Calculus III

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Contents

Chapter 1

Lab 7

1.1 Work

Question 1

$$\vec{r} \in \mathbb{R} \therefore t \geq 0 \because e^{\sqrt{t}} \notin \mathbb{R}\ t|t < 0$$

This corresponds with answer choice D

Question 2

$$\forall t > 4, \quad \vec{r} \cdot \hat{j} \notin \mathbb{R}$$

$\ln(t-1)$ is not defined $\forall t \leq 1$

This corresponds with answer choice D

Question 3

$$\vec{r} \cdot \hat{j} \in [-4, 4] \quad \wedge \quad \vec{r} \cdot \hat{i} \in [-3, 3]$$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle -3\sin t, 4\cos t \rangle$$

$$\left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{t=\frac{\pi}{2}} = \langle -3, 0 \rangle \quad \vec{r}(0) = \langle 3, 0 \rangle \qquad dotted$$

This corresponds with answer choice B

Question 4

This can be directly evaluated to:

$$\langle 28, -49 \rangle$$

This corresponds with answer choice D

Question 5

$\vec{r} \cdot \hat{j}$ is not defined for $t = 1$ but the limit as $t \to 6$ can be evaluated without affecting the proceess

$\vec{r} \cdot \hat{i}$ is not defined for $t = 1$ but the limit as $t \to 6$ can be evaluated without affecting the proceess

Using directevaluation:

$$\langle \frac{5}{35}, -\frac{36+12-3}{5} \rangle = \langle \tfrac{1}{7}, -9 \rangle$$

This corresponds with answer choice B

Question 6

$$\implies \langle 0, -6 \rangle$$

This corresponds with answer choice B

Question 7

$$e^{-\ln 6} = 6^{-1} = \tfrac{1}{6}$$

$$\implies \lim_{t \to \ln 6} \langle 6e^{-t}, 3e^{-t} \rangle = \langle 1, \tfrac{1}{2} \rangle$$

This corresponds witha nswer choice B

Question 8

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle -14t, \tfrac{1}{3}t^2 \rangle$$

This corresponds with answer choice C

Question 9

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle -\csc^2 t, -\cot t \csc t \rangle$$

This corresponds with answer choice A

Question 10

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 8te^{t^2}, -3, 2t \rangle$$

This corresponds with answer choice B

Question 11

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 18\frac{1}{6t}, 6t^2\rangle$$
$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \langle\frac{-108}{36t^2}, 12t\rangle\langle\frac{-3}{t^2}, 12t\rangle$$

This corresponds with answer choice C

Question 12

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 15t^4, -60t^4, 20t^4\rangle$$
$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\|$$
$$= \sqrt{15^2 \cdot t^8 + 60^2 \cdot t^8 + 20^2 \cdot t^8}$$
$$= t^4\sqrt{15^2 + 60^2 + 20^2} = 65t^4$$
$$\hat{T} = \langle\tfrac{15}{65}, \tfrac{-60}{65}, \tfrac{20}{65}\rangle$$

This corresponds with answer choice B

Question 13

$$\vec{r}(t) = 6\sin^3(2t)\,\hat{i} + 6\cos^3(2t)\,\hat{j}$$
$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 36\sin^2(2t)\cos(2t),\ -36\cos^2(2t)\sin(2t)\rangle$$
$$= 36\sin(2t)\cos(2t)\,(\sin(2t)\,\hat{i} - \cos(2t)\,\hat{j})$$

$$\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\| = \sqrt{(36\sin^2(2t)\cos(2t))^2 + (-36\cos^2(2t)\sin(2t))^2}$$
$$= 36\sqrt{\sin^4(2t)\cos^2(2t) + \cos^4(2t)\sin^2(2t)}$$
$$= 36\sqrt{\sin^2(2t)\cos^2(2t)}$$
$$= 36\,|\sin(2t)\cos(2t)|$$
$$= 18\,|\sin(4t)|$$

$$\hat{T}(t) = \frac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|} = \frac{36\sin(2t)\cos(2t)(\sin(2t)\,\hat{i} - \cos(2t)\,\hat{j})}{36|\sin(2t)\cos(2t)|}$$
$$= \sin(2t)\,\hat{i} - \cos(2t)\,\hat{j}$$

This corresponds with answer choice B.

Question 14

Let $\vec{C}$ encode the components of the constant of integration such that $\vec{C} \in \mathbb{R}^3$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle t^4 + 5t^2, 4t \rangle$$
$$\int \mathrm{d}\vec{r} = \int \langle t^4 + 5t^2, 4t \rangle \mathrm{d}t$$
$$\vec{r}(t) = \langle \tfrac{1}{5}t^5 + \tfrac{5}{3}t^3, 2t^2 \rangle + \vec{C}$$
$$\vec{C} = \vec{r}(0) = \langle 5, -6 \rangle$$

This corresponds with answer choice C.

Question 15

$$\implies \int_0^3 \left\langle \frac{2}{\sqrt{1+t}}, -9t^2, \frac{6t}{(1+t^2)^2} \right\rangle \mathrm{d}t$$
$$\implies \int_0^3 \left\langle 2(1+t)^{-\frac{1}{2}}, -9t^2, \frac{6t}{(1+t^2)^2} \right\rangle \mathrm{d}t$$
$$\implies \left[\langle 4\sqrt{t+1}, -3t^3, \frac{-3}{1+t^2} \right]_0^3 = \langle 8-4, -81, \tfrac{-3}{10} + 3 \rangle = \langle 4, -81, \tfrac{27}{10} \rangle$$

This corresponds with answer choice C.

Question 16

$$u := t^2 + 1 \quad \mathrm{d}u = 2t\mathrm{d}t \qquad \implies \left\langle \left[8t^3 - 3t \right]_0^1, \int_0^1 \frac{4}{u}\mathrm{d}u, -\int_0^1 \frac{1}{2\sqrt{u}}\mathrm{d}u \right\rangle$$
$$\left[\left\langle 8t^3 - 3t, 4\ln(t^2+1), -\sqrt{t^2+1} \right\rangle \right]_0^1 = \langle 8-3, 4\ln 2 - 0, -\sqrt{2} + 1 \rangle$$

This corresponds with answer choice A.

Chapter 2

Lab 8

2.1 Work

Question 1

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \langle -14t, \tfrac{1}{7}t^2 \rangle$$

This corresponds with answer choice B.

Question 2

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \langle -3\sin(3t), 5\cos t \rangle$$
$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \vec{a} = \langle -9\cos(3t), -5\sin t \rangle$$

This corresponds with answer choice D.

Question 3

$$\vec{v}(t) = \langle 18t + 4, -15t^2, -2t \rangle$$
$$\vec{v}(3) = \langle 58, -135, -6 \rangle$$

This correspon ds with answer choice B.

Question 4

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \langle -2\sin(2t), \frac{7}{t-3}, -\tfrac{1}{3}t^2 \rangle$$
$$\vec{v}(0) = \langle 0, \tfrac{-7}{3}, 0 \rangle$$

This corresponds with answer choice D.

Question 5

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \langle 8\cos(2t),\ 10\sin(2t),\ -6\csc(2t)\cot(2t)\rangle$$
$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \vec{a} = \langle -16\sin(2t),\ 20\cos(2t),\ 12(\csc(2t)\cot^2(2t) + \csc^3(2t))\rangle$$
$$\vec{a}\left(\tfrac{\pi}{4}\right) = \langle -16\sin\left(\tfrac{\pi}{2}\right),\ 20\cos\left(\tfrac{\pi}{2}\right),\ 12(1\cdot 0^2 + 1^3)\rangle$$
$$= \langle -16,\ 0,\ 12\rangle$$

This corresponds with answer choice B.

Question 6

$$\vec{v}(0) = \langle \sqrt{2}, 0, \sqrt{2}\rangle, \qquad \vec{a}(0) = \langle 0, 0, \tfrac{\pi}{2}\rangle,$$
$$\vec{a}\cdot\vec{v} = 0\cdot\sqrt{2} + 0\cdot 0 + \tfrac{\pi}{2}\cdot\sqrt{2} = \tfrac{\pi\sqrt{2}}{2},$$
$$\|\vec{a}\| = \tfrac{\pi}{2}, \qquad \|\vec{v}\| = \sqrt{(\sqrt{2})^2 + 0^2 + (\sqrt{2})^2} = \sqrt{4} = 2,$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{v}}{\|\vec{a}\|\|\vec{v}\|} = \frac{\frac{\pi\sqrt{2}}{2}}{\frac{\pi}{2}\cdot 2} = \frac{\sqrt{2}}{2},$$
$$\theta = \arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

Question 7

$$\vec{v}_0 = \langle 75\cos\alpha, 75\sin\alpha\rangle \quad \vec{a} = \langle 0, -32\rangle$$

This corresponds with answer choice A.

Question 8

$$\vec{v}_0 = \langle 800; 34^\circ\rangle \quad \vec{a} = \langle 0, -9.8\rangle$$
$$\vec{r}\cdot\hat{i} = 800\cos 34^\circ t \implies \frac{20000}{800\cdot\cos 34^\circ} = t_h = 30.1554487126\text{ s}$$
$$\vec{r}(t_h)\cdot\hat{j} = 800\sin 34^\circ 30.1554487126 - \tfrac{1}{2}9.8\cdot 30.1554487126^2$$
$$\vec{r}(t_h)\cdot\hat{j} = 9034.35001026 \geq 0$$

The answer is B.

Question 9

$$\vec{v}(t) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 585\tfrac{\sqrt{2}}{2}, 585\tfrac{\sqrt{2}}{2} - 9.8t\rangle$$
$$\vec{v}(t)\cdot\hat{j} = 0 \implies t_h = \frac{585\frac{\sqrt{2}}{2}}{9.8} \implies t_h = 42.2099456116$$
$$\vec{r}(t_h)\cdot\hat{j} = 585\tfrac{\sqrt{2}}{2}t_h - \tfrac{1}{2}9.8t_h^2 = 8730.22959184 \text{ m}$$

This corresponds with answer choice D.

Question 10

$$12 = \|\vec{v}\|\cos 33^\circ t \quad 0 = \|\vec{v}\|\sin 33^\circ t - \tfrac{1}{2}9.8t^2$$
$$\text{Computationally solving this system of equations: } \|\vec{v}\| = 11.34589$$

This corresponds with answewr choice D.

Question 11

$$\vec{r}(t)\cdot\hat{j} = -\tfrac{1}{2}32t^2 + 115\sin 44^\circ t + 6.6$$
$$\vec{r}(4) = 70.1428504111 \text{ ft}$$

This corresponds with answer choice C.

Question 12

$$\vec{r}(t)\cdot\hat{j} = -\tfrac{1}{2}32t^2 + 47\sin 39^\circ t + 6.1$$
$$\vec{r}(t_f)\cdot\hat{j} = 0\ t_f = 2.03589 \text{ s}$$
$$\vec{r}(t_f)\cdot\hat{i} = 2.03589\cdot 47\cos 39^\circ = 74.3626334991 \text{ ft}$$

This corresponds with answer choice C.

Question 13

$$0 = 30 + 39\sin 26^\circ t - \tfrac{1}{2}32t^2$$
$$\implies t = 2.00411 \text{ s}$$

This corresponds with answer choice B.

Question 14

$$x_{max} = \frac{\|\vec{v}_0\|^2\sin 2\alpha}{g} = 14000 = \frac{380^2\sin 2\alpha}{9.8}$$
$$\implies \alpha = \tfrac{1}{2}\arcsin\frac{14000\cdot 9.8}{380^2}$$

This corresponds with answer choice C.

Chapter 3

Lab 9

3.1 Work

Question 1

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle -2, 6, 9 \rangle \quad \|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{2^2 + 6^2 + 9^2} = \sqrt{4 + 36 + 81} = 11$$
$$L = \int_a^b \|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| \mathrm{d}t = \int_{-10}^{-3} 11 \mathrm{d}t = 11 \, [t]_{-10}^{-3} = 11(-3 + 10) = 77$$

This corresponds with answer choice C.

Question 2

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 2\sin t + 2t\cos t - 2\sin t, 2\cos t - 2t\sin t - 2\cos t \rangle$$
$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{(2t)^2\cos^2 t + (2t)^2\sin^2 t} = 2t$$
$$L = \int_a^b \|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| \mathrm{d}t = \int_{-1}^{7} 2t \mathrm{d}t = \left[t^2\right]_{-1}^{7} = [49 - 1] = 48$$

This corresponds with answer choice C.

Question 3

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 30t^2, 33t^2, 6t^2 \rangle$$
$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{30^2 \cdot t^4 + 33^2 \cdot t^4 + 6^2 \cdot t^4} = t^2\sqrt{30^2 + 33^2 + 6^2} = 45t^2$$
$$L = \int_a^b \|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| \mathrm{d}t = \int_{-1}^{2} 45t^2 = 15 \left[t^2\right]_{-1}^{2} = 15(4 - 1) = 45$$

This corresponds with answer choice B.

Question 4

$$\frac{d\vec{r}}{dt} = \langle 0, -18 \cdot 7\cos^2 7t \sin 7t, 18 \cdot 7 \sin^2 7t \cos 7t\rangle = \langle 0, -126\cos^2 7t \sin 7t, 126 \sin^2 7t \cos 7t\rangle$$

$$\|\frac{d\vec{r}}{dt}\| = \sqrt{126^2 \cos^4 7t \sin^2 7t + 126^2 \sin^4 7t \cos^2 27}$$

$$\implies 126\sqrt{(\sin^2 7t \cos^2 7t)(\cos^2 7t + \sin^2 7t)} = 126\sqrt{\sin^2 7t \cos^2 7t} = 126 \sin 7t \cos 7t$$

$$L = \int_a^b \|\frac{d\vec{r}}{dt}\| dt = \int_{\frac{6\pi}{7}}^{\frac{13\pi}{14}} 126 \sin 7t \cos 7t dt = 63 \int_{\frac{6\pi}{7}}^{\frac{13\pi}{14}} \sin 14t dt$$

$$u = 14t \quad du = 14dt \qquad \implies L = \frac{-63}{14}\left[\cos 14t\right]_{\frac{6\pi}{7}}^{\frac{13\pi}{14}} = \frac{-9}{2} = \frac{-9}{2}(-1-1) = 9$$

This corresponds with answer choice B.

Question 5

$$\frac{d\vec{r}}{dt} = \langle e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, 5e^t\rangle$$

$$\|\frac{d\vec{r}}{dt}\| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 25e^{2t}} = e^t\sqrt{1 + \sin 2t + 1 - \sin 2t + 25} = e^t\sqrt{27} = 3\sqrt{3}e^t$$

$$L = \int_a^b \|\frac{d\vec{r}}{dt}\| dt = 3\sqrt{3}\int_{-\ln 2}^{0} e^t dt = 3\sqrt{3}(1 - \tfrac{1}{2}) = \frac{3\sqrt{3}}{2}$$

This corresponds with answer choice D.

Question 6

$$\frac{d\vec{r}}{dt} = \langle 6t^2, 6t^2, -3t^2\rangle$$

$$\|\frac{d\vec{r}}{dt}\| = \sqrt{36t^4 + 36t^4 + 9t^4} = t^2\sqrt{36 + 36 + 9} = 9t^2$$

$$L = \int_a^b \|\frac{d\vec{r}}{dt}\| dt = 3\int_{-1}^{2} 3t^2 dt$$

$$\implies 3\left[t^3\right]_{-1}^{2} = 3(8 + 1) = 27$$

This corresponds with answer choice B.

Question 7

$$\frac{d\vec{r}}{dt} = \langle 4\cos 2t, -4\sin 2t, 3\rangle$$

$$\|\frac{d\vec{r}}{dt}\| = \sqrt{16\cos^2 2t + 16 \sin^2 2t + 9} = \sqrt{16 + 9} = 5$$

$$\vec{T}(t) = \tfrac{1}{5}\langle 4\cos 2t, -4\sin 2t, 3\rangle$$

This corresponds with answer choice A.

Question 8

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 84t^6, -28t^6, 21t^6 \rangle$$

$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = t^6\sqrt{84^2 + 28^2 + 21^2} = 91t^6$$

$$\vec{T}(t) = \tfrac{1}{91}\langle 84t^6, -28t^6, 21t^6 \rangle$$

This corresponds with answer choice B.

Question 9

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 0, 6\cos t - 6t\sin t - 6\cos t, 6t\cos t + 6\sin t - 6\sin t \rangle$$

$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{36t^2\sin^2 t + 36t^2\cos^2 t} = 6t$$

$$\vec{T}(t) = \frac{1}{6t}\langle 0, -6t\sin t, 6t\cos t \rangle = \langle 0, -\sin t, \cos t \rangle$$

This corresponds with answer choice D.

Question 10

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle -54\sin 9t, -54\cos 9t, 0 \rangle$$

$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = 54\sqrt{\sin^2 9t + \cos^2 9t} = 54$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \langle -486\cos 9t, 486\sin 9t, 0 \rangle$$

$$\kappa = \frac{\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\|}{\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\|^3}$$

$$\implies \kappa = \frac{\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -54\sin 9t & -54\cos 9t & 0 \\ -486\cos 9t & 486\sin 9t & 0 \end{vmatrix}\right\|}{54^3} = -26244 \cdot \tfrac{1}{157464}(\cos^2 9t + \sin^2 9t)$$

$$\implies \kappa = \frac{1}{6}$$

This corresponds with answer choice C.

Question 11

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 1, 0, \frac{\cancel{\sec t}\tan t}{\cancel{\sec t}} \rangle$$

$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{1+\tan^2 t} = \sqrt{\sec^2 t} = \sec t$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \langle 0, 0, \sec^2 t \rangle$$

$$\kappa = \frac{\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\|}{\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\|^3}$$

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \tan t \\ 0 & 0 & \sec^2 t \end{vmatrix} = -\hat{j}[\sec^2 t] = \langle 0, -\sec^2 t, 0 \rangle$$

$$\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\| = \sec^2 t$$

$$\implies \kappa = \frac{\sec^2 t}{\sec^3 t} = \cos t$$

Chapter 4

Lab 10

4.1 Work

Question 1

$$\begin{aligned}
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 1, 0, \frac{-\sin t}{\cos t} \rangle = \langle 1, 0, -\tan t \rangle \\
\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\| = \sqrt{1 + \tan^2 t} = \sec t \\
\vec{T}(t) = \cos t \cdot \langle 1, 0, -\tan t \rangle = \langle \cos t, 0, -\sin t \rangle \\
\implies \vec{T}\prime(t) = \langle -\sin t, 0, -\cos t \rangle \\
\implies \|\vec{T}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1 \\
\vec{N}(t) = \langle -\sin t, 0, -\cos t \rangle
\end{aligned}$$

Corresponds with answer choice D.

Question 2

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 0, 2t, 2\rangle$$
$$\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\| = 2\sqrt{t^2+1}$$
$$\vec{T}(t) = \frac{\langle 0, t, 1\rangle}{\sqrt{t^2+1}}$$
$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \left\langle 0,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{\sqrt{t^2+1}}\right),\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\sqrt{t^2+1}}\right)\right\rangle$$
$$= \left\langle 0,\ (t^2+1)^{-1/2} - \frac{t^2}{(t^2+1)^{3/2}},\ -\frac{t}{(t^2+1)^{3/2}}\right\rangle$$
$$= (t^2+1)^{-3/2}\langle 0, 1, -t\rangle$$
$$\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| = (t^2+1)^{-1}$$
$$\vec{N}(t) = \frac{\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}}{\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\|} = \frac{(t^2+1)^{-3/2}\langle 0,1,-t\rangle}{(t^2+1)^{-1}} = \frac{\langle 0,1,-t\rangle}{\sqrt{t^2+1}}.$$

This corresponds with answer choice B.

Question 3

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \langle 2t\cos t + \cancel{2\sin t - 2\sin t}, 0, \cancel{2\cos t} - 2t\sin t - \cancel{2\cos t}\rangle$$
$$\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\| = 2t\sqrt{\cos^2 t + \sin^2 t} = 2t$$
$$\vec{T}(t) = \langle \cos t, 0, -\sin t\rangle$$
$$\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} = \langle -\sin t, 0, -\cos t\rangle$$
$$\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| = 1$$
$$\vec{N}(t) = \langle -\sin t, 0, -\cos t\rangle$$

This corresponds with answer choice C.

Question 4

$$
\begin{aligned}
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \langle 4\cos 2t, -4\sin 2t, 3\rangle \\
\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\| &= \sqrt{4^2(\cos^2 2t + \sin^2 2t) + 3^2} = 5 \\
\vec{T}(t) &= \langle \tfrac{4}{5}\cos 2t, -\tfrac{4}{5}\sin 2t, \tfrac{3}{5}\rangle \\
\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} &= \langle -\tfrac{8}{5}\sin 2t, -\tfrac{8}{5}\cos 2t, 0\rangle \\
\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| &= \frac{8}{5} \\
\vec{N}(t) &= \langle -\sin 2t, -\cos 2t, 0\rangle
\end{aligned}
$$

This corresponds with answer choice C.

Question 5

$$
\begin{aligned}
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \langle 6\cos t, -6\sin t, -1\rangle \\
\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right\| &= \sqrt{6^2 + 1} = \sqrt{37} \\
\vec{T}(t) &= \frac{1}{\sqrt{37}}\langle 6\cos t, -6\sin t, -1\rangle \\
\frac{\mathrm{d}\vec{T}}{\mathrm{d}t} &= \frac{1}{\sqrt{37}}\langle -6\sin t, -6\cos t, 0\rangle \\
\left\|\frac{\mathrm{d}\vec{T}}{\mathrm{d}t}\right\| &= \sqrt{\frac{6^2}{\sqrt{37}^2}(\sin^2 t + \cos^2 t)} = \frac{6}{\sqrt{37}} \\
\vec{N}(t) &= \frac{1}{\sqrt{37}}\langle -\sin t, -\cos t, 0\rangle \\
\vec{B}(t) = \vec{T}(t)\times\vec{N}(t) &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{6}{\sqrt{37}}\cos t & \frac{-6}{\sqrt{37}}\sin t & \frac{-1}{\sqrt{37}} \\ \frac{-1}{\sqrt{37}}\sin t & \frac{-1}{\sqrt{37}}\cos t & 0 \end{vmatrix} \\
\vec{B}(t) &= \frac{-\langle \cos t, -\sin t, 6\rangle}{\sqrt{37}}
\end{aligned}
$$

This corresponds with answer choice D.

Question 6

$$
\begin{aligned}
\frac{d\vec{r}}{dt} &= \langle t, -7, 0\rangle \\
\left\|\frac{d\vec{r}}{dt}\right\| &= \sqrt{t^2+49} \\
\vec{T}(t) &= \langle \frac{t}{\sqrt{t^2+49}}, \frac{-7}{\sqrt{t^2+49}}, 0\rangle \\
\frac{d\vec{T}}{dt} &= \left\langle \frac{\sqrt{t^2+49} - t\cdot \frac{t}{\sqrt{t^2+49}}}{t^2+49}, \frac{0-(-7)\cdot \frac{t}{\sqrt{t^2+49}}}{t^2+49}, 0\right\rangle \\
&= \left\langle \frac{49}{(t^2+49)^{3/2}}, \frac{7t}{(t^2+49)^{3/2}}, 0\right\rangle \\
\left\|\frac{d\vec{T}}{dt}\right\| &= \frac{\sqrt{49^2+(7t)^2}}{(t^2+49)^{3/2}} = \frac{7\sqrt{t^2+49}}{(t^2+49)^{3/2}} = \frac{7}{t^2+49} \\
\vec{N}(t) &= \frac{\frac{d\vec{T}}{dt}}{\left\|\frac{d\vec{T}}{dt}\right\|} = \frac{\langle 49, 7t, 0\rangle/(t^2+49)^{3/2}}{7/(t^2+49)} = \frac{\langle 7, t, 0\rangle}{\sqrt{t^2+49}} \\
\vec{B}(t) &= \vec{T}(t)\times\vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t/\sqrt{t^2+49} & -7/\sqrt{t^2+49} & 0 \\ 7/\sqrt{t^2+49} & t/\sqrt{t^2+49} & 0 \end{vmatrix} \\
&= \frac{1}{t^2+49}\langle 0, 0, t^2+49\rangle = \langle 0, 0, 1\rangle
\end{aligned}
$$

This corresponds with answer choice D.

Question 7

$$
\begin{aligned}
\frac{d\vec{r}}{dt} &= \langle 1, \frac{\cancel{\sec t}\tan t}{\cancel{\sec t}}, 0\rangle \\
\left\|\frac{d\vec{r}}{dt}\right\| &= \sqrt{1+\tan^2 t} = \sec t \\
\vec{T}(t) &= \langle \cos t, \sin t, 0\rangle \\
\frac{d\vec{T}}{dt} &= \langle -\sin t, \cos t, 0\rangle \\
\left\|\frac{d\vec{T}}{dt}\right\| &= 1 \\
\vec{N}(t) &= \langle -\sin t, \cos t, 0\rangle \\
\vec{B}(t) &= \vec{T}(t)\times\vec{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \end{vmatrix} \\
&= \hat{k}\left[\cos^2 t + \sin^2 t\right] \\
&= \hat{k}
\end{aligned}
$$

This corresponds with answer choice C.

Definition 4.1.1: Torsion

$$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\left\|\vec{r}' \times \vec{r}''\right\|^2}$$

Question 8

$$\begin{aligned}
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \langle 4\cos\tfrac{2}{5}t, 3, -4\sin\tfrac{2}{5}t\rangle \\
\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= \langle -\tfrac{8}{5}\sin\tfrac{2}{5}t, 0, -\tfrac{8}{5}\cos\tfrac{2}{5}t\rangle \\
\frac{\mathrm{d}^3\vec{r}}{\mathrm{d}t^3} &= \langle -\tfrac{16}{25}\cos\tfrac{2}{5}t, 0, \tfrac{16}{25}\sin\tfrac{2}{5}t\rangle \\
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4\cos\frac{2}{5}t & 3 & -4\sin\frac{2}{5}t \\ -\frac{8}{5}\sin\frac{2}{5}t & 0 & -\frac{8}{5}\cos\frac{2}{5}t \end{vmatrix} \\
&= \hat{i}\left[-\tfrac{24}{5}\cos\tfrac{2}{5}t\right] - \hat{j}\left[-\tfrac{32}{5}\cos^2\tfrac{2}{5}t - \tfrac{32}{5}\sin^2\tfrac{2}{5}t\right] + \hat{k}\left[\tfrac{24}{5}\sin\tfrac{2}{5}t\right] \\
\left\|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right\| &= \sqrt{(\tfrac{24}{5})^2 + (\tfrac{32}{5})^2} = 8 \\
\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\left\|\vec{r}' \times \vec{r}''\right\|^2} &= \frac{\langle -\frac{24}{5}\cos\frac{2}{5}t, \frac{32}{5}, \frac{24}{5}\sin\frac{2}{5}t\rangle \cdot \langle -\frac{-16}{25}\cos\frac{2}{5}t, 0, \frac{16}{25}\sin\frac{2}{5}t\rangle}{8^2} \\
&= \left(-\frac{24}{5}\cos\tfrac{2}{5}t\right)\left(-\frac{16}{25}\cos\tfrac{2}{5}t\right) + \left(\frac{32}{5}\right)(0) + \left(\frac{24}{5}\sin\tfrac{2}{5}t\right)\left(\frac{16}{25}\sin\tfrac{2}{5}t\right) \\
&= \frac{384}{125}\cos^2\tfrac{2}{5}t + \frac{384}{125}\sin^2\tfrac{2}{5}t \\
&= \frac{384}{125} \\
\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2} &= \frac{384/125}{8^2} = \frac{384}{10000} = \frac{6}{125}.
\end{aligned}$$

This corresponds with answer choice A.

Question 9

$$\begin{aligned}
\vec{r}(t) &= \langle 2t-7,\ t^2-2,\ 7\rangle \\
\vec{r}'(t) &= \langle 2,\ 2t,\ 0\rangle \\
\vec{r}''(t) &= \langle 0,\ 2,\ 0\rangle \\
\vec{r}'''(t) &= \langle 0,\ 0,\ 0\rangle
\end{aligned}$$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = \langle 0, 0, 4\rangle$$

$$(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' = \langle 0,0,4\rangle \cdot \langle 0,0,0\rangle = 0$$

$$\tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{\|\vec{r}' \times \vec{r}''\|^2} = \frac{0}{\|\langle 0,0,4\rangle\|^2} = 0.$$

This corresponds with answer choice A.

Question 10

$$
\begin{aligned}
\vec{r}(t) &= \langle 3\sin t,\ 3\cos t,\ -t\rangle \\
\vec{r}'(t) &= \langle 3\cos t,\ -3\sin t,\ -1\rangle \\
\vec{r}''(t) &= \langle -3\sin t,\ -3\cos t,\ 0\rangle \\
\vec{r}'''(t) &= \langle -3\cos t,\ 3\sin t,\ 0\rangle \\
\vec{r}' \times \vec{r}'' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & -3\sin t & -1 \\ -3\sin t & -3\cos t & 0 \end{vmatrix} = \langle -3\cos t,\ 3\sin t,\ -9\rangle \\
\|\vec{r}' \times \vec{r}''\|^2 &= (-3\cos t)^2 + (3\sin t)^2 + (-9)^2 = 9 + 81 = 90 \\
(\vec{r}' \times \vec{r}'') \cdot \vec{r}''' &= (-3\cos t)(-3\cos t) + (3\sin t)(3\sin t) + (-9)(0) = 9(\cos^2 t + \sin^2 t) = 9 \\
\tau &= \frac{9}{90} = \frac{1}{10}.
\end{aligned}
$$

This corresponds with answer choice B.