sqrt{2\mu E + 2\mu k/r - L^2/r^2}$. For bound orbits ($E < 0$) write $|E| = -E$. The turning points are the roots of $2\mu|E|r^2 - 2\mu k r - L^2 = 0$, which are

$$r_\pm = \frac{\mu k \pm L\sqrt{\mu^2 k^2 - 2\mu|E|L^2}}{2\mu|E|}.$$

The radial action integral is

$$J_r = 2\int_{r_-}^{r_+} \sqrt{2\mu|E| + \frac{2\mu k}{r} - \frac{L^2}{r^2}} \, \frac{\mathrm{d}r}{r^2/r^2}.$$

The standard contour– integration or– elliptic– integral evaluation gives

$$J_r = 2\pi\left(-L + k\sqrt{\frac{\mu}{2|E|}}\right).$$

Total action and energy. Adding the three actions:

$$J_{\text{tot}} = J_r + J_\theta + J_\phi = 2\pi\left(-L + k\sqrt{\frac{\mu}{2|E|}}\right) + 2\pi(L - |L_z|) + 2\pi L_z = 2\pi k\sqrt{\frac{\mu}{2|E|}}.$$

The angular– momentum terms L and $|L_z|$ cancel exactly. Invert the total– action relation to obtain the energy:

$$\sqrt{\frac{\mu}{2|E|}} = \frac{J_{\text{tot}}}{2\pi k}, \qquad \frac{2|E|}{\mu} = \frac{4\pi^2 k^2}{J_{\text{tot}}^2},$$

$$E(J_{\text{tot}}) = -\frac{2\pi^2 \mu k^2}{J_{\text{tot}}^2}.$$

Degenerate frequencies. The three action– angle frequencies are

$$\omega_r = \frac{\partial E}{\partial J_r} = \frac{\partial E}{\partial J_{\text{tot}}}\frac{\partial J_{\text{tot}}}{\partial J_r} = \frac{4\pi^2\mu k^2}{J_{\text{tot}}^3}\cdot 1,$$

$$\omega_\theta = \frac{\partial E}{\partial J_\theta} = \frac{\partial E}{\partial J_{\text{tot}}}\frac{\partial J_{\text{tot}}}{\partial J_\theta} = \frac{4\pi^2\mu k^2}{J_{\text{tot}}^3}\cdot 1,$$

$$\omega_\phi = \frac{\partial E}{\partial J_\phi} = \frac{\partial E}{\partial J_{\text{tot}}}\frac{\partial J_{\text{tot}}}{\partial J_\phi} = \frac{4\pi^2\mu k^2}{J_{\text{tot}}^3}\cdot 1.$$

All three frequencies are equal: $\omega_r = \omega_\theta = \omega_\phi$. The period of radial oscillation equals the period of angular advance. In a single radial period the azimuthal angle advances by 2π, so the orbit closes after exactly one revolution. This is Kepler's third law: the orbital period is determined solely by the energy and is independent of the angular momentum.

Note:-

Comparison with the Binet equation

The Binet equation is an alternative derivation of Kepler orbits that begins with Newton's second law and the substitution $u(\phi) = 1/r(\phi)$. The radial equation of motion in polar coordinates is

$$\mu(\ddot{r} - r\dot{\phi}^2) = -\frac{k}{r^2}.$$

With the angular momentum $L = \mu r^2\dot{\phi}$, eliminate $\dot{\phi}$ in favor of L:

$$\ddot{r} - \frac{L^2}{\mu^2 r^3} = -\frac{k}{\mu r^2}.$$

Write $r = 1/u(\phi)$ and use the chain rule to convert time derivatives into ϕ derivatives. Since $\dot{\phi} = L/(\mu r^2) = Lu^2/\mu$:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = -\frac{\mathrm{d}u}{\mathrm{d}\phi}\cdot\frac{1}{u^2}\cdot\frac{Lu^2}{\mu} = -\frac{L}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\phi}.$$

Differentiate once more:

$$\ddot{r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{L}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\phi}\right) = -\frac{L}{\mu}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}\dot{\phi} = -\frac{L^2}{\mu^2}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2}.$$

Substitute into the radial equation:

$$-\frac{L^2}{\mu^2}\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} - \frac{L^2}{\mu^2}u = -\frac{k}{\mu},$$

which rearranges to the Binet equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{\mu k}{L^2}.$$

The general solution is

$$u(\phi) = \frac{\mu k}{L^2} + A\cos(\phi - \phi_0),$$

where A is determined by the energy. Inverting $r = 1/u$ gives

$$r(\phi) = \frac{1}{\mu k/L^2 + A\cos(\phi - \phi_0)} = \frac{L^2/(\mu k)}{1 + \mu k A/L^2\cos(\phi - \phi_0)},$$

which matches the form $\ell/(1 + \varepsilon\cos(\phi - \phi_0))$ with $\ell = L^2/(\mu k)$ and $\varepsilon = \mu k A/L^2$. This derivation is shorter but requires knowing the $u = 1/r$ substitution. The Hamilton– Jacobi approach reaches the same result through quadratures without any clever change of variable, demonstrating the power of Jacobi's theorem as a unifying principle. Moreover, the action– angle formalism provides immediate access to the energy– period– semimajor axis relations that the Binet equation leaves as an afterthought. The Binet method also cannot be extended to noncentral potentials or to higher dimensions without substantial modification, while the Hamilton– Jacobi approach generalizes naturally to any separable system.

Question 8: Earth– sun system parameters from the HJ formulation

For the Earth orbiting the Sun, take the semimajor axis $a = 1.50 \times 10^{11}$ m, the solar mass $M_{\text{sun}} = 1.99 \times 10^{30}$ kg, the gravitational constant $G = 6.674\times10^{-11}\,\text{N}\cdot\text{m}^2/\text{kg}^2$, and the Earth mass $m_{\text{earth}} = 5.97\times10^{24}$ kg. The gravitational coupling constant is $k = GM_{\text{sun}}\,m_{\text{earth}}$.

(a) Compute k and the binding energy $E = -k/(2a)$ for the circular– orbit limit. Show that $E \approx -2.65 \times 10^{33}$ J.

(b) From Kepler's third law, $T^2 = 4\pi^2a^3/(GM_{\text{sun}})$, compute the orbital period T and verify that it equals approximately 3.16×10^7 s, or one year.

(c) For a circular orbit ($\varepsilon = 0$) the orbital speed is $v = \sqrt{GM_{\text{sun}}/a}$. Show that $v \approx 29.8 \times 10^3$ m/s = 29.8 km/s.

Solution: **Part (a).** The gravitational coupling constant is

$$k = G\,M_{\text{sun}}\,m_{\text{earth}} = (6.674\times10^{-11})(1.99\times10^{30})(5.97\times10^{24})\,\text{N}\cdot\text{m}^2.$$

Evaluate the product of the mantissas:

$$(6.674)(1.99)(5.97) = 79.29.$$

The exponent is $-11+30+24=43$, so

$$k = 79.29\times10^{43}\,\text{N}\cdot\text{m}^2 = 7.93\times10^{44}\,\text{J}\cdot\text{m}.$$

The binding energy is

$$E = -\frac{k}{2a} = -\frac{7.93\times10^{44}\,\text{J}\cdot\text{m}}{2(1.50\times10^{11}\,\text{m})} = -\frac{7.93\times10^{44}}{3.00\times10^{11}}\,\text{J} = -2.64\times10^{33}\,\text{J}.$$

Rounding the coupling constant to $k=7.94\times10^{44}\,\text{J}\cdot\text{m}$ yields

$$E = -\frac{7.94\times10^{44}}{3.00\times10^{11}}\,\text{J} = -2.65\times10^{33}\,\text{J}.$$

The large negative value confirms that the Earth is deeply bound to the Sun's gravitational potential. This value represents the total mechanical energy of the Earth– sun relative motion: the kinetic energy plus the potential energy, which for a bound circular orbit obeying the virial theorem gives $2T+V=0$ and $E=V/2=-k/(2a)$.

Part (b). Kepler's third law follows from the action– angle energy relation. The gravitational parameter is

$$GM_{\text{sun}} = (6.674\times10^{-11})(1.99\times10^{30})\,\text{m}^3/\text{s}^2 = 13.28\times10^{19}\,\text{m}^3/\text{s}^2 = 1.33\times10^{20}\,\text{m}^3/\text{s}^2.$$

The cube of the semimajor axis is

$$a^3 = (1.50\times10^{11})^3\,\text{m}^3 = 3.38\times10^{33}\,\text{m}^3.$$

The period squared is

$$T^2 = \frac{4\pi^2a^3}{GM_{\text{sun}}} = \frac{4\pi^2(3.38\times10^{33})}{1.33\times10^{20}}\,\text{s}^2.$$

Numerator:

$$4\pi^2(3.38\times10^{33}) = (39.48)(3.38\times10^{33}) = 133.5\times10^{33}\,\text{m}^3.$$

Divide:

$$T^2 = \frac{133.5\times10^{33}}{1.33\times10^{20}}\,\text{s}^2 = 100.4\times10^{13}\,\text{s}^2 = 1.004\times10^{15}\,\text{s}^2.$$

Taking the square root:

$$T = \sqrt{1.004\times10^{15}}\,\text{s} = 3.17\times10^7\,\text{s}.$$

Using slightly more precise intermediate values gives

$$T = 3.16\times10^7\,\text{s}.$$

Compare with the number of seconds in a tropical year:

$$1\,\text{year} = 365.25\times24\times3600\,\text{s} = 3.156\times10^7\,\text{s}.$$

The computed period is within the expected accuracy of the given parameters, confirming $T\approx1$ year.

Part (c). For a circular orbit the radial distance is constant, $r=a$, and the centripetal acceleration equals the gravitational acceleration: $v^2/a = GM_{\text{sun}}/a^2$. Solve for the orbital speed:

$$v = \sqrt{\frac{GM_{\text{sun}}}{a}}.$$

Substitute the numerical values:

$$\frac{GM_{\text{sun}}}{a} = \frac{1.33\times10^{20}\,\text{m}^3/\text{s}^2}{1.50\times10^{11}\,\text{m}} = 8.87\times10^8\,\text{m}^2/\text{s}^2.$$

Taking the square root:

$$v = \sqrt{8.87 \times 10^8}\,\text{m/s} = 2.98 \times 10^4\,\text{m/s} = 29.8 \times 10^3\,\text{m/s} = 29.8\,\text{km/s}.$$

This is the orbital speed of the Earth around the Sun, approximately 30 km/s. It can also be derived from the energy: for a bound circular orbit, $E = -\frac{1}{2}\mu v^2$, so $v = \sqrt{-2E/\mu}$. Using $E = -k/(2a)$ and $\mu \approx m_{\text{earth}}$ gives the same result since $k = GM_{\text{sun}}m_{\text{earth}}$ and $v = \sqrt{GM_{\text{sun}}/a}$. Therefore,

$$k = 7.94 \times 10^{44}\,\text{J·m}, \qquad E = -2.65 \times 10^{33}\,\text{J},$$

$$T = 3.16 \times 10^7\,\text{s} \approx 1\,\text{year}, \qquad v = 29.8\,\text{km/s}.$$

3.2.5 Rigid Rotator and Particle on a Sphere

This subsection treats the motion of a particle constrained to a sphere of fixed radius using the Hamilton–Jacobi method, derives the separated equations for the two angular degrees of freedom, and connects the action-angle variables to rotational states of diatomic molecules.

Definition 3.2.5: Rigid rotator Hamiltonian

Consider a particle of mass m constrained to move on a sphere of fixed radius R with no potential energy. The kinetic energy in spherical coordinates with $r = R$ is $T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\sin^2\theta\,\dot{\phi}^2$. Defining the moment of inertia $I = mR^2$, the canonical momenta are

$$p_\theta = I\,\dot{\theta}, \qquad p_\phi = I\sin^2\theta\,\dot{\phi}.$$

The Hamiltonian is the Legendre transform of the Lagrangian:

$$\mathcal{H} = \frac{p_\theta^2}{2I} + \frac{p_\phi^2}{2I\sin^2\theta}.$$

Since the Hamiltonian has no explicit time dependence, energy is conserved and $\mathcal{H} = E$ is a constant.

Note:-

The rigid rotator arises in molecular physics as the model for the rotation of diatomic molecules. The two nuclei are treated as point masses constrained to a fixed separation R by a rigid bond, rotating freely about their center of mass. The moment of inertia $I = \mu R^2$ uses the reduced mass μ of the two-atom system. Because there is no potential energy, the problem is purely kinematic and governed by the geometry of the sphere.

The Hamiltonian depends only on θ and the two momenta, and it contains no explicit dependence on the azimuthal angle ϕ. Therefore ϕ is a cyclic coordinate and its conjugate momentum is conserved.

Theorem 3.2.6 Separation of the rigid-rotator Hamilton– Jacobi equation

With $\mathcal{H} = p_\theta^2/(2I) + p_\phi^2/(2I\sin^2\theta)$, the time-independent Hamilton–Jacobi equation $\mathcal{H}\big(\theta, \phi, \frac{\partial \mathcal{S}}{\partial\theta}, \frac{\partial \mathcal{S}}{\partial\phi}\big) = E$ separates as follows. Because ϕ is cyclic, $\frac{\partial \mathcal{S}}{\partial\phi} = L_z$, a constant. Setting $\mathcal{S} = W_\theta(\theta) + L_z\phi - Et$ reduces the equation to

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = 2IE \equiv L^2,$$

where L is the total angular momentum and $L^2 = 2IE$ is the second separation constant. The θ-equation is

$$\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta} = \pm\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}},$$

which integrates to

$$W_\theta(\theta) = \int \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}\,\mathrm{d}\theta.$$

Derivation of the separated equations: The time-independent Hamilton–Jacobi equation is obtained by substituting $p_\theta = \frac{\partial \mathcal{S}}{\partial \theta}$ and $p_\phi = \frac{\partial \mathcal{S}}{\partial \phi}$ into the Hamiltonian:

$$\frac{1}{2I}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{2I\sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 = E.$$

Multiply both sides by $2I$:

$$\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 = 2IE.$$

The coordinate ϕ is absent from the Hamiltonian, so ϕ is cyclic. The contribution of the cyclic coordinate to the characteristic function is linear:

$$\frac{\partial \mathcal{S}}{\partial \phi} = L_z,$$

where L_z is the conserved z-component of the angular momentum. Substituting this constant into the HJ equation gives

$$\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = 2IE.$$

Seeking an additive separation $\mathcal{S} = W_\theta(\theta) + L_z\phi$, the partial derivative $\frac{\partial \mathcal{S}}{\partial \theta}$ becomes the ordinary derivative $\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}$:

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = 2IE.$$

Define the separation constant $L^2 = 2IE$, which has the dimensions of angular-momentum squared and equals the square of the total angular momentum. Solving for the derivative:

$$\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta} = \pm\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}.$$

The right-hand side vanishes at the turning points where $L^2 = L_z^2/\sin^2\theta$, or equivalently $\sin\theta = |L_z|/L$. Between these turning points, the particle oscillates in θ, tracing a cone on the surface of the sphere. The polar angle sweeps between $\theta_{\min} = \arcsin(|L_z|/L)$ and $\theta_{\max} = \pi - \theta_{\min}$, while ϕ advances monotonically. The trajectory is a closed orbit when the ratio of the azimuthal advance to the θ-oscillation is rational. 😊

Note:-

Geometric interpretation of the orbit

On the surface of the sphere, the angular momentum vector is fixed in space with magnitude L and z-component L_z. The fixed polar angle that this vector makes with the z-axis is $\theta_{\text{cone}} = \arccos(|L_z|/L)$. The instantaneous position vector of the particle precesses around the angular-momentum vector, so the trajectory on the sphere is the intersection of the sphere with the cone defined by $\theta = \theta_{\text{cone}}$. In Hamilton's equations, the azimuthal rate is $\dot{\phi} = L_z/(I\sin^2\theta)$, which varies as θ oscillates. Near the turning points the denominator is small compared to the pole, so the particle slows in ϕ and spends more time near the maximum polar excursion.

Corollary 3.2.2 Equatorial orbit

When the total angular momentum equals the absolute value of its z-component, $L = |L_z|$, the square root in the θ-equation vanishes identically except at $\sin\theta = 1$. The radial momentum $p_\theta = \frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}$ vanishes everywhere except on the equator $\theta = \pi/2$, where the denominator of $L_z^2/\sin^2\theta$ exactly matches the separation constant. The motion is therefore confined to the equatorial plane. From Hamilton's equations, the azimuthal velocity is

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{I\sin^2\theta} = \frac{L_z}{I}$$

on the equator where $\sin\theta = 1$. The azimuthal angle advances linearly in time:

$$\phi(t) = \frac{L_z}{I}t + \phi_0,$$

representing uniform circular motion at constant angular speed $\omega = |L_z|/I$.

Note:-

Action-angle variables for the rigid rotator

The two independent action variables are computed by integrating the conjugate momenta over their respective cycles. For the cyclic coordinate ϕ:

$$J_\phi = \oint p_\phi \, d\phi = \int_0^{2\pi} L_z \, d\phi = 2\pi L_z.$$

For the oscillating coordinate θ, the integral runs between the two turning points:

$$J_\theta = \oint p_\theta \, d\theta = 2\int_{\theta_{\min}}^{\theta_{\max}} \sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}} \, d\theta = 2\pi\big(L - |L_z|\big).$$

Inverting these relations gives $L_z = J_\phi/(2\pi)$ and $L = (J_\theta + |J_\phi|)/(2\pi)$. The Hamiltonian expressed in terms of actions is

$$\mathcal{H}(J_\theta, J_\phi) = \frac{L^2}{2I} = \frac{\big(J_\theta + |J_\phi|\big)^2}{8\pi^2 I}.$$

The frequencies follow from $\omega_i = \frac{\partial \mathcal{H}}{\partial J_i}$. They are generally unequal, so the motion is quasiperiodic unless $L = |L_z|$ (equatorial orbit, $J_\theta = 0$).

Note:-

Connection to quantum mechanics

In the Bohr–Sommerfeld semiclassical quantization, the action variables are quantized as integer multiples of Planck's constant:

$$J_\phi = m\,h, \qquad J_\theta = (l - |m|)\,h,$$

where l and m are integers satisfying $l \geq |m| \geq 0$. Using $L_z = J_\phi/(2\pi) = m\hbar$ and $L = (J_\theta + |J_\phi|)/(2\pi) = l\hbar$, the semiclassical energy is $E = l^2\hbar^2/(2I)$. The fully quantized result from the Schrodinger equation is $E = \hbar^2 l(l+1)/(2I)$. The two agree in the limit of large l, since $l(l+1) \approx l^2$ for $l \gg 1$. For small l, the $+l$ correction in $l(l+1)$ represents a shift with no classical counterpart.

Example 3.2.2 (Action-angle frequencies for the rigid rotator)

Using the Hamiltonian in action variables,

$$\mathcal{H} = \frac{\big(J_\theta + |J_\phi|\big)^2}{8\pi^2 I},$$

the two frequencies are

$$\omega_\theta = \frac{\partial \mathcal{H}}{\partial J_\theta} = \frac{J_\theta + |J_\phi|}{4\pi^2 I} = \frac{L}{2\pi I}, \qquad \omega_\phi = \frac{\partial \mathcal{H}}{\partial J_\phi} = \pm\frac{J_\theta + |J_\phi|}{4\pi^2 I} = \pm\frac{L}{2\pi I}.$$

Here the signs conventionally match the chosen signs of the actions, so J_ϕ can be negative and the $\pm$ sign on the right-hand side is the sign of J_ϕ. These two frequencies are equal in magnitude, confirming the degeneracy. For the special case $J_\theta = 0$ (equatorial orbit), there is no θ oscillation and the motion is purely azimuthal at the single frequency $\omega = L/(2\pi I)$.

Question 9: Diatomic molecule as a rigid rotator

A diatomic molecule is modeled as a rigid rotator with moment of inertia

$$I = 1.46 \times 10^{-46}\,\text{kg}\cdot\text{m}^2.$$

The total angular momentum is $L = 2\hbar$, where $\hbar = 1.055 \times 10^{-34}$ J·s.

(a) Write the Hamilton– Jacobi equation for the rigid rotator in spherical coordinates with fixed $r = R$. Identify the cyclic coordinate and state the corresponding conserved quantity.

(b) Use the classical action-angle result $E = L^2/(2I)$ to compute the rotational energy of the molecule in SI units. Compare this to the quantum result $E = \hbar^2 l(l+1)/(2I)$ with $l = 2$.

(c) Find the absolute difference between the classical and quantum energy values and express it as a percentage of the quantum value.

Solution: **Part (a).** The Hamiltonian of the rigid rotator is $\mathcal{H} = p_\theta^2/(2I) + p_\phi^2/(2I\sin^2\theta)$. The full Hamilton–Jacobi equation follows by replacing p_θ with $\frac{\partial \mathcal{S}}{\partial \theta}$, p_ϕ with $\frac{\partial \mathcal{S}}{\partial \phi}$, and appending the time derivative:

$$\frac{1}{2I}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{2I\sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

For time-independent motion, the action separates as $\mathcal{S} = W(\theta, \phi) - Et$. The Hamiltonian does not depend explicitly on ϕ, so ϕ is the cyclic coordinate. Its conjugate momentum is conserved:

$$\frac{\partial \mathcal{S}}{\partial \phi} = L_z,$$

which is the z-component of the angular momentum.

Part (b). The classical action-angle result for the rigid rotator is $E = L^2/(2I)$. With $L = 2\hbar$, we have $L^2 = 4\hbar^2$. First compute $\hbar^2$:

$$\hbar^2 = \left(1.055 \times 10^{-34}\right)^2 \text{ J}^2\cdot\text{s}^2 = 1.113 \times 10^{-68} \text{ J}^2\cdot\text{s}^2.$$

The classical energy is

$$E_{\text{class}} = \frac{4\hbar^2}{2I} = \frac{2\hbar^2}{I} = \frac{2(1.113 \times 10^{-68})}{1.46 \times 10^{-46}} \text{ J} = 1.525 \times 10^{-22} \text{ J}.$$

The quantum energy with $l = 2$ is

$$E_{\text{quant}} = \frac{\hbar^2\, l(l+1)}{2I} = \frac{(1.113 \times 10^{-68})(6)}{2(1.46 \times 10^{-46})} \text{ J} = \frac{6.678 \times 10^{-68}}{2.92 \times 10^{-46}} \text{ J} = 2.287 \times 10^{-22} \text{ J}.$$

The quantum energy exceeds the classical value. The ratio is

$$\frac{E_{\text{quant}}}{E_{\text{class}}} = \frac{6}{4} = 1.50.$$

Part (c). The absolute difference between the two energies is

$$\Delta E = E_{\text{quant}} - E_{\text{class}} = 2.287 \times 10^{-22} - 1.525 \times 10^{-22} \text{ J} = 7.62 \times 10^{-23} \text{ J}.$$

Expressed as a percentage of the quantum value:

$$\frac{\Delta E}{E_{\text{quant}}} \times 100\% = \frac{7.62 \times 10^{-23}}{2.287 \times 10^{-22}} \times 100\% = 33.3\%.$$

Analytically, since $E_{\text{class}} = l^2\hbar^2/(2I)$ and $E_{\text{quant}} = l(l+1)\hbar^2/(2I)$, the fractional difference is

$$\frac{\Delta E}{E_{\text{quant}}} = \frac{l(l+1) - l^2}{l(l+1)} = \frac{l}{l(l+1)} = \frac{1}{l+1}.$$

For $l = 2$ this gives $1/3 = 33.3\%$, which matches the numerical calculation. The discrepancy arises entirely from the quantum $+l$ correction in $l(l+1)$ relative to the classical l^2.

Therefore, the Hamilton– Jacobi equation is

$$\frac{1}{2I}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{2I\sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 + \frac{\partial \mathcal{S}}{\partial t} = 0,$$

the cyclic coordinate is ϕ with $p_\phi = L_z = \text{const}$, and the energies are

$$E_{\text{class}} = 1.53\times 10^{-22}\,\text{J}, \qquad E_{\text{quant}} = 2.29\times 10^{-22}\,\text{J}, \qquad \frac{\Delta E}{E_{\text{quant}}} = 33.3\%.$$

3.3 Electromagnetism Problems via HJ

3.3.1 Charged Particle in Uniform Electric Field

This subsection solves the Hamilton–Jacobi equation for a charged particle in a uniform electric field, showing that Jacobi's theorem reproduces the parabolic motion dictated by the constant electric force $\vec{F} = q\vec{E}$.

Definition 3.3.1: Hamiltonian for a charged particle in a uniform electric field

A particle of mass m and charge q in a uniform electric field $\vec{E} = E_0\,\hat{\mathbf{z}}$ (with $\vec{B} = 0$) is described by the scalar potential $\varphi = -E_0 z$ and zero vector potential. The Hamiltonian is

$$\mathcal{H} = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - qE_0 z.$$

The coordinates x and y are absent from $\mathcal{H}$, so they are cyclic and the conjugate momenta p_x, p_y are constants of the motion. The Hamilton–Jacobi equation for $\mathcal{S}(\vec{r}, t)$ is

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z}\right)^2\right] - qE_0 z + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Note:-

This problem is the electromagnetic analogue of projectile motion. The gravitational acceleration g is replaced by the electric acceleration qE_0/m and the direction of $\hat{\mathbf{y}}$ by $\hat{\mathbf{z}}$. The two problems are formally equivalent under the substitution $g \longrightarrow -qE_0/m$.

Theorem 3.3.1 Complete integral and trajectory from Jacobi's theorem

The complete integral of the Hamilton–Jacobi equation for a charged particle in the uniform field $\vec{E} = E_0\,\hat{\mathbf{z}}$ is

$$\mathcal{S}(x, y, z, t) = p_x x + p_y y + \frac{2\sqrt{2m}}{3qE_0}\left(E_z + qE_0 z\right)^{3/2} - Et,$$

where p_x and p_y are the conserved transverse momenta and $E_z = E - (p_x^2 + p_y^2)/(2m)$. Jacobi's theorem with respect to the energy, $\frac{\partial \mathcal{S}}{\partial E} = \beta_E$, yields the trajectory along the field direction:

$$z(t) = v_{0z}\,t + \frac{1}{2}\frac{qE_0}{m}\,t^2,$$

a parabola identical in form to the kinematic equation for constant acceleration.

Separation of the HJ equation and extraction of $z(t)$: Because the Hamiltonian has no explicit time dependence, use the ansatz $\mathcal{S} = p_x x + p_y y + W_z(z) - Et$, where p_x, p_y, and E are the separation constants. The partial derivatives are

$$\frac{\partial \mathcal{S}}{\partial x} = p_x, \qquad \frac{\partial \mathcal{S}}{\partial y} = p_y, \qquad \frac{\partial \mathcal{S}}{\partial z} = \frac{\mathrm{d}W_z}{\mathrm{d}z}, \qquad \frac{\partial \mathcal{S}}{\partial t} = -E.$$

Substitute into the HJ PDE:

$$\frac{1}{2m}\left(p_x^2 + p_y^2 + \left(\frac{\mathrm{d}W_z}{\mathrm{d}z}\right)^2\right) - qE_0 z = E.$$

Define the energy associated with z-motion, $E_z = E - (p_x^2 + p_y^2)/(2m)$, and solve for the derivative:

$$\frac{\mathrm{d}W_z}{\mathrm{d}z} = \sqrt{2m\left(E_z + qE_0 z\right)}.$$

Integrate with respect to z. Set $u = E_z + qE_0 z$, so $\mathrm{d}u = qE_0\,\mathrm{d}z$:

$$W_z(z) = \frac{\sqrt{2m}}{qE_0}\int \sqrt{u}\,\mathrm{d}u = \frac{2\sqrt{2m}}{3qE_0}\left(E_z + qE_0 z\right)^{3/2}.$$

Reassemble the principal function:

$$\mathcal{S}(x, y, z, t) = p_x x + p_y y + \frac{2\sqrt{2m}}{3qE_0}\left(E_z + qE_0 z\right)^{3/2} - Et.$$

Jacobi's theorem with respect to E gives

$$\frac{\partial \mathcal{S}}{\partial E} = \frac{2\sqrt{2m}}{3qE_0}\cdot\frac{3}{2}\left(E_z + qE_0 z\right)^{1/2} - t = \beta_E,$$

since $\frac{\partial E_z}{\partial E} = 1$ with p_x and p_y held fixed. Simplify:

$$\frac{\sqrt{2m}}{qE_0}\sqrt{E_z + qE_0 z} - t = \beta_E.$$

The square root equals $p_z(z)/\sqrt{2m} = mv_z$ divided by $\sqrt{2m}$, so multiplying by $qE_0/\sqrt{2m}$ gives

$$v_z(t) = \frac{qE_0}{m}\,(t + \beta_E).$$

At $t = 0$, set $v_z(0) = v_{0z}$. Then $\beta_E = v_{0z}\, m/(qE_0)$ and

$$v_z(t) = v_{0z} + \frac{qE_0}{m}\,t.$$

Integrating once more with $z(0) = 0$:

$$z(t) = v_{0z}\,t + \frac{1}{2}\frac{qE_0}{m}\,t^2.$$

This is the trajectory along the field. The transverse coordinates evolve uniformly, with Jacobi's theorem applied to p_x and p_y giving $x(t) = (p_x/m)t + \beta_x$ and $y(t) = (p_y/m)t + \beta_y$. ☺

Note:-

Verification against the Lorentz force

Newton's second law with $\vec{F} = q\vec{E} = qE_0\,\hat{\mathbf{z}}$ gives the component equation $m\,\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = qE_0$. Integrating twice subject to $z(0) = 0$ and $\dot{z}(0) = v_{0z}$ yields

$$z(t) = v_{0z}\,t + \frac{1}{2}\frac{qE_0}{m}\,t^2,$$

which matches the Hamilton–Jacobi trajectory exactly. The constant acceleration $a_z = qE_0/m$ depends on the charge-to-mass ratio and the field strength. For an electron ($q < 0$) the acceleration opposes the field direction, just as a positively charged particle accelerates along the field. The equivalence between HJ and the force-law approach holds for any time-independent potential.

Question 10: Electron in a uniform electric field

An electron ($q = -e = -1.60\times10^{-19}$ C, $m = 9.11\times10^{-31}$ kg) moves in a uniform electric field $\vec{E} = 1000\,\mathrm{N/C}$ directed along $+\hat{\mathbf{z}}$. The electron is released from rest at the origin at $t = 0$.

(a) Write the Hamilton–Jacobi equation for this system. Show that x and y are cyclic coordinates and identify the corresponding separation constants.

(b) For an electron with $p_x = p_y = 0$ and $v_{0z} = 0$, find $z(t)$ using Jacobi's theorem and give the canonical z-momentum p_z as a function of time.

(c) At $t = 1.0\,\mathrm{ns} = 1.0\times10^{-9}$ s, compute the position $z(t)$ and kinetic energy. Compare to the $F = ma$ prediction.

Solution: **Part (a).** The Hamiltonian is

$$\mathcal{H} = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - qE_0 z,$$

with $E_0 = 1000\,\mathrm{N/C}$. The Hamilton–Jacobi equation reads

$$\frac{1}{2m}\left[\left(\frac{\partial\mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial y}\right)^2 + \left(\frac{\partial\mathcal{S}}{\partial z}\right)^2\right] - qE_0 z + \frac{\partial\mathcal{S}}{\partial t} = 0.$$

Neither x nor y appears explicitly in the Hamiltonian, so both are cyclic. Their conjugate momenta

$$\frac{\partial\mathcal{S}}{\partial x} = \alpha_x, \qquad \frac{\partial\mathcal{S}}{\partial y} = \alpha_y,$$

are conserved separation constants. The complete integral is $\mathcal{S} = \alpha_x x + \alpha_y y + W_z(z) - Et$.

Part (b). With $p_x = p_y = 0$ we have $\alpha_x = \alpha_y = 0$ and the action reduces to $\mathcal{S} = W_z(z) - Et$. The HJ equation gives

$$\frac{1}{2m}\left(\frac{\mathrm{d}W_z}{\mathrm{d}z}\right)^2 - qE_0 z = E.$$

Solve for the canonical momentum:

$$p_z(z) = \frac{\mathrm{d}W_z}{\mathrm{d}z} = \sqrt{2m\big(E + qE_0 z\big)}.$$

Jacobi's theorem yields the velocity $v_z(t) = \dfrac{qE_0}{m}(t + \beta_E)$. The particle starts from rest, so $v_z(0) = 0$ fixes $\beta_E = 0$ and

$$v_z(t) = \frac{qE_0}{m}\,t, \qquad p_z(t) = qE_0\,t.$$

Integrating $v_z(t)$ with $z(0) = 0$:

$$z(t) = \frac{1}{2}\,\frac{qE_0}{m}\,t^2.$$

Because $q = -e < 0$ and $E_0 > 0$, the acceleration is negative and the electron moves in the $-z$ direction.

Part (c). Compute the acceleration:

$$a = \frac{qE_0}{m} = \frac{(-1.60\times10^{-19})(1000)}{9.11\times10^{-31}}\,\mathrm{m/s^2} = -1.76\times10^{14}\,\mathrm{m/s^2}.$$

At $t = 1.0\times10^{-9}$ s the position is

$$z = \frac{1}{2}\,a\,t^2 = \frac{1}{2}(-1.76\times10^{14})(1.0\times10^{-18})\,\mathrm{m} = -8.8\times10^{-5}\,\mathrm{m}.$$

The speed is $|v_z| = |a|\,t = (1.76\times10^{14})(1.0\times10^{-9})\,\mathrm{m/s} = 1.76\times10^{5}\,\mathrm{m/s}$. The kinetic energy is

$$K = \tfrac{1}{2}\,m\,v_z^2 = \tfrac{1}{2}\,(9.11\times10^{-31})(1.76\times10^{5})^2\,\mathrm{J} = 1.4\times10^{-20}\,\mathrm{J}.$$

In electron volts, $K = (1.4 \times 10^{-20})/(1.60 \times 10^{-19})\,\text{eV} = 0.088\,\text{eV}$. From the $F = ma$ approach, $\vec{F} = q\vec{E} = (-1.60 \times 10^{-19})(1000)\,\hat{\mathbf{z}}\,\text{N} = -1.60 \times 10^{-16}\,\hat{\mathbf{z}}\,\text{N}$. The resulting acceleration $a = -1.76 \times 10^{14}\,\text{m/s}^2$ is identical and the integrated kinematics $z = \frac{1}{2}at^2$ reproduce both the position and energy exactly.

Therefore,

$$z(1.0\,\text{ns}) = -8.8 \times 10^{-5}\,\text{m}, \qquad K = 1.4 \times 10^{-20}\,\text{J} = 0.088\,\text{eV}.$$

3.3.2 Cyclotron Motion

This subsection solves for a charged particle moving in a uniform magnetic field through the Hamilton–Jacobi equation, derives the helical trajectory by quadrature, and computes the action-angle variables that recover the cyclotron frequency.

Definition 3.3.2: Hamilton–Jacobi formulation of cyclotron motion

A particle of mass m and charge q moves in a uniform magnetic field $\vec{B} = B_0\,\hat{\mathbf{z}}$. Choose the Landau gauge for the vector potential $\vec{A} = (0, B_0 x, 0)$ and set the scalar potential $\varphi = 0$. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left[p_x^2 + \left(p_y - qB_0x\right)^2 + p_z^2\right].$$

In this gauge, the coordinate x appears explicitly in $\mathcal{H}$ while y and z are absent, so y and z are cyclic: their conjugate momenta p_y and p_z are constants of the motion. The Hamilton–Jacobi equation for $\mathcal{S}(\vec{r}, t)$ reads

$$\frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial y} - qB_0x\right)^2 + \left(\frac{\partial \mathcal{S}}{\partial z}\right)^2\right] + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Note:-

The Landau gauge $\vec{A} = (0, B_0x, 0)$ breaks rotational symmetry and makes y cyclic but x not. Other gauges exist (for instance $A = (-B_0y, 0, 0)$), but they lead to mathematically equivalent Hamiltonian systems. The physics of cyclotron motion – circular gyration at the Larmor frequency – is gauge-independent, as the magnetic field $\vec{B} = \nabla \times \vec{A} = B_0\hat{\mathbf{z}}$ is the same in every gauge.

Theorem 3.3.2 Complete integral for cyclotron motion

The cyclotron frequency is $\omega_c = qB_0/m$. The guiding-center x-coordinate is $X_c = \alpha_y/(qB_0)$ and the gyroradius is $R = \sqrt{2mE_\perp}/(qB_0)$, where α_y is the conserved canonical y-momentum and $E_\perp$ is the transverse energy. The complete integral of the Hamilton–Jacobi equation is

$$\mathcal{S} = W_x(x) + \alpha_y y + \alpha_z z - Et,$$

where the x-part of the characteristic function is

$$W_x(x) = \frac{E_\perp}{\omega_c}\arcsin\left(\frac{x - X_c}{R}\right) + \frac{qB_0}{2}(x - X_c)\sqrt{R^2 - (x - X_c)^2}.$$

Here $E_\perp = E - \alpha_z^2/(2m)$ is the energy of motion in the xy-plane alone. The complete integral is defined for $|x - X_c| < R$.

Separation of the HJ equation and integration of the x-dependent part: Because the potentials are time-independent, separate as $\mathcal{S}(\vec{r}, t) = W(\vec{r}) - Et$. Because y and z are cyclic coordinates, set $\frac{\partial \mathcal{S}}{\partial y} = \alpha_y$ and $\frac{\partial \mathcal{S}}{\partial z} = \alpha_z$. The remaining dependence on x is carried by a single function $W_x(x)$, so $W = W_x(x) + \alpha_y y + \alpha_z z$ and the time-independent Hamilton–Jacobi equation reads

$$\frac{1}{2m}\left[\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\alpha_y - qB_0x\right)^2 + \alpha_z^2\right] = E.$$

Define the transverse energy $E_\perp = E - \alpha_z^2/(2m)$. The x-equation simplifies to

$$\left(\frac{\mathrm{d}W_x}{\mathrm{d}x}\right)^2 + \left(\alpha_y - qB_0 x\right)^2 = 2mE_\perp.$$

Solve for the spatial derivative:

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \pm\sqrt{2mE_\perp - \left(\alpha_y - qB_0 x\right)^2}.$$

The square root is real when $|\alpha_y - qB_0 x| \le \sqrt{2mE_\perp}$. Introduce the guiding-center coordinate

$$X_c = \frac{\alpha_y}{qB_0}.$$

Then $\alpha_y - qB_0 x = -qB_0(x - X_c)$, and the radicand factors as

$$2mE_\perp - q^2B_0^2(x - X_c)^2 = q^2B_0^2\left(\frac{2mE_\perp}{q^2B_0^2} - (x - X_c)^2\right).$$

Define the gyroradius $R = \sqrt{2mE_\perp}/(qB_0)$. The derivative of W_x becomes

$$\frac{\mathrm{d}W_x}{\mathrm{d}x} = \pm qB_0\sqrt{R^2 - (x - X_c)^2}.$$

This has the same square-root structure as the simple harmonic oscillator. Integrate by the trigonometric substitution $x - X_c = R\sin\theta$, giving $\mathrm{d}x = R\cos\theta\,\mathrm{d}\theta$:

$$W_x = \int qB_0\sqrt{R^2 - R^2\sin^2\theta}\cdot R\cos\theta\,\mathrm{d}\theta = qB_0R^2\int\cos^2\theta\,\mathrm{d}\theta.$$

The antiderivative of $\cos^2\theta$ is $\frac{1}{2}(\theta + \sin\theta\cos\theta)$, so

$$W_x = \frac{qB_0R^2}{2}\left(\theta + \sin\theta\cos\theta\right).$$

Evaluate the constant prefactor using $R^2 = 2mE_\perp/(q^2B_0^2)$:

$$\frac{qB_0R^2}{2} = \frac{qB_0}{2}\cdot\frac{2mE_\perp}{q^2B_0^2} = \frac{mE_\perp}{qB_0} = \frac{E_\perp}{\omega_c}.$$

Now express the trigonometric factors in terms of x:

$$\theta = \arcsin\left(\frac{x - X_c}{R}\right), \qquad \sin\theta = \frac{x - X_c}{R}, \qquad \cos\theta = \frac{\sqrt{R^2 - (x - X_c)^2}}{R}.$$

The product $\sin\theta\cos\theta$ is $(x - X_c)\sqrt{R^2 - (x - X_c)^2}/R^2$. Substituting back,

$$W_x(x) = \frac{E_\perp}{\omega_c}\arcsin\left(\frac{x - X_c}{R}\right) + \frac{E_\perp}{\omega_c}\cdot\frac{(x - X_c)\sqrt{R^2 - (x - X_c)^2}}{R^2}.$$

The coefficient of the second term simplifies as

$$\frac{E_\perp}{\omega_c R^2} = \frac{E_\perp}{\omega_c}\cdot\frac{q^2B_0^2}{2mE_\perp} = \frac{q^2B_0^2}{2m\omega_c} = \frac{qB_0}{2},$$

since $\omega_c = qB_0/m$. Therefore,

$$W_x(x) = \frac{E_\perp}{\omega_c}\arcsin\left(\frac{x - X_c}{R}\right) + \frac{qB_0}{2}(x - X_c)\sqrt{R^2 - (x - X_c)^2}.$$

The full complete integral is $\mathcal{S} = W_x(x) + \alpha_y y + \alpha_z z - Et$. 😊

Corollary 3.3.1 Helical trajectory from Jacobi's theorem

Jacobi's theorem states that differentiating the complete integral with respect to each separation constant produces a constant fixed by the initial conditions. Differentiate $\mathcal{S}$ with respect to $E_\perp$ at fixed x:

$$\frac{\partial \mathcal{S}}{\partial E_\perp} = \frac{\partial W_x}{\partial E_\perp} - t.$$

Write $\chi = (x - X_c)/R$ and $U = \sqrt{R^2 - (x - X_c)^2}$. The partial derivative of W_x with respect to $E_\perp$ is

$$\frac{\partial W_x}{\partial E_\perp} = \frac{1}{\omega_c} \arcsin \chi + \chi \cos \chi \cdot \frac{\partial E_\perp/\omega_c}{\partial E_\perp} \cdot \frac{1}{(E_\perp/\omega_c)} + \frac{qB_0}{2}(x - X_c) \cdot \frac{1}{2U}\frac{\partial R^2}{\partial E_\perp}.$$

Because $R^2 = 2mE_\perp/(q^2B_0^2)$, one has $\frac{\partial R^2}{\partial E_\perp} = 2m/(q^2B_0^2)$. The last two terms are

$$-\frac{1}{2\omega_c}\frac{\chi}{\sqrt{1-\chi^2}} = -\frac{x - X_c}{2\omega_c R\sqrt{1-\chi^2}},$$

$$\frac{qB_0}{2}(x - X_c) \cdot \frac{m}{q^2B_0^2U} = \frac{m}{qB_0} \cdot \frac{x - X_c}{2U} = \frac{x - X_c}{2\omega_c R\sqrt{1-\chi^2}},$$

which exactly cancel, as in the harmonic oscillator case. Hence,

$$\frac{\partial W_x}{\partial E_\perp} = \frac{1}{\omega_c} \arcsin\left(\frac{x - X_c}{R}\right).$$

Set $\frac{\partial \mathcal{S}}{\partial E_\perp} = \beta$ (constant):

$$\frac{1}{\omega_c} \arcsin\left(\frac{x - X_c}{R}\right) - t = \beta,$$

or equivalently,

$$x(t) = X_c + R\sin\big(\omega_c(t + \beta)\big).$$

Define the initial phase $\phi_0 = \omega_c\beta$.
Differentiate $\mathcal{S}$ with respect to the separation constant α_y:

$$\frac{\partial \mathcal{S}}{\partial \alpha_y} = \frac{\partial W_x}{\partial \alpha_y} + y = \beta_y.$$

Since $X_c = \alpha_y/(qB_0)$, the chain rule gives $\frac{\partial W_x}{\partial \alpha_y} = -\frac{\partial W_x}{\partial X_c} \cdot (1/qB_0)$. From the structure of W_x, this derivative evaluates to $-(x - X_c)/R \cdot (E_\perp/(\omega_c R)) - \frac{1}{2}\sqrt{R^2 - (x - X_c)^2} \cdot (qB_0/qB_0)$, but the result is more easily found from the canonical relation $v_y = (\alpha_y - qB_0 x)/m$:

$$v_y(t) = \frac{\alpha_y - qB_0 x(t)}{m} = \frac{qB_0\big(X_c - x(t)\big)}{m} = -\omega_c R\sin(\omega_c t + \phi_0).$$

Integrating with respect to time,

$$y(t) = Y_c + R\cos(\omega_c t + \phi_0),$$

where Y_c is an integration constant set by the initial conditions. Meanwhile, for the z-direction, $p_z = \alpha_z$ is constant, giving $z(t) = (\alpha_z/m)t + z_0 = v_z t + z_0$. The full trajectory is helical:

$$x(t) = X_c + R\sin(\omega_c t + \phi_0), \qquad y(t) = Y_c + R\cos(\omega_c t + \phi_0), \qquad z(t) = v_z t + z_0.$$

The projection onto the xy-plane is a circle of radius R centered at (X_c, Y_c), traversed at the constant angular speed ω_c. Superimposed is uniform motion along the field direction at speed v_z.

Proposition 3.3.1 Action-angle variables for cyclotron motion

The action-angle formalism applied to cyclotron motion yields the following results:

① The action variable associated with the transverse motion is the phase-space area enclosed by one gyration:

$$J = \oint p_x \, dx = 2 \int_{X_c - R}^{X_c + R} qB_0 \sqrt{R^2 - (x - X_c)^2} \, dx.$$

The integral is twice the area of a semicircle of radius R (multiplied by qB_0), so

$$J = qB_0 \cdot \pi R^2 = qB_0 \cdot \pi \cdot \frac{2mE_\perp}{q^2 B_0^2} = \frac{2\pi m E_\perp}{qB_0} = \frac{2\pi E_\perp}{\omega_c}.$$

Geometrically, $J/(qB_0)$ is the area of the circular orbit in the xy-plane.

② Inverting the action–energy relation, the transverse energy as a function of the action is

$$E_\perp(J) = \frac{\omega_c J}{2\pi}.$$

The Hamiltonian expressed in terms of the action variables is $E = \omega_c J/(2\pi) + \alpha_z^2/(2m)$, linear in J and quadratic in α_z.

③ The Hamilton–Jacobi frequency is $\hat{\omega} = \frac{\partial E_\perp}{\partial J} = \omega_c/(2\pi)$. The physical angular frequency is $\omega = 2\pi\hat{\omega} = \omega_c = qB_0/m$, which depends only on the charge-to-mass ratio and the field strength. It is independent of the transverse energy $E_\perp$ and the gyroradius R. This amplitude independence is the hallmark of uniform circular motion in a constant magnetic field.

④ The angle variable advances linearly in time: $w = \hat{\omega} t + w_0 = (\omega_c/2\pi)t + w_0$. The phase of the circular gyration, $\omega_c t + \phi_0$, equals $2\pi w$ up to a constant phase. The action-angle variables provide a clean canonical description even though the original Cartesian coordinates exhibit coupled oscillatory dynamics.

Note:-

Comparison with the Lorentz force

The Lorentz force law $m\,\ddot{\vec{r}} = q\,\dot{\vec{r}} \times \vec{B}$ with $\vec{B} = B_0\hat{\mathbf{z}}$ gives the component equations

$$\ddot{x} = \frac{qB_0}{m}\,\dot{y}, \qquad \ddot{y} = -\frac{qB_0}{m}\,\dot{x}, \qquad \ddot{z} = 0.$$

Introducing $v_x = \dot{x}$ and $v_y = \dot{y}$, these become $\dot{v}_x = \omega_c v_y$ and $\dot{v}_y = -\omega_c v_x$, whose solutions are

$$v_x = v_\perp \cos(\omega_c t + \phi_0), \qquad v_y = -v_\perp \sin(\omega_c t + \phi_0).$$

Integrating once more gives the same circular trajectory with radius $R = v_\perp/\omega_c = \sqrt{2mE_\perp}/(qB_0)$, and uniform z-motion. The Hamilton–Jacobi approach arrives at the identical values of ω_c and R through a radically different route: solving a first-order nonlinear PDE by separation and quadrature, then differentiating the complete integral. The agreement reaffirms the consistency of the Hamiltonian and Newtonian formulations.

Question 11: Proton cyclotron motion from the HJ complete integral

A proton of mass $m = 1.67 \times 10^{-27}$ kg and charge $q = e = 1.60 \times 10^{-19}$ C moves in a uniform magnetic field $\vec{B} = (1.5\,\mathrm{T})\,\hat{\mathbf{z}}$. The transverse (perpendicular-to-field) kinetic energy is $E_\perp = 1.0\,\mathrm{keV} = 1.60 \times 10^{-16}$ J.

(a) Compute the cyclotron angular frequency $\omega_c = qB_0/m$ and the gyration period $T = 2\pi/\omega_c$.

(b) Find the gyroradius $R = \sqrt{2mE_\perp}/(qB_0)$ in meters.

(c) Compute the action variable $J = 2\pi E_\perp/\omega_c$ in SI units and verify by differentiation that $\frac{\partial E_\perp}{\partial J} = \omega_c/(2\pi)$, recovering the cyclotron frequency.

Solution: **Part (a).** The cyclotron angular frequency is

$$\omega_c = \frac{qB_0}{m}.$$

Substitute the given values:

$$q = 1.60 \times 10^{-19}\,\mathrm{C}, \qquad B_0 = 1.5\,\mathrm{T}, \qquad m = 1.67 \times 10^{-27}\,\mathrm{kg}.$$

Form the ratio:

$$\omega_c = \frac{(1.60 \times 10^{-19})(1.5)}{1.67 \times 10^{-27}}\,\mathrm{rad/s} = \frac{2.40 \times 10^{-19}}{1.67 \times 10^{-27}}\,\mathrm{rad/s}.$$

This gives

$$\omega_c = 1.437 \times 10^8\,\mathrm{rad/s}.$$

Rounding to two significant figures (consistent with the field strength 1.5 T),

$$\omega_c = 1.4 \times 10^8\,\mathrm{rad/s}.$$

The gyration period is

$$T = \frac{2\pi}{\omega_c} = \frac{2\pi}{1.437 \times 10^8}\,\mathrm{s} = 4.37 \times 10^{-8}\,\mathrm{s}.$$

In more convenient units,

$$T = 4.4 \times 10^{-8}\,\mathrm{s} = 44\,\mathrm{ns}.$$

Part (b). The gyroradius is

$$R = \frac{\sqrt{2mE_\perp}}{qB_0}.$$

Evaluate the numerator:

$$2mE_\perp = 2(1.67 \times 10^{-27}\,\mathrm{kg})(1.60 \times 10^{-16}\,\mathrm{J}) = 5.34 \times 10^{-43}\,\mathrm{kg^2\,m^2/s^2}.$$

Taking the square root:

$$\sqrt{2mE_\perp} = \sqrt{5.34 \times 10^{-43}}\,\mathrm{kg\,m/s} = 7.31 \times 10^{-22}\,\mathrm{kg\,m/s}.$$

The denominator is

$$qB_0 = (1.60 \times 10^{-19}\,\mathrm{C})(1.5\,\mathrm{T}) = 2.40 \times 10^{-19}\,\mathrm{C\,T}.$$

Therefore,

$$R = \frac{7.31 \times 10^{-22}}{2.40 \times 10^{-19}}\,\mathrm{m} = 3.05 \times 10^{-3}\,\mathrm{m}.$$

In more convenient units,

$$R = 3.05\,\mathrm{mm}.$$

Part (c). The action variable for the transverse cyclotron motion is

$$J = \frac{2\pi E_\perp}{\omega_c}.$$

Substitute the numerical values:

$$J = \frac{2\pi(1.60 \times 10^{-16}\,\mathrm{J})}{1.437 \times 10^8\,\mathrm{rad/s}} = \frac{1.005 \times 10^{-15}}{1.437 \times 10^8}\,\mathrm{J{\cdot}s}.$$

This gives

$$J = 7.0 \times 10^{-24}\,\mathrm{J{\cdot}s}.$$

Now verify the energy–action relation $E_\perp(J) = \omega_c J/(2\pi)$. Differentiating with respect to J:

$$\frac{\partial E_\perp}{\partial J} = \frac{\omega_c}{2\pi}.$$

The physical angular frequency is recovered as $\omega = 2\pi(\frac{\partial E_\perp}{\partial J}) = \omega_c$. For the numerical values,

$$E_\perp(J) = \frac{\omega_c J}{2\pi} = \frac{(1.437\times10^{8}\,\text{rad/s})(7.0\times10^{-24}\,\text{J·s})}{2\pi} = 1.60\times10^{-16}\,\text{J},$$

which is exactly the original transverse energy of 1.0 keV. The relation $\frac{\partial E_\perp}{\partial J} = \omega_c/(2\pi)$ holds both algebraically and numerically, confirming that the action-angle formalism reproduces the cyclotron frequency exactly.

Therefore,

$$\omega_c = 1.4\times10^{8}\,\text{rad/s},\qquad T = 44\,\text{ns},\qquad R = 3.1\,\text{mm},\qquad J = 7.0\times10^{-24}\,\text{J·s}.$$

3.3.3 E × B Drift

This subsection shows how a uniform electric field crossed with a uniform magnetic field produces a constant guiding-centre drift, derivable from the Hamilton–Jacobi equation by recognizing the harmonic nature of the transverse motion.

Definition 3.3.3: Crossed-field Hamiltonian in the Landau gauge

A particle of mass m and charge q moves in a uniform electric field $\vec{E} = E_0\,\hat{\mathbf{y}}$ and a uniform magnetic field $\vec{B} = B_0\,\hat{\mathbf{z}}$. We choose the Landau gauge for the vector potential, $\vec{A} = (-B_0 y, 0, 0)$, and the scalar potential $\varphi = -E_0 y$. The curl of the vector potential,

$$\nabla\times\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} = B_0\,\hat{\mathbf{z}},$$

reproduces the magnetic field, and the gradient of the scalar potential gives $\vec{E} = -\nabla\varphi = E_0\,\hat{\mathbf{y}}$. The electromagnetic Hamiltonian for a charged particle,

$$\mathcal{H} = \frac{1}{2m}\left|\vec{p} - q\vec{A}\right|^2 + q\varphi,$$

becomes explicitly

$$\mathcal{H} = \frac{1}{2m}\left[\left(p_x + qB_0 y\right)^2 + p_y^2 + p_z^2\right] - qE_0 y.$$

The coordinates x and z do not appear in $\mathcal{H}$, so they are cyclic and their conjugate momenta p_x and p_z are conserved.

Note:-

Choice of gauge does not affect physical observables. The Landau gauge $\vec{A} = (-B_0 y, 0, 0)$ breaks translational symmetry in y but preserves it in x, making p_x the conserved momentum. An alternative symmetric gauge $\vec{A} = \frac{1}{2}B_0(-y, x, 0)$ would be natural for purely rotational problems but obscures the drift structure we want to expose here.

Theorem 3.3.3 *$E\times B$ drift velocity from the guiding centre*

For crossed fields $\vec{E} = E_0\,\hat{\mathbf{y}}$ and $\vec{B} = B_0\,\hat{\mathbf{z}}$, the guiding centre of the orbit lies at

$$y_c = \frac{mE_0/B_0 - \alpha_x}{qB_0},$$

where α_x is the conserved canonical x-momentum. The time-averaged x-velocity (drift velocity) is

$$v_d = \frac{\alpha_x + qB_0 y_c}{m} = \frac{E_0}{B_0},$$

pointing in the $\hat{\mathbf{x}} = \hat{\mathbf{E}} \times \hat{\mathbf{B}}$ direction. The drift velocity is universal: every particle, regardless of charge or mass, drifts at this same speed.

Derivation of the drift velocity from the Hamilton–Jacobi equation: Because the Hamiltonian has no explicit time dependence, energy is conserved: $\mathcal{H} = E$. The time variable separates as $\mathcal{S} = W(x, y, z) - Et$. The coordinates x and z are cyclic in $\mathcal{H}$, so their conjugate momenta are constants:

$$\frac{\partial W}{\partial x} = \alpha_x, \qquad \frac{\partial W}{\partial z} = \alpha_z.$$

The full action takes the additive form

$$\mathcal{S}(x, y, z, t) = \alpha_x x + W_y(y) + \alpha_z z - Et.$$

The time-independent Hamilton–Jacobi equation $\mathcal{H}(\vec{r}, \nabla\mathcal{S}) = E$ becomes

$$\frac{1}{2m}\left[\left(\alpha_x + qB_0 y\right)^2 + \left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + \alpha_z^2\right] - qE_0 y = E.$$

The term involving the only remaining unknown is isolated by solving for $\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2$:

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 = 2mE - \alpha_z^2 + 2mqE_0 y - \left(\alpha_x + qB_0 y\right)^2.$$

Expand the square $\left(\alpha_x + qB_0 y\right)^2 = \alpha_x^2 + 2\alpha_x qB_0 y + q^2 B_0^2 y^2$ and collect terms by powers of y:

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 = -q^2 B_0^2 y^2 + 2mqE_0 y - 2\alpha_x qB_0 y + 2mE - \alpha_x^2 - \alpha_z^2.$$

Factor the linear-y terms:

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 = -q^2 B_0^2 y^2 + 2qB_0\left(mE_0/B_0 - \alpha_x\right)y + 2mE - \alpha_x^2 - \alpha_z^2.$$

Complete the square on the right-hand side. Factor out $-q^2 B_0^2$ from the quadratic and linear terms in y:

$$-q^2 B_0^2\left[y^2 - \frac{2}{qB_0}\left(\frac{mE_0}{B_0} - \alpha_x\right)y\right].$$

Add and subtract the square of half the coefficient of y inside the bracket:

$$-q^2 B_0^2\left[y - \frac{1}{qB_0}\left(\frac{mE_0}{B_0} - \alpha_x\right)\right]^2 + q^2 B_0^2\left[\frac{1}{qB_0}\left(\frac{mE_0}{B_0} - \alpha_x\right)\right]^2.$$

The shift in brackets is the guiding-centre coordinate:

$$y_c = \frac{mE_0/B_0 - \alpha_x}{qB_0}.$$

Substituting back, the full equation becomes

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + q^2 B_0^2\left(y - y_c\right)^2 = 2mE - \alpha_z^2 - \alpha_x^2 + q^2 B_0^2 y_c^2.$$

The right-hand side is a constant determined by the energy and separation constants. Define $C = 2mE - \alpha_z^2 - \alpha_x^2 + q^2 B_0^2 y_c^2$. Then

$$\left(\frac{\mathrm{d}W_y}{\mathrm{d}y}\right)^2 + q^2 B_0^2 (y - y_c)^2 = C.$$

This is precisely the Hamilton–Jacobi equation for a harmonic oscillator in the shifted variable $Y = y - y_c$, with frequency

$$\omega_c = \frac{|q| B_0}{m}.$$

The y-motion oscillates sinusoidally about y_c with the cyclotron frequency. The time average of the position is the guiding centre, $\langle y \rangle = y_c$.

Now compute the kinematic x-velocity. From Hamilton's equation, $\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x}$:

$$v_x = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{m}\big(p_x + q B_0 y\big).$$

The canonical momentum $p_x = \frac{\partial \mathcal{S}}{\partial x} = \alpha_x$ is constant. Averaging over one cyclotron orbit, $\langle y \rangle = y_c$, so the drift velocity is

$$\langle v_x \rangle = \frac{\alpha_x + q B_0 y_c}{m}.$$

Substitute the explicit expression for the guiding centre:

$$q B_0 y_c = q B_0 \cdot \frac{m E_0 / B_0 - \alpha_x}{q B_0} = \frac{m E_0}{B_0} - \alpha_x.$$

The separation constant α_x cancels out:

$$\alpha_x + q B_0 y_c = \alpha_x + \frac{m E_0}{B_0} - \alpha_x = \frac{m E_0}{B_0}.$$

Dividing by m gives the drift velocity:

$$\langle v_x \rangle = \frac{E_0}{B_0}.$$

The drift is positive in the $+x$ direction, equal to $(E_0/B_0)\,\hat{\mathbf{x}} = (\vec{E} \times \vec{B})/B_0^2$, and depends on neither the particle's mass nor its charge.

😄

Corollary 3.3.2 Comparison with the Lorentz-force prediction

The steady-state solution of the Lorentz-force equation $m\dot{\vec{v}} = q(\vec{E} + \vec{v} \times \vec{B})$ for zero acceleration, $\dot{\vec{v}} = 0$, requires

$$q\vec{E} + q\vec{v} \times \vec{B} = 0, \qquad \text{or} \qquad \vec{v} \times \vec{B} = -\vec{E}.$$

Take the cross product of both sides with $\vec{B}$ from the right:

$$(\vec{v} \times \vec{B}) \times \vec{B} = -\vec{E} \times \vec{B}.$$

Using the vector identity $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$:

$$(\vec{v} \cdot \vec{B})\vec{B} - B^2 \vec{v} = -\vec{E} \times \vec{B}.$$

Since $\vec{E} \perp \vec{B}$, the velocity component parallel to $\vec{B}$ does not contribute to the drift, and choosing $\vec{v} \cdot \vec{B} = 0$ gives

$$\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2}.$$

With $\vec{B} = B_0\,\hat{\mathbf{z}}$ and $\vec{E} = E_0\,\hat{\mathbf{y}}$:

$$\vec{v}_d = \frac{E_0 B_0\,\hat{\mathbf{x}}}{B_0^2} = \frac{E_0}{B_0}\,\hat{\mathbf{x}}.$$

This matches the Hamilton–Jacobi result exactly. The cross product $\vec{E} \times \vec{B}$ determines the drift direction and division by B^2 converts the magnitude into a velocity. Both formalisms predict the same drift regardless of the particle's charge or mass.

Note:-

A charge-sign reversal changes both the sense of Larmor rotation and the location of the guiding centre y_c, but these two effects exactly compensate in the averaged x-velocity. An electron and a proton spiralling in the same crossed fields therefore share the same guiding-centre drift, even though their individual gyroradii and rotation frequencies differ enormously. This universality is what makes the $E \times B$ drift so important in plasma physics: bulk plasma drifts as a coherent fluid.

Proposition 3.3.2 Properties of the $E \times B$ drift

The drift velocity $\vec{v}_d = \vec{E} \times \vec{B}/B^2$ satisfies the following properties:

1. Independence of charge sign. Positive and negative charges drift in the same direction at the same speed. The sign of q cancels between the force $q\vec{E}$ and the Lorentz deflection $q\vec{v} \times \vec{B}$.
2. Independence of mass. Heavy ions and light electrons drift side by side at the same velocity. The mass appears in neither $\vec{E} \times \vec{B}$ nor B^2.
3. Direction perpendicular to both fields. The drift points along $\hat{\mathbf{E}}\times\hat{\mathbf{B}}$, orthogonal to the plane containing the two fields.
4. Magnitude depends on the field ratio. $v_d = E_0/B_0$ grows with stronger electric field and weaker magnetic field. Doubling both fields leaves the drift unchanged.
5. The guiding centre y_c itself depends on q, m, and α_x, but these dependencies cancel in the drift velocity $\langle v_x \rangle$.
6. The transverse y-motion is a harmonic oscillation with cyclotron frequency $\omega_c = |q|B_0/m$ about y_c. The full trajectory is a trochoid: the superposition of circular Larmor motion and uniform drift.

Example 3.3.1 (Trochoidal orbit geometry)

When the transverse kinetic energy is large compared to the electric-field energy scale E_0 times the gyroradius, the particle traces a cycloid-like path in the xy plane while drifting in x. If the drift speed exceeds the thermal speed, the orbit is a prolate trochoid with open loops; at equal speeds it is a common cycloid with cusps; and at slower drift the orbit folds back on itself as a curtate trochoid. In every case the guiding centre advances uniformly at $v_d = E_0/B_0$.

Question 12: Electron in crossed electric and magnetic fields

An electron travels through a region with crossed fields $\vec{E} = (500\,\mathrm{V/m})\,\hat{\mathbf{y}}$ and $\vec{B} = (0.01\,\mathrm{T})\,\hat{\mathbf{z}}$. The electron has mass $m_e = 9.11 \times 10^{-31}$ kg and charge $q_e = -e = -1.60 \times 10^{-19}$ C.

(a) Compute the drift velocity $\vec{v}_d = \vec{E} \times \vec{B}/B^2$ and show it equals 50 000 m/s in the $\hat{\mathbf{x}}$ direction.

(b) Using the guiding-centre formula $y_c = (mE_0/B_0 - \alpha_x)/(qB_0)$, verify that the drift $\langle v_x \rangle = (\alpha_x + qB_0 y_c)/m$ is independent of both m and q.

(c) For a proton ($m_p = 1.67 \times 10^{-27}$ kg, $q_p = +e = +1.60 \times 10^{-19}$ C) in the same fields, confirm that the drift velocity is identical to the electron's.

Solution: **Part (a).** The cross product of the field vectors is

$$\vec{E} \times \vec{B} = \big(500\,\mathrm{V/m}\big)\big(0.01\,\mathrm{T}\big)\,\hat{\mathbf{y}} \times \hat{\mathbf{z}} = 5.00\,\mathrm{V{\cdot}T/m}\,\hat{\mathbf{x}}.$$

The unit check: $1\,\mathrm{V\cdot T/m} = 1\,(\mathrm{V/m})/(\mathrm{T}) = 1\,\mathrm{m/s}$, because $\mathrm{T} = \mathrm{V\cdot s/m^2}$. The squared magnetic field strength is

$$B^2 = (0.01\,\mathrm{T})^2 = 1.00 \times 10^{-4}\,\mathrm{T}^2.$$

The drift velocity is

$$\vec{v}_d = \frac{\vec{E} \times \vec{B}}{B^2} = \frac{5.00\,\mathrm{V\cdot T/m}}{1.00 \times 10^{-4}\,\mathrm{T}^2}\,\hat{\mathbf{x}}.$$

Equivalently, using the scalar ratio directly:

$$\frac{E_0}{B_0} = \frac{500\,\mathrm{V/m}}{0.01\,\mathrm{T}} = 50\,000\,\mathrm{m/s}.$$

Therefore,

$$\vec{v}_d = 50\,000\,\mathrm{m/s}\,\hat{\mathbf{x}}.$$

The drift is in the $\hat{\mathbf{x}}$ direction, perpendicular to both $\vec{E}$ ($\hat{\mathbf{y}}$) and $\vec{B}$ ($\hat{\mathbf{z}}$), consistent with the $\hat{\mathbf{E}} \times \hat{\mathbf{B}}$ rule.

Part (b). The guiding-centre position is

$$y_c = \frac{mE_0/B_0 - \alpha_x}{qB_0}.$$

Substitute into the drift formula for the time-averaged x-velocity:

$$\langle v_x \rangle = \frac{\alpha_x + qB_0 y_c}{m}.$$

First evaluate the product $qB_0 y_c$:

$$qB_0 y_c = qB_0 \cdot \frac{mE_0/B_0 - \alpha_x}{qB_0} = \frac{mE_0}{B_0} - \alpha_x.$$

The factors of qB_0 cancel cleanly. Now the numerator of the drift is

$$\alpha_x + qB_0 y_c = \alpha_x + \left(\frac{mE_0}{B_0} - \alpha_x\right) = \frac{mE_0}{B_0}.$$

Dividing by m:

$$\langle v_x \rangle = \frac{mE_0/B_0}{m} = \frac{E_0}{B_0}.$$

Both the charge q and the mass m have cancelled algebraically. The drift depends only on the field ratio E_0/B_0. For the numerical values of this problem:

$$\frac{E_0}{B_0} = \frac{500\,\mathrm{V/m}}{0.01\,\mathrm{T}} = 50\,000\,\mathrm{m/s},$$

which agrees with part (a).

Part (c). For the proton, the guiding centre is

$$y_c^{(\mathrm{p})} = \frac{m_p E_0/B_0 - \alpha_x^{(\mathrm{p})}}{q_p B_0}.$$

The proton mass $m_p = 1.67 \times 10^{-27}$ kg is about 1837 times the electron mass, and the proton charge $q_p = +e$ has the opposite sign. The numerical value of $y_c^{(\mathrm{p})}$ therefore differs substantially from the electron's guiding centre. However, the cancellation in the drift formula is purely algebraic and does not depend on the numerical values of m or q. The proton drift is

$$\langle v_x^{(\mathrm{p})} \rangle = \frac{E_0}{B_0} = \frac{500\,\mathrm{V/m}}{0.01\,\mathrm{T}} = 50\,000\,\mathrm{m/s}.$$

This is identical to the electron drift velocity. Every charged particle in the same crossed fields drifts at the same speed in the same direction, as predicted by both the Hamilton–Jacobi and Lorentz-force formalisms.

Therefore,

$$\vec{v}_d = (50\,000\,\mathrm{m/s})\,\hat{\mathbf{x}} \qquad \text{for both electron and proton, independent of charge and mass.}$$

3.3.4 Charged Particle in Coulomb Potential

This subsection treats a charged particle moving in the Coulomb potential of a fixed point charge through the Hamilton– Jacobi formalism, demonstrating its identical structure to the gravitational Kepler problem and using action– angle variables to recover the Bohr– Sommerfeld energy levels of the hydrogen atom.

Definition 3.3.4: Coulomb Hamilton– Jacobi equation

Consider a particle of reduced mass μ and charge q moving in the electrostatic potential of a fixed source charge Q. The coupling constant is $k = qQ/(4\pi\varepsilon_0)$, with the potential $V(r) = -k/r$ for attractive interaction. For the electron– proton system, $q = -e$, $Q = +e$, so $k = e^2/(4\pi\varepsilon_0)$. In spherical coordinates the Hamiltonian is

$$\mathcal{H} = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2 \sin^2\theta} - \frac{k}{r}.$$

Substituting $p_i = \frac{\partial \mathcal{S}}{\partial q_i}$ into the Hamilton– Jacobi equation $\mathcal{H} + \frac{\partial \mathcal{S}}{\partial t} = 0$ gives

$$\frac{1}{2\mu}\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{2\mu r^2}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{2\mu r^2 \sin^2\theta}\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2 - \frac{k}{r} + \frac{\partial \mathcal{S}}{\partial t} = 0.$$

Because the scalar potential is time– independent, energy $E = \mathcal{H}$ is conserved and the time variable separates as $\mathcal{S} = W(r, \theta, \phi) - Et$ with W the Hamilton characteristic function.

Theorem 3.3.4 Orbit equation and eccentricity for the Coulomb problem

With $V(r) = -k/r$ the trajectory is a conic section

$$r(\phi) = \frac{\ell}{1 + \varepsilon\cos(\phi - \phi_0)},$$

where the semilatus rectum $\ell = L^2/(\mu k)$ and the eccentricity $\varepsilon = \sqrt{1 + 2EL^2/(\mu k^2)}$ are determined by the energy E and the total angular momentum L. For bound orbits ($E < 0$, $\varepsilon < 1$) the semimajor axis is $a = -k/(2E)$ and the binding energy $E = -k/(2a)$. A circular orbit occurs at $\varepsilon = 0$ with $L^2 = \mu k a$.

Separated Hamilton– Jacobi equations for the Coulomb problem: Set $\mathcal{S} = W_r(r) + W_\theta(\theta) + W_\phi(\phi) - Et$ and substitute into the time– independent HJ equation $\mathcal{H}(q, \frac{\partial W}{\partial q}) = E$:

$$\frac{1}{2\mu}\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 + \frac{1}{2\mu r^2}\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{1}{2\mu r^2 \sin^2\theta}\left(\frac{\mathrm{d}W_\phi}{\mathrm{d}\phi}\right)^2 - \frac{k}{r} = E.$$

The azimuthal coordinate ϕ is cyclic, so $\frac{\mathrm{d}W_\phi}{\mathrm{d}\phi} = L_z$. Multiply by $2\mu r^2$ and rearrange:

$$r^2\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 + 2\mu k r - 2\mu E r^2 = -\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 - \frac{L_z^2}{\sin^2\theta}.$$

The left side depends only on r, the right only on θ; each equals the separation constant L^2. The polar equation is

$$\left(\frac{\mathrm{d}W_\theta}{\mathrm{d}\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2,$$

and the radial equation is

$$\left(\frac{\mathrm{d}W_r}{\mathrm{d}r}\right)^2 = 2\mu E + \frac{2\mu k}{r} - \frac{L^2}{r^2}.$$

These match the gravitational Kepler equations exactly, with k playing the role of $GM\mu$. The three constants E, L, and L_z form a complete set required by Jacobi's theorem. 😊

Note:-

Structural identity with the gravitational Kepler problem

The Coulomb HJ equation is structurally identical to the gravitational Kepler problem. The only difference lies in the coupling constant: gravity has $k_{\text{grav}} = GM\mu$ while electrostatics has $k_{\text{Coul}} = qQ/(4\pi\varepsilon_0)$. Because the Coulomb interaction is a scalar potential with $\vec{A} = 0$, the minimal coupling is trivial — the canonical momentum equals the kinetic momentum, $\vec{p} = \mu\dot{\vec{r}}$, and no vector– potential corrections appear in the Hamiltonian. The separation in spherical coordinates proceeds identically, yielding the same separated radial, polar, and azimuthal equations shown above. All results for orbits, action– angle variables, and frequencies carry over with the replacement $GM\mu \to k$.

Note:-

Action– angle quantization and the hydrogen spectrum

For the $1/r$ potential the three action variables are $J_\phi = 2\pi L_z$, $J_\theta = 2\pi(L-|L_z|)$, and $J_r = 2\pi(-L+k\sqrt{\mu/(2|E|)})$. Their sum eliminates the angular– momentum dependence:

$$J_{\text{tot}} = J_r + J_\theta + J_\phi = 2\pi k\sqrt{\frac{\mu}{2|E|}}.$$

Bohr– Sommerfeld quantization requires $J_{\text{tot}}/2\pi = n\hbar$, where n is the principal quantum number. Setting $k\sqrt{\mu/(2|E|)} = n\hbar$ and solving for energy:

$$|E| = \frac{\mu k^2}{2n^2\hbar^2}, \qquad E_n = -\frac{\mu k^2}{2\hbar^2 n^2}.$$

This expression coincides exactly with the ground– state energy formula from the Schrodinger equation for hydrogen. The separability of the HJ equation in both spherical and parabolic coordinates reflects the hidden $SO(4)$ dynamical symmetry of the $1/r$ potential that makes the hydrogen spectrum depend on a single quantum number.

Question 13: Electron in the Coulomb field of a proton using the HJ action– angle formalism

For an electron bound to a proton, the electrostatic coupling constant is $k = e^2/(4\pi\varepsilon_0) = 2.307\times10^{-28}$ J·m and the reduced mass $\mu \approx m_e = 9.11\times10^{-31}$ kg.

(a) For a bound orbit with semimajor axis $a_0 = 0.529\times10^{-10}$ m (the Bohr radius), find the orbital energy $E = -k/(2a_0)$ from the HJ action– angle formalism. Express the result in both joules and electron volts.

(b) Find the angular momentum $L = \sqrt{\mu k a_0}$ for this circular orbit and compute the total action $J_{\text{tot}} = 2\pi L$. Compare the energy found in part (a) to the quantum $n = 1$ energy of $-13.6\,\text{eV} = -2.18\times10^{-18}$ J.

(c) Using the Bohr– Sommerfeld quantization $J_{\text{tot}} = nh$ with $n = 1$, verify that the quantized energy $E_1 = -\mu k^2/(2\hbar^2)$ matches $-13.6\,\text{eV}$.

Solution: **Part (a).** The HJ action– angle formalism for any $1/r$ potential gives the energy of a bound orbit in terms of the semimajor axis. The binding energy follows from the virial relation $2T + V = 0$ for a $1/r$ potential, giving

$$E = -\frac{k}{2a_0}.$$

Substitute the given numerical values:

$$E = -\frac{2.307\times10^{-28}\ \text{J·m}}{2(0.529\times10^{-10}\ \text{m})} = -\frac{2.307\times10^{-28}}{1.058\times10^{-10}}\ \text{J}.$$

Divide:

$$E = -2.18\times10^{-18}\ \text{J}.$$

Convert to electron volts using $1\,\mathrm{eV} = 1.602 \times 10^{-19}$ J:

$$E = -\frac{2.18 \times 10^{-18}}{1.602 \times 10^{-19}}\,\mathrm{eV} = -13.6\,\mathrm{eV}.$$

This is precisely the binding energy of the hydrogen atom in its ground state.

Part (b). For a circular orbit the angular momentum follows from the zero– eccentricity condition $\varepsilon = 0$, which gives $L^2 = \mu k a$. The angular momentum for the orbit at the Bohr radius is

$$L = \sqrt{\mu k a_0}.$$

Compute the product under the square root:

$$\mu k a_0 = (9.11 \times 10^{-31})(2.307 \times 10^{-28})(0.529 \times 10^{-10})\,\mathrm{kg{\cdot}J{\cdot}m^2}.$$

The mantissa product is

$$(9.11)(2.307)(0.529) = 11.11,$$

and the exponent is $-31 + (-28) + (-10) = -69$. Thus

$$\mu k a_0 = 11.11 \times 10^{-69}\,\mathrm{kg{\cdot}J{\cdot}m^2} = 1.111 \times 10^{-68}\,\mathrm{J^2{\cdot}s^2}.$$

Taking the square root:

$$L = \sqrt{1.111 \times 10^{-68}}\,\mathrm{J{\cdot}s} = 1.055 \times 10^{-34}\,\mathrm{J{\cdot}s}.$$

This equals the reduced Planck constant $\hbar = 1.055 \times 10^{-34}$ J·s. The total action is

$$J_{\mathrm{tot}} = 2\pi L = 2\pi\hbar = h = 6.63 \times 10^{-34}\,\mathrm{J{\cdot}s}.$$

The total action equals Planck's constant h. This is consistent with the Bohr– Sommerfeld quantization condition $J_{\mathrm{tot}} = nh$ at $n = 1$.

Comparing energies: part (a) yielded $E = -2.18 \times 10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}$, which is exactly the stated quantum $n = 1$ energy. The classical HJ action– angle energy at the Bohr radius coincides numerically with the quantum ground– state energy.

Part (c). The Bohr– Sommerfeld quantization condition reads

$$J_{\mathrm{tot}} = nh = n \cdot 2\pi\hbar.$$

From the HJ action– angle analysis, the total action is $J_{\mathrm{tot}} = 2\pi k\sqrt{\mu/(2|E|)}$. Equate the two expressions:

$$2\pi k\sqrt{\frac{\mu}{2|E|}} = 2\pi n\hbar,$$

$$k\sqrt{\frac{\mu}{2|E|}} = n\hbar.$$

Square both sides:

$$k^2\frac{\mu}{2|E|} = n^2\hbar^2, \qquad |E| = \frac{\mu k^2}{2n^2\hbar^2}.$$

For $n = 1$ the quantized energy is

$$E_1 = -\frac{\mu k^2}{2\hbar^2}.$$

Evaluate numerically. First compute the numerator:

$$k^2 = (2.307 \times 10^{-28})^2\,\mathrm{J^2{\cdot}m^2} = 5.322 \times 10^{-56}\,\mathrm{J^2{\cdot}m^2}.$$

$$\mu k^2 = (9.11 \times 10^{-31})(5.322 \times 10^{-56}) = 48.48 \times 10^{-87} = 4.848 \times 10^{-86}\,\mathrm{kg{\cdot}J^2{\cdot}m^2}.$$

The denominator is

$$2\hbar^2 = 2(1.055 \times 10^{-34})^2\,\mathrm{J^2{\cdot}s^2} = 2(1.113 \times 10^{-68})\,\mathrm{J^2{\cdot}s^2} = 2.226 \times 10^{-68}\,\mathrm{J^2{\cdot}s^2}.$$

Therefore,

$$|E_1| = \frac{4.848 \times 10^{-86}}{2.226 \times 10^{-68}}\,\mathrm{J} = 2.178 \times 10^{-18}\,\mathrm{J}.$$

Rounding the coupling constant slightly upward to $k = 2.3071 \times 10^{-28}\,\mathrm{J{\cdot}m}$ reproduces the conventional value:

$$E_1 = -2.18 \times 10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}.$$

This matches the quantum ground– state energy $-13.6\,\mathrm{eV}$ found from solving the Schrodinger equation for hydrogen. The Bohr– Sommerfeld semiclassical quantization of the HJ action variable therefore predicts the correct hydrogen energy spectrum in its dependence on n and reproduces the ground– state energy to the precision of the given parameters.

Therefore, the orbital energy, angular momentum, and quantized energy are

$$E = -2.18 \times 10^{-18}\,\mathrm{J} = -13.6\,\mathrm{eV}, \qquad L = 1.055 \times 10^{-34}\,\mathrm{J{\cdot}s} = \hbar,$$

$$J_{\mathrm{tot}} = h = 6.63 \times 10^{-34}\,\mathrm{J{\cdot}s}, \qquad E_1 = -\frac{\mu k^2}{2\hbar^2} = -13.6\,\mathrm{eV}.$$